Available online at www.sciencedirect.com

SCIENCE DIRECT

Journal of Algebra 295 (2005) 387–414

JOURNAL OF Algebra

www.elsevier.com/locate/jalgebra

# The smallest Mealy automaton of intermediate growth

L. Bartholdi [a,*], I.I. Reznykov [b], V.I. Sushchansky [c]

[a] *École Polytechnique Fédérale de Lausanne (EPFL), Institut de Mathématiques B (IMB), CH-1015 Lausanne, Switzerland*

[b] *IKC5 Ltd., 5, Krasnogvardeyskaya str., office 2, Kyiv 02094, Ukraine*

[c] *Institut of Mathematics, Silesian University of Technology, Ul. Kaszubska, 23, 44-100 Gliwice, Poland*

Received 22 March 2004

Available online 9 December 2005

Communicated by Corrado de Concini

**Abstract**

In this paper we study the automaton $I_2$, the smallest Mealy automaton of intermediate growth, first considered by the last two authors [I.I. Reznykov, V.I. Sushchansky, The two-state Mealy automata over the two-symbol alphabet of the intermediate growth, Mat. Zametki 72 (2002) 102–117]. We describe the automatic transformation monoid defined by $I_2$, give a formula for the generating series for the (ball volume) growth function of $I_2$, and give sharp asymptotics for the growth function of $I_2$, namely

$$\gamma_{I_2}(n) \sim 2^{5/2} 3^{3/4} \pi^{-2} n^{1/4} \exp\left(\pi\sqrt{n/6}\right),$$

with the ratios of left- to right-hand side tending to 1 as $n \to \infty$.


* Corresponding author.
*E-mail addresses:* laurent.bartholdi@google.com (L. Bartholdi), illya.reznykov@ikc5.com.ua (I.I. Reznykov), wital.suszczanski@polsl.pl (V.I. Sushchansky).

0021-8693/$ – see front matter 
doi:10.1016/j.jalgebra.2004.08.040

**Contents**



## 1. Introduction

The growth of a Mealy automaton is defined as the growth of the number of pairwise inequivalent internal states of iterates of that automaton. This notion of growth was introduced by R.I. Grigorchuk in [6], for related growth notions see [19]. The growth function of an arbitrary Mealy automaton coincides with the spherical growth function of the automatic transformation semigroup it defines, and actually the growth of automata are calculated by investigating the growth of the corresponding automatic transformation semigroups.

The automatic transformation groups defined by invertible 2-state Mealy automata over the 2-symbol alphabet were described in [7]. The automatic transformation semigroups defined by all 2-state Mealy automata over the 2-symbol alphabet were investigated in [14] and in the papers [15–17].

Among these semigroups there are twelve finite semigroups, seven semigroups of polynomial growth, one semigroup of intermediate growth, and eight semigroups of exponential growth, including the free semigroup. There are four pairwise similar (in the sense of Definition 8) 2-state Mealy automata over the 2-symbol alphabet of intermediate growth order, and these automata define isomorphic automatic transformation semigroups. One of these automata was considered in [14,17]. There, an automatic transformation semigroup of intermediate growth was constructed, with an exact formula for the growth function, expressed as an infinite sum. Its growth order was estimated between $[e^{\sqrt[4]{n}}]$ and $[e^{\sqrt{n}}]$.

In this paper we consider the automaton of intermediate growth $I_2$ and the semigroup of automatic transformations $S_{I_2}$ that it defines. In Theorem 1 we describe the semigroup $S_{I_2}$ and its quotient semigroups, in Theorem 2 we exhibit the growth series of the automaton and the semigroup, and in Theorem 3 we derive sharp asymptotics for the growth functions. The first part of Theorem 1 was proved in [14,17], but we give here a shorter proof, and a

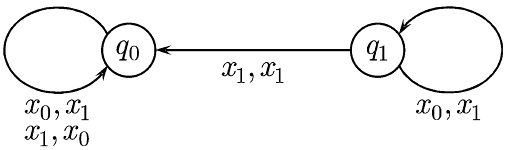


Fig. 1. The automaton $I_2$.

new proof of the minimality of the system of defining relations. Moreover, the other results are new.

There are various motivations for the precise study of growth functions of semigroups generated by automata. The first, and in some sense only, known examples of groups of intermediate growth come from automata [5], and these groups' structure can at least partly be understood through their growth. Also, the natural algebraic object associated to a Mealy automaton is a semigroup, which is a group only under an additional assumption. Furthermore, it seems beyond reach to obtain as sharp results as those of this paper for even the simplest known groups of intermediate growth.

Finally, a word should be added as to what is meant by deriving an "exact formula" for the growth of a semigroup, that is not tautological. The formulae we obtain in this paper have the merits of being easily and quickly computable, and of being expressible algebraically in terms of the partition function. This is certainly the most that can be hoped from a transcendental generating series.

## 2. Main results

Let $I_2$ be the 2-state Mealy automaton over the 2-symbol alphabet whose Moore diagram is shown on Fig. 1. Let us denote the semigroup defined by $I_2$ by the symbol $S_{I_2}$, and the growth functions of $I_2$ and $S_{I_2}$ by the symbols $\gamma_{I_2}$ and $\gamma_{S_{I_2}}$, respectively. Let us denote for each $n \in \mathbb{N}$ the quotient semigroup given by the representation of $I_2$ as maps from $\{x_0, x_1\}^n$ to itself by the symbol $W_n$. The following theorem holds:

**Theorem 1.**

(1) *The semigroup $S_{I_2}$ is a monoid, and has the following presentation* [14,17]:

$$S_{I_2} = \langle f_0, f_1 \mid f_0^2 = 1;\ f_1(f_0 f_1)^p (f_1 f_0)^p f_1^2 = f_1(f_0 f_1)^p (f_1 f_0)^p,\ p \geqslant 0\rangle. \quad (1)$$

*The monoid $S_{I_2}$ is infinitely presented, and its word problem is solvable in polynomial time.*

(2) *The semigroup $W_n$, $n \in \mathbb{N}$, has the presentation*

$$W_n = \left\langle f_0, f_1 \;\middle|\; \begin{array}{l} f_0^2 = 1; \\ f_1(f_0 f_1)^p (f_1 f_0)^p f_1^2 = f_1(f_0 f_1)^p (f_1 f_0)^p,\ 0 \leqslant p \leqslant n-2; \\ f_1(f_0 f_1)^{n-1} f_1 = f_1(f_0 f_1)^{n-1} f_0 = f_1(f_0 f_1)^{n-1} \end{array} \right\rangle.$$

The following corollary follows (for relevant definitions see Section 3.6):

**Corollary 1.** *The semigroup $S_{I_2}$ has Hausdorff dimension* 0.

**Theorem 2.**

(1) *The word growth series $\Delta_{S_{I_2}}(X) = \sum_{n\geqslant 0} \delta_{S_{I_2}}(n)X^n$ of $S_{I_2}$ admits the description*

$$\Delta_{S_{I_2}}(X) = (1+X)\left(1+\frac{X}{1-X}\prod_{n\geqslant 0}\bigl(1+X^{2n+1}\bigr)\right).$$

(2) *The growth series $\Gamma_{I_2}(X) = \sum_{n\geqslant 0} \gamma_{I_2}(n)X^n$ of $I_2$ admits the description*

$$\Gamma_{I_2}(X) = \frac{1}{1-X}\left(1+\frac{X}{1-X}\prod_{n\geqslant 0}\bigl(1+X^{2n+1}\bigr)\right).$$

(3) *The growth series $\Gamma_{S_{I_2}}(X) = \sum_{n\geqslant 0} \gamma_{S_{I_2}}(n)X^n$ of $S_{I_2}$ admits the description*

$$\Gamma_{S_{I_2}}(X) = \frac{1+X}{1-X}\left(1+\frac{X}{1-X}\prod_{n\geqslant 0}\bigl(1+X^{2n+1}\bigr)\right).$$

Let us denote the number of all partitions of a positive integer $n$ into $k$ odd parts by the symbol $q(n)$.

**Theorem 3.** *The growth functions have the following sharp estimates*:

$$\delta_{S_{I_2}}(n) \sim \frac{4\sqrt{6}}{\pi}\sqrt{n}\cdot q(n) \sim \frac{2^2 3^{1/4}}{\pi} n^{-1/4}\exp\left(\pi\sqrt{\frac{n}{6}}\right);$$

$$\gamma_{I_2}(n) \sim \frac{24}{\pi^2} n\cdot q(n) \sim \frac{2^{5/2}3^{3/4}}{\pi^2} n^{1/4}\exp\left(\pi\sqrt{\frac{n}{6}}\right);$$

$$\gamma_{S_{I_2}}(n) \sim \frac{48}{\pi^2} n\cdot q(n) \sim \frac{2^{7/2}3^{3/4}}{\pi^2} n^{1/4}\exp\left(\pi\sqrt{\frac{n}{6}}\right).$$

**Corollary 2.** *The growth orders of the growth functions of $I_2$ and $S_{I_2}$ are equal, and*

$$[\gamma_{I_2}] = [\gamma_{S_{I_2}}] = \bigl[\exp\bigl(\sqrt{n}\bigr)\bigr].$$

## 3. Preliminaries

By $\mathbb{N}$ we mean the set of nonnegative integers $\mathbb{N} = \{0, 1, 2, \ldots\}$.

### 3.1. Growth functions

Let us consider the set of positive nondecreasing functions of a natural argument $\gamma : \mathbb{N} \to \mathbb{N}$; in the sequel such functions will be called *growth functions*.

**Definition 1.** For $i = 1, 2$ let $\gamma_i : \mathbb{N} \to \mathbb{N}$ be growth functions. The function $\gamma_1$ has *no greater growth order* (notation $\gamma_1 \preccurlyeq \gamma_2$) than the function $\gamma_2$, if there exist numbers $C_1, C_2, N_0 \in \mathbb{N}$ such that

$$\gamma_1(n) \leqslant C_1 \gamma_2(C_2 n)$$

for any $n \geqslant N_0$.

**Definition 2.** The growth functions $\gamma_1$ and $\gamma_2$ are equivalent or have *the same growth order* (notation $\gamma_1 \sim \gamma_2$), if the following inequalities hold:

$$\gamma_1 \preccurlyeq \gamma_2 \quad \text{and} \quad \gamma_2 \preccurlyeq \gamma_1.$$

The equivalence class of the function $\gamma$ is called its *growth order* and is denoted by the symbol $[\gamma]$. The relation $\preccurlyeq$ induces a partial order relation, written $<$, on equivalence classes. The growth order $[\gamma]$ is called *intermediate* if $[n^d] < [\gamma] < [e^n]$ for any $d > 0$.

### 3.2. Mealy automata

For $m \geqslant 2$ let $X_m$ be the $m$-symbol alphabet, $X_m = \{x_0, x_1, \ldots, x_{m-1}\}$. Let us denote the set of all finite words over $X_m$, including the empty word $\varepsilon$, by the symbol $X_m^*$, and denote the set of all infinite (to the right) words by the symbol $X_m^\omega$.

Let $A = (X_m, Q_n, \pi, \lambda)$ be a *noninitial Mealy automaton* [11] with finite set of states $Q_n = \{q_0, q_1, \ldots, q_{n-1}\}$; input and output alphabets are the same and are equal to $X_m$, and $\pi : X_m \times Q_n \to Q_n$ and $\lambda : X_m \times Q_n \to X_m$ are its transition and output functions, respectively.

The function $\lambda$ can be extended in a natural way to a mapping $\lambda : X_m^* \times Q_n \to X_m^*$ or to a mapping $\lambda : X_m^\omega \times Q_n \to X_m^\omega$. Set indeed $\lambda(aw, \mathsf{q}) = \lambda(a, \mathsf{q})\lambda(w, \pi(a, \mathsf{q}))$ for $a \in X_m$, $w \in X_m^\omega$ or $X_m^*$.

**Definition 3.** For any state $\mathsf{q} \in Q_n$ the transformation $f_{\mathsf{q},A} : X_m^* \to X_m^*$, respectively $f_{\mathsf{q},A} : X_m^\omega \to X_m^\omega$, defined by

$$f_{\mathsf{q},A}(u) = \lambda(u, \mathsf{q}),$$

where $u \in X_m^*$, respectively $u \in X_m^\omega$, is called the *automatic transformation* defined by $A$ at the state $\mathsf{q}$.

We write a function $f : X_m \to X_m$ as $(f(x_0), f(x_1), \dots, f(x_{m-1}))$. Let us consider the transformation $\sigma_{\mathsf{q}}$ over the alphabet $X_m$, $\mathsf{q} \in Q_n$, defined by the output function $\lambda$:

$$\sigma_{\mathsf{q}} = \big(\lambda(x_0, \mathsf{q}), \lambda(x_1, \mathsf{q}), \dots, \lambda(x_{m-1}, \mathsf{q})\big).$$

Interpreting an automatic transformation as an endomorphism of the rooted $m$-regular tree (see, for example, [7]), we see the following. Let $\mathsf{q}$ be an arbitrary state. The image of the word $u = u_0u_1u_2 \ldots \in X_m^{\omega}$ under the action of the automatic transformation $f_{\mathsf{q},A}$ can be written in the following way:

$$f_{\mathsf{q},A}(u_0u_1u_2\ldots) = \lambda(u_0, \mathsf{q}) \cdot f_{\pi(u_0,\mathsf{q}),A}(u_1u_2\ldots) = \sigma_{\mathsf{q}}(u_0) \cdot f_{\pi(u_0,\mathsf{q}),A}(u_1u_2\ldots).$$

This means that $f_{\mathsf{q},A}$ acts on the first symbol of the word $u$ by the transformation $\sigma_{\mathsf{q}}$ over the alphabet $X_m$, and acts on the remainder of the word without its first symbol by the transformation $f_{\pi(u_0,\mathsf{q}),A}$. Therefore the transformations defined by the automaton $A$ can be written in *unrolled form*:

$$f_{q_i} = (f_{\pi(x_0,q_i)}, f_{\pi(x_1,q_i)}, \dots, f_{\pi(x_{m-1},q_i)})\sigma_{q_i},$$

where $i = 0, 1, \dots, n-1$.

Let us illustrate this notion. Let $I_2$ be the automaton, shown on Fig. 1, and let us construct the unrolled forms of its automatic transformations. As $\pi(x_0, q_0) = \pi(x_1, q_0) = q_0$ and $\sigma_{q_0} = (x_0, x_1)$, the unrolled form of $f_{q_0}$ is written as

$$f_{q_0} = (f_{q_0}, f_{q_0})(x_0, x_1).$$

Similarly, we have $\pi(x_0, q_1) = q_1$, $\pi(x_1, q_1) = q_0$ and $\sigma_{q_1} = (x_1, x_1)$. Hence the unrolled form of $f_{q_1}$ is

$$f_{q_1} = (f_{q_1}, f_{q_0})(x_1, x_1).$$

Let $u = x_0x_0x_1x_0x_0x_1 \ldots = (x_0x_0x_1)^*$ be an infinite word, and let us consider the action of $f_{q_0}$ and $f_{q_1}$ on it. We have

$$\begin{aligned} f_{q_0}(u) &= \sigma_{q_0}(x_0) \cdot f_{q_0}(x_0x_1x_0x_0x_1\ldots) = x_1 \cdot \sigma_{q_0}(x_0) \cdot f_{q_0}(x_1x_0x_0x_1\ldots) = \\ &= x_1x_1 \cdot \sigma_{q_0}(x_1) \cdot f_{q_0}(x_0x_0x_1\ldots) = x_1x_1x_0 \cdot f_{q_0}(u) = \cdots = \\ &= x_1x_1x_0x_1x_1x_0\ldots = (x_1x_1x_0)^*, \end{aligned}$$

and

$$\begin{aligned} f_{q_1}(u) &= \sigma_{q_1}(x_0) \cdot f_{q_1}(x_0x_1x_0x_0x_1\ldots) = x_1 \cdot \sigma_{q_1}(x_0) \cdot f_{q_1}(x_1x_0x_0x_1\ldots) = \\ &= x_1x_1 \cdot \sigma_{q_1}(x_1) \cdot f_{q_0}(x_0x_0x_1\ldots) = x_1x_1x_1 \cdot f_{q_0}(u) = \\ &= x_1x_1x_1 \cdot (x_1x_1x_0)^*. \end{aligned}$$

The Mealy automaton $A = (X_m, Q_n, \pi, \lambda)$ defines the set $F_A = \{f_{q_0}, f_{q_1}, \dots, f_{q_{n-1}}\}$ of automatic transformations over $X_m^*$. The Mealy automaton $A$ is called *invertible* if all transformations from the set $F_A$ are bijections. It is easy to show (see, for example, [7]) that $A$ is invertible if and only if the transformation $\sigma_{\mathsf{q}}$ is a permutation of $X_m$ for each state $\mathsf{q} \in Q_n$.

**Definition 4** [4]. The Mealy automata $A_i = (X_m, Q_n, \pi_i, \lambda_i)$ for $i = 1, 2$ are called *isomorphic* if there exist permutations $\xi, \psi \in \mathrm{Sym}(X_m)$ and $\theta \in \mathrm{Sym}(Q_n)$ such that

$$\theta\pi_1(\mathsf{x}, \mathsf{q}) = \pi_2(\xi\mathsf{x}, \theta\mathsf{q}), \qquad \psi\lambda_1(\mathsf{x}, \mathsf{q}) = \lambda_2(\xi\mathsf{x}, \theta\mathsf{q})$$

for all $\mathsf{x} \in X_m$ and $\mathsf{q} \in Q_n$.

**Definition 5** [4]. The Mealy automata $A_i = (X_m, Q_{n_i}, \pi_i, \lambda_i)$ for $i = 1, 2$ are called *equivalent* if $F_{A_1} = F_{A_2}$.

**Proposition 1** [4]. *Each equivalence class of Mealy automata over the alphabet $X_m$ contains, up to isomorphism, a unique automaton that is minimal with respect to the number of states* (*such an automaton is called* reduced).

The minimal automaton can be found using the standard algorithm of minimization.

**Definition 6** [3]. For $i = 1, 2$ let $A_i = (X_m, Q_{n_i}, \pi_i, \lambda_i)$ be arbitrary Mealy automata. The automaton $A = (X_m, Q_{n_1} \times Q_{n_2}, \pi, \lambda)$ such that its transition and output functions are defined in the following way:

$$\begin{gathered}\pi\big(\mathsf{x}, (\mathsf{q}_1, \mathsf{q}_2)\big) = \big(\pi_1\big(\lambda_2(\mathsf{x}, \mathsf{q}_2), \mathsf{q}_1\big), \pi_2(\mathsf{x}, \mathsf{q}_2)\big),\\ \lambda\big(\mathsf{x}, (\mathsf{q}_1, \mathsf{q}_2)\big) = \lambda_1\big(\lambda_2(\mathsf{x}, \mathsf{q}_2), \mathsf{q}_1\big),\end{gathered}$$

where $\mathsf{x} \in X_m$ and $(\mathsf{q}_1, \mathsf{q}_2) \in Q_{n_1} \times Q_{n_2}$, is called the *product* of the automata $A_1$ and $A_2$.

**Proposition 2** [3]. *For any states $\mathsf{q}_1 \in Q_{n_1}$ and $\mathsf{q}_2 \in Q_{n_2}$ and an arbitrary word $u \in X_m^*$ the following equality holds*:

$$f_{(\mathsf{q}_1, \mathsf{q}_2), A}(u) = f_{\mathsf{q}_1, A_1}\big(f_{\mathsf{q}_2, A_2}(u)\big).$$

It follows from Proposition 2 that for the transformations $f_{\mathsf{q}_1, A_1}$ and $f_{\mathsf{q}_2, A_2}$, with $\mathsf{q}_1 \in Q_{n_1}$ and $\mathsf{q}_2 \in Q_{n_2}$, the unrolled form of the product $f_{(\mathsf{q}_1, \mathsf{q}_2), A_1 \times A_2}$ is defined by:

$$f_{(\mathsf{q}_1, \mathsf{q}_2), A_1 \times A_2} = f_{\mathsf{q}_1, A_1} f_{\mathsf{q}_2, A_2} = (g_0, g_1, \dots, g_{m-1})\sigma_{\mathsf{q}_1, A_1}\sigma_{\mathsf{q}_2, A_2},$$

where $g_i = f_{\pi_1(\sigma_{\mathsf{q}_2, A_2}(x_i), \mathsf{q}_1), A_1} f_{\pi_2(x_i, \mathsf{q}_2), A_2}$ for $i = 0, 1, \dots, m-1$.

The power $A^n$ is defined for any automaton $A$ and any positive integer $n$. Let us denote $A^{(n)}$ the minimal Mealy automaton equivalent to $A^n$. It follows from Definition 6 that $|Q_{A^{(n)}}| \leqslant |Q_A|^n$.

**Definition 7** [6]. The function $\gamma_A$ of a natural argument $n \geqslant 1$, defined by

$$\gamma_A(n) = |Q_{A^{(n)}}|,$$

is called the *growth function* of the Mealy automaton $A$.

**Definition 8** [14]. The Mealy automata $A_i = (X_m, Q_n, \pi_i, \lambda_i)$, for $i = 1, 2$, are called *similar* if they are isomorphic in the sense of Definition 4, for permutations $\xi, \psi \in \mathrm{Sym}(X_m)$ satisfying furthermore $\psi = \xi$.

*3.3. Semigroups*

Let $S$ be a semigroup with the finite set of generators $G = \{s_0, s_1, \dots, s_{k-1}\}$. Let us denote the free semigroup with the set $G$ of generators by the symbol $G^+$. It is easy to see (for example, in [9]) that if the semigroup $S$ does not contain the identity, then $S$ is a homomorphic image of the free semigroup $G^+$. Similarly, the monoid $S = \mathrm{sg}(G)$ is a homomorphic image of the free monoid $G^*$.

The elements of the free semigroup $G^+$ are called *semigroup words*. In the sequel, we identify them with corresponding elements of $S$. The semigroup words $\mathsf{s}_1$ and $\mathsf{s}_2$ are called *equivalent relative to the system $G$ of generators in the semigroup $S$*, if in $S$ the equality $\mathsf{s}_1 = \mathsf{s}_2$ holds [9].

**Definition 9.** Let $\mathsf{s}$ be an arbitrary element of $S$. The length $\ell(\mathsf{s})$ of $\mathsf{s}$ is the minimal possible number $\ell > 0$ of generators in a factorization

$$\mathsf{s} = s_{i_1} s_{i_2} s_{i_3} \dots s_{i_\ell},$$

where $s_{i_j} \in G$ for all $1 \leqslant j \leqslant \ell$.

Obviously for any $\mathsf{s} \in S$ the length $\ell(\mathsf{s})$ is greater than 0; but let us assume $\ell(1) = 0$, if $S$ is a monoid.

Let us order the generators of $S$ according to their index; and introduce a linear order on the set of elements of $G^+$: semigroup words are ranked by length, and then words of the same length are arranged lexicographically. The *representative* of a class in the equivalence relation introduced above is the minimal semigroup word in the sense of this order.

**Definition 10.** Let $\mathsf{s} \in S$ be an arbitrary element. The *normal form* of this element is the representative of the equivalence class of semigroup words mapped to the element $\mathsf{s}$.

**Definition 11.** The function $\gamma_S$ of a natural argument $n \in \mathbb{N}$ defined by

$$\gamma_S(n) = \left|\left\{s \in S \mid \ell(s) \leqslant n\right\}\right|$$

is called the *growth function of $S$ relative to the system $G$ of generators*.

**Definition 12.** The function $\widehat{\gamma}_S$ of a natural argument $n \in \mathbb{N}$ defined by

$$\widehat{\gamma}_S(n) = \left|\left\{s \in S \mid s = s_{i_1} s_{i_2} \dots s_{i_n},\ s_{i_j} \in G,\ 1 \leqslant j \leqslant n\right\}\right|$$

is called the *spherical growth function of S relative to the system G of generators*.

**Definition 13.** The function $\delta_S$ of a natural argument $n \in \mathbb{N}$ defined by

$$\delta_S(n) = \left|\left\{s \in S \mid \ell(s) = n\right\}\right|$$

is called the *word growth function of S relative to the system G of generators*.

If we denote by $\pi : G^+ \to S$ the natural epimorphism from the free semigroup $G^+$ to $S$, these functions can be expressed as follows:

$$\gamma_S(n) = \left|\bigcup_{i=0}^{n} \pi\left(G^i\right)\right|,$$

$$\widehat{\gamma}_S(n) = \left|\pi\left(G^n\right)\right|,$$

$$\delta_S(n) = \left|\pi\left(G^n\right) \Big\backslash \bigcup_{i=0}^{n-1} \pi\left(G^i\right)\right|.$$

The following proposition is well-known, and is proved in many papers (see, for example, [7,12]):

**Proposition 3.** *Let $S$ be an arbitrary finitely generated semigroup, and let $G_1$ and $G_2$ be systems of generators of $S$. Let us denote the growth function of $S$ relative to the set $G_i$ of generators by the symbol $\gamma_{S_i}$, for $i = 1, 2$. Then $[\gamma_{S_1}] = [\gamma_{S_2}]$.*

From Definitions 11, 12 and 13, the following inequalities hold for all $n \in \mathbb{N}$:

$$\delta_S(n) \leqslant \widehat{\gamma}_S(n) \leqslant \gamma_S(n) = \sum_{i=0}^{n} \delta_S(i). \tag{2}$$

**Proposition 4.** *Let $S$ be an arbitrary finitely generated monoid. Then*

$$[\delta_S] \leqslant [\widehat{\gamma}_S] = [\gamma_S].$$

Let $S$ be a semigroup without identity. Then the growth function and the spherical growth function may have different growth orders. For example, let $S = \mathbb{N}$ be the additive semigroup $S = \mathrm{sg}(1)$. Then $\gamma_S(n) = n$, $\widehat{\gamma}_S(n) = 1$, and these functions have different growth orders, $[1] < [n]$.

There are many results concerning the growth of groups. For references see the survey [7], or the book [8].

### 3.4. Growth series

It is often convenient to encode the growth function of a semigroup in a generating series:

**Definition 14.** Let $S$ be a semigroup generated by a finite set $G$. The *growth series* of $S$ is the formal power series

$$\Gamma_S(X) = \sum_{n \geqslant 0} \gamma_S(n) X^n.$$

The power series $\Delta_S(X) = \sum_{n \geqslant 0} \delta_S(n) X^n$ can also be introduced; we then have $\Delta_S(X) = (1 - X)\Gamma_S(X)$. The series $\Delta_S$ is called the *word growth series* of the semigroup $S$.

The growth series of a Mealy automaton is introduced similarly:

**Definition 15.** Let $A$ be an arbitrary Mealy automaton. The *growth series* of $A$ is the formal power series

$$\Gamma_A(X) = \sum_{n \geqslant 0} \gamma_A(n) X^n.$$

The radius of convergence, and behavior of $\Gamma_S$ near its singularities, encode the asymptotics of $\gamma_S$. The semigroup $S$ has subexponential growth if and only if $\Gamma_S$ converges in the open unit disk.

Sharper results of this flavor are often called *tauberian* and *abelian* theorems. We quote two such results [13]:

**Theorem 4.** *If $\Gamma_S$ converges in the open unit disk, and $\log \gamma_S(n) \sim 2\sqrt{\alpha n}$ for some $\alpha > 0$, then*

$$\log \Gamma_S(X) \sim \frac{\alpha}{1 - X}$$

*as $X \to 1^-$, i.e., as $X \to 1$ from the left.*

*If $\Delta_S(n) \sim \frac{c}{1-X}$ as $X \to 1^-$, then $\gamma_S(n) \sim cn$.*

### 3.5. Growth of Mealy automata and of automatic transformation semigroups they define

**Definition 16.** Let $A = (X_m, Q_n, \pi, \lambda)$ be a Mealy automaton. The semigroup

$$S_A = \mathrm{sg}(f_{q_0}, f_{q_1}, \ldots, f_{q_{n-1}})$$

is called the *semigroup of automatic transformations defined by $A$*.

For an invertible Mealy automaton, let us examine the group of transformations it defines. Let $A$ be a Mealy automaton, let $S_A$ be the semigroup defined by $A$, and let us denote the growth function and the spherical growth function of $S_A$ by the symbols $\gamma_{S_A}$ and $\widehat{\gamma}_{S_A}$, respectively. From Definition 16 we have

**Proposition 5** [6]. *For any $n \in \mathbb{N}$ the value $\gamma_A(n)$ equals the number of those elements of $S_A$ that can be presented as a product of length $n$ in the generators $\{f_{q_0}, f_{q_1}, \dots, f_{q_{n-1}}\}$, i.e.,*

$$\gamma_A(n) = \widehat{\gamma}_{S_A}(n), \quad \text{for all } n \in \mathbb{N}.$$

From this proposition and (2) it follows that $\gamma_A(n) \leqslant \gamma_{S_A}(n)$ for any $n \in \mathbb{N}$.

**Proposition 6** [14]. *Let $A_i = (X_m, Q_n, \pi_i, \lambda_i)$ for $i = 1, 2$ be two similar Mealy automata. Then these automata define isomorphic automatic transformation semigroups and have the same growth function.*

### 3.6. Hausdorff dimension

We introduce now the Hausdorff dimension of semigroups acting on trees. This topic was already extensively studied for groups [1,2].

Let $S$ be a semigroup acting on a tree $X_m^*$. This action extends to an action on the boundary $X_m^\omega$ of the tree. This space has the topology of a Cantor set, and can be given the natural metric

$$d(v, w) = \sup\{m^{-n}\colon v_n \neq w_n\},$$

where $v = v_0 v_1 v_2 \dots, w = w_0 w_1 w_2 \dots \in X_m^\omega$. This metric induces the Cantor topology on $X_m^\omega$, and turns it into a compact space of diameter 1.

The semigroup $S$ is a subset of the semigroup of tree endomorphisms of $X_m^*$, and $\mathrm{End}(X_m^*)$ has the natural function (compact-open) topology. The natural metric on $\mathrm{End}(X_m^*)$ is

$$d(g, h) = \sup\{\left|\mathrm{End}(X_m^n)\right|^{-1}\colon \text{there exists } v \in X_m^n \text{ with } v^g \neq v^h\},$$

where $g, h$ are arbitrary endomorphisms and $X_m^n$ denotes the first $n$ levels of the tree $X_m^*$. This induces on $\mathrm{End}(X_m^*)$, and therefore on $S$, the Cantor topology, and turns $\mathrm{End}(X_m^*)$ into a compact space of diameter 1.

Furthermore, $\mathrm{End}(X_m^*)$ has Hausdorff dimension 1, since it is covered by $|\mathrm{End}(X_m^n)|$ subspaces of diameter $|\mathrm{End}(X_m^n)|^{-1}$. Let $W_n$ denote the image of $S$ in $\mathrm{End}(X_m^n)$; then we define the Hausdorff dimension of $S$ as

$$\operatorname{Hdim} S = \liminf_{n \to \infty} \frac{\log |W_n|}{\log |\mathrm{End}(X_m^n)|}.$$

This is a number in the interval [0, 1] which measures the proportion of $\mathrm{End}(X_m^*)$ occupied by $S$.

Let us compute $|\mathrm{End}(X_m^n)|$: such an endomorphism is determined by an endomorphism of $X_m$ (there are $m^m$ of them), and $m$ endomorphisms in $\mathrm{End}(X_m^{n-1})$; we arrive at the recursive formula

$$\left|\mathrm{End}\left(X_m^n\right)\right| = m^m \left|\mathrm{End}\left(X_m^{n-1}\right)\right|^m = m^{m\frac{m^n-1}{m-1}}. \tag{$*$}$$

## 4. The semigroup $S_{I_2}$

### 4.1. Properties of automatic transformations

For $i = 0, 1$ let us denote the automatic transformation $f_{q_i, I_2}$ by the symbol $f_i$. The unrolled forms of the automatic transformations $f_0$ and $f_1$ are the following:

$$f_0 = (f_0, f_0)(x_1, x_0), \qquad f_1 = (f_1, f_0)(x_1, x_1). \tag{3}$$

From (3) the following equalities hold:

$$\begin{aligned} f_0^2 &= \left(f_0^2, f_0^2\right)(x_0, x_1), & f_0 f_1 &= \left(f_0 f_1, f_0^2\right)(x_0, x_0), \\ f_1^2 &= \left(f_0 f_1, f_0^2\right)(x_1, x_1), & f_1 f_0 &= \left(f_0^2, f_1 f_0\right)(x_1, x_1); \end{aligned} \tag{4}$$

whence we have

**Lemma 1.** *The automatic transformation $f_0$ is an involution.*

From Lemma 1 and (4), the following equalities hold for any $p \geqslant 1$:

$$(f_0 f_1)^p = \left((f_0 f_1)^p, (f_0 f_1)^{p-1}\right)(x_0, x_0), \tag{5a}$$

$$(f_1 f_0)^p = \left((f_1 f_0)^{p-1}, (f_1 f_0)^p\right)(x_1, x_1), \tag{5b}$$

$$(f_0 f_1)^p f_1 = \left((f_0 f_1)^{p-1} f_1, (f_0 f_1)^{p-1} f_0\right)(x_0, x_0). \tag{5c}$$

Here we assume $f^0 = 1$ for an arbitrary automatic transformation $f$.

**Lemma 2.** *In the semigroup $S_{I_2}$ the following relations hold*:

$$r_p\colon \quad f_1(f_0 f_1)^p (f_1 f_0)^p f_1^2 = f_1 (f_0 f_1)^p (f_1 f_0)^p, \tag{6}$$

*for all $p \geqslant 0$.*

**Proof.** Let us prove the lemma by induction on $p$. For $p = 0$ from (4) follows

$$f_1^3 = (f_0 f_1, 1)(x_1, x_1) \cdot (f_1, f_0)(x_1, x_1) = (f_1, f_0)(x_1, x_1) = f_1.$$

For $p > 1$ from (4), (5a) and (5b) we have

$$\begin{aligned} &f_1(f_0 f_1)^p (f_1 f_0)^p \\ &\quad = (f_1, f_0)(x_1, x_1) \cdot \big((f_0 f_1)^p, (f_0 f_1)^{p-1}\big)(x_0, x_0) \cdot \big((f_1 f_0)^{p-1}, (f_1 f_0)^p\big)(x_1, x_1) \\ &\quad = \big(f_1(f_0 f_1)^{p-1}(f_1 f_0)^{p-1}, f_1(f_0 f_1)^{p-1}(f_1 f_0)^p\big)(x_1, x_1), \end{aligned}$$

and

$$\begin{aligned} f_1(f_0 f_1)^p (f_1 f_0)^p f_1^2 &= \big(f_1(f_0 f_1)^{p-1}(f_1 f_0)^p f_0 f_1, f_1(f_0 f_1)^{p-1}(f_1 f_0)^p\big)(x_1, x_1) \\ &= \big(f_1(f_0 f_1)^{p-1}(f_1 f_0)^{p-1} f_1^2, f_1(f_0 f_1)^{p-1}(f_1 f_0)^p\big)(x_1, x_1). \end{aligned}$$

By the induction hypothesis, the right-hand sides of both equalities define the same automatic transformation, so the lemma holds. □

**Remark 1.** Application of any defining relation to an arbitrary semigroup word changes the length of this word by an even number.

**Remark 2.** The relation $r_p$ for all $p \geqslant 1$ can be written in the following way

$$r_p\colon \quad f_1(f_0 f_1)^p f_1(f_0 f_1)^p f_1 = f_1(f_0 f_1)^p f_1(f_0 f_1)^{p-1} f_0.$$

In the sequel, we will use both presentations of the relations $r_p$.

**Lemma 3.** *For any $n \in \mathbb{N}$ the element $f_1(f_0 f_1)^{n-1}$ is a left-side zero in the semigroup $W_n$. That is, the relations*

$$f_1(f_0 f_1)^{n-1} f_0 = f_1(f_0 f_1)^{n-1}, \qquad f_1(f_0 f_1)^{n-1} f_1 = f_1(f_0 f_1)^{n-1}, \tag{7}$$

*hold in the semigroup $W_n$.*

**Proof.** It is enough to show that the image of an arbitrary word $u \in X_2^n$ under the action of $f_1(f_0 f_1)^{n-1}$ does not depend on $u$. Indeed, from (5a) for any $p > 0$ follows

$$f_1(f_0 f_1)^p = \big(f_1(f_0 f_1)^p, f_1(f_0 f_1)^{p-1}\big)(x_1, x_1).$$

Let us write the word $u$ as

$$u = x_0^{t_1} x_1^{t_2} x_0^{t_3} x_1^{t_4} \ldots x_0^{t_{2k-1}} x_1^{t_{2k}},$$

where $k > 0$, $t_1, t_{2k} \geqslant 0$, $t_i > 0$, $2 \leqslant i \leqslant 2k-1$, $\sum_{i=1}^{2k} t_i = n$. For $u = x_1^n$ we have:

$$f_1(f_0 f_1)^{n-1}(u) = x_1^{n-1} \cdot f_1(x_1) = x_1^n.$$

Otherwise, $\sum_{i=1}^{k} t_{2i} < n$ and the equalities hold:

$$\begin{aligned} f_1(f_0 f_1)^{n-1}(u) &= x_1^{t_1} \cdot f_1(f_0 f_1)^{n-1}\left(x_1^{t_2} x_0^{t_3} x_1^{t_4} \ldots x_0^{t_{2k-1}} x_1^{t_{2k}}\right) \\ &= x_1^{t_1+t_2} \cdot f_1(f_0 f_1)^{n-1-t_2}\left(x_0^{t_3} x_1^{t_4} \ldots x_0^{t_{2k-1}} x_1^{t_{2k}}\right) \\ &= \cdots = x_1^{t_1+t_2+t_3+\cdots+t_{2k-1}} \cdot f_1(f_0 f_1)^{n-1-t_2-t_4-\cdots-t_{2k-2}}\left(x_1^{t_{2k}}\right) = x_1^n. \end{aligned}$$

Therefore $f_1(f_0 f_1)^{n-1}(u) = x_1^n$ and the lemma holds. $\square$

## 4.2. Normal forms

**Proposition 7** [17]. *Every $\mathsf{s} \in S_{I_2}$ admits a unique minimal-length representation as a word of the form* $1$, $f_0$, *or*

$$f_0^{\varepsilon_1} f_1(f_0 f_1)^{p_1} f_1(f_0 f_1)^{p_2} f_1 \ldots (f_0 f_1)^{p_k} f_1(f_0 f_1)^{p_{k+1}} f_0^{\varepsilon_2}, \tag{8}$$

*where $\varepsilon_1, \varepsilon_2 \in \{0,1\}$, $k \geqslant 0$, $0 \leqslant p_1 < p_2 < \cdots < p_k$, and $p_{k+1} \geqslant 0$.*

**Proof.** Let $\mathsf{s} \in S_{I_2}$ be an arbitrary semigroup element, written in the following way:

$$f_0^{p_0} f_1^{p_1} f_0^{p_2} f_1^{p_3} \ldots f_0^{p_{2k}} f_1^{p_{2k+1}},$$

where $k \geqslant 0$, $p_0 \geqslant 0$, $p_{2k+1} \geqslant 0$, $p_i > 0$, $i = 1, 2, \ldots, 2k$. The relation $f_0^2 = 1$ implies that there can never be two consecutive $f_0$'s in a reduced word, and the relation $r_0$ is $f_1^3 = f_1$, so there can never be three consecutive $f_1$'s.

If the representation of $\mathsf{s}$ contains at least one symbol $f_1$, then it can be written in the form

$$\mathsf{s} = f_0^{\varepsilon_1} f_1(f_0 f_1)^{p_1} f_1(f_0 f_1)^{p_2} f_1 \ldots (f_0 f_1)^{p_k} f_1(f_0 f_1)^{p_{k+1}} f_0^{\varepsilon_2}, \tag{9}$$

where $\varepsilon_1, \varepsilon_2 \in \{0,1\}$, $k \geqslant 0$, $p_1, p_{k+1} \geqslant 0$, $p_i > 0$, $2 \leqslant i \leqslant k$. Furthermore if $p_i \geqslant p_{i+1}$ for some $i \in \{1, 2, \ldots, k-1\}$, we have the relation

$$r_{p_{i+1}}\colon \quad f_1(f_0 f_1)^{p_{i+1}} f_1(f_0 f_1)^{p_{i+1}} f_1 = f_1(f_0 f_1)^{p_{i+1}} f_1(f_0 f_1)^{p_{i+1}-1} f_0,$$

and therefore the representation can be shortened. Then the semigroup word $\mathsf{s}$ is irreducible if and only if for all $i = 1, 2, \ldots, k-1$ the inequality $p_i < p_{i+1}$ holds, that is $0 \leqslant p_1 < p_2 < \cdots < p_k$. $\square$

In [17] an algorithm of reducing an arbitrary semigroup word to normal form is considered. Let $\mathsf{s}$ be an arbitrary semigroup word over the alphabet $\{f_0, f_1\}$. It can be reduced to normal form by the following steps:

(1) The word s is reduced by the defining relation $f_0^2 = 1$;
(2) The word s is reduced by the defining relation $r_0$;
(3) After steps (1) and (2) the word is written as (9);
(4) If for all $i = 1, 2, \dots, k-1$ the numbers $p_i$ in (9) satisfy the inequalities $p_i < p_{i+1}$, then the algorithm finishes, otherwise it goes to the next step;
(5) For the first pair of exponents $p_j$ and $p_{j+1}$, with $1 \leqslant j \leqslant k-1$, such that $p_j \geqslant p_{j+1}$, the subword $f_1^2$ of length 2 is canceled in s, by the application of the relation $r_{p_{j+1}}$;
(6) Go to step 1.

**Proposition 8** [17]. *The algorithm with steps* (1)–(6) *reduces an arbitrary semigroup word* s *to its normal form in no more than* $[|\mathsf{s}|/2]$ *steps.*

**Lemma 4.** *For any $n \geqslant 1$ an arbitrary element* s *of $W_n$ equals $1$, $f_0$, or can be written in normal form*

$$f_0^{\varepsilon_1} f_1 (f_0 f_1)^{p_1} f_1 (f_0 f_1)^{p_2} f_1 \dots (f_0 f_1)^{p_k} f_1 (f_0 f_1)^{p_{k+1}} f_0^{\varepsilon_2}, \tag{10}$$

*where $\varepsilon_1, \varepsilon_2 \in \{0, 1\}$, $0 \leqslant k$, $0 \leqslant p_1 < p_2 < \dots < p_k < n-1$, and $0 \leqslant p_{k+1} + \varepsilon_2 \leqslant n-1$.*

**Proof.** Let us fix a number $n \geqslant 1$. Let s be an arbitrary word of normal form (8):

$$\mathsf{s} = f_0^{\varepsilon_1} f_1 (f_0 f_1)^{p_1} f_1 (f_0 f_1)^{p_2} f_1 \dots (f_0 f_1)^{p_k} f_1 (f_0 f_1)^{p_{k+1}} f_0^{\varepsilon_2},$$

where $\varepsilon_1, \varepsilon_2 \in \{0, 1\}$, $0 \leqslant k$, $0 \leqslant p_1 < p_2 < \dots < p_k$, and $0 \leqslant p_{k+1}$. If $p_i \geqslant n-1$ for some $i$, then the semigroup word may be shortened by using the relations (7):

$$\begin{aligned} \mathsf{s} &= f_0^{\varepsilon_1} f_1 (f_0 f_1)^{p_1} f_1 (f_0 f_1)^{p_2} f_1 \dots (f_0 f_1)^{p_i} f_1 \dots (f_0 f_1)^{p_k} f_1 (f_0 f_1)^{p_{k+1}} f_0^{\varepsilon_2} \\ &= f_0^{\varepsilon_1} f_1 (f_0 f_1)^{p_1} f_1 \dots (f_0 f_1)^{p_{i-1}} f_1 (f_0 f_1)^{n-1} (f_0 f_1)^{p_i - n+1} f_1 \dots (f_0 f_1)^{p_k} f_1 (f_0 f_1)^{p_{k+1}} f_0^{\varepsilon_2} \\ &= f_0^{\varepsilon_1} f_1 (f_0 f_1)^{p_1} f_1 (f_0 f_1)^{p_2} f_1 \dots (f_0 f_1)^{p_{i-1}} f_1 (f_0 f_1)^{n-1}. \end{aligned}$$

This gives the requirements $0 \leqslant p_1 < p_2 < \dots < p_k < n-1$. Similarly, the end of s, the subword $(f_0 f_1)^{p_{k+1}} f_0^{\varepsilon_2}$, should be no longer than $(f_0 f_1)^{n-1}$. Hence, the requirement $p_{k+1} + \varepsilon_2 \leqslant n-1$ should be satisfied. □

## 4.3. Proof of Theorem 1

Let $k \in \mathbb{N}$ be an arbitrary positive integer, and let us denote its remainder modulo 2 by the symbol $[\![k]\!]$.

**Proposition 9.** *Let* $\mathsf{s} \in S_{I_2}$ *be an arbitrary element such that*

$$\mathsf{s} = f_0 f_1 (f_0 f_1)^{p_1} f_1 (f_0 f_1)^{p_2} f_1 \dots (f_0 f_1)^{p_k} f_1,$$

*where* $k \geqslant 1$, $0 \leqslant p_1 < p_2 < \cdots < p_k$. *Then*

$$\mathsf{s}\big(x_0^*\big) = x_0^{p_1+1} x_1^{p_2-p_1} x_0^{p_3-p_2} \dots x_{1-[\![k]\!]}^{p_k-p_{k-1}} x_{[\![k]\!]}^*.$$

**Proof.** Let $u \in X_2^\omega$ be an arbitrary word, and let $t_2 \geqslant t_1 \geqslant 0$ be arbitrary integers. Then from (5c) we have

$$\begin{aligned}(f_0 f_1)^{t_1} f_1\big(x_0^{t_2} x_1 u\big) &= x_0 \cdot (f_0 f_1)^{t_1-1} f_1\big(x_0^{t_2-1} x_1 u\big) = \cdots \\ &= x_0^{t_1} \cdot f_1\big(x_0^{t_2-t_1} x_1 u\big) = x_0^{t_1} x_1^{t_2-t_1+1} \cdot f_0(u).\end{aligned}$$

Let us prove the lemma by induction on $k$. For $k = 1$ from (5c) follows

$$(f_0 f_1)^{p_1+1} f_1\big(x_0^*\big) = x_0^{p_1+1} \cdot f_1\big(x_0^*\big) = x_0^{p_1+1} x_1^*.$$

For $k > 1$ we have

$$\begin{aligned}\mathsf{s}\big(x_0^*\big) &= f_0 f_1 (f_0 f_1)^{p_1} f_1\big(f_0 f_1 (f_0 f_1)^{p_2-1} f_1 (f_0 f_1)^{p_3} f_1 \dots (f_0 f_1)^{p_k} f_1\big(x_0^*\big)\big) \\ &= f_0 f_1 (f_0 f_1)^{p_1} f_1\big(x_0^{p_2} x_1^{p_3-p_2+1} x_0^{p_4-p_3} \dots x_{1-[\![k-1]\!]}^{p_k-p_{k-1}} x_{[\![k-1]\!]}^*\big) \\ &= x_0^{p_1+1} x_1^{p_2-p_1} \cdot f_0\big(x_1^{p_3-p_2} x_0^{p_4-p_3} \dots x_{[\![k]\!]}^{p_k-p_{k-1}} x_{1-[\![k]\!]}^*\big) \\ &= x_0^{p_1+1} x_1^{p_2-p_1} x_0^{p_3-p_2} \dots x_{1-[\![k]\!]}^{p_k-p_{k-1}} x_{[\![k]\!]}^*,\end{aligned}$$

and the lemma holds. □

**Corollary 3.** *Let* $n \in \mathbb{N}$ *be any, and let* $\mathsf{s}$ *be an semigroup element, written in the form* (10):

$$\mathsf{s} = f_0 f_1 (f_0 f_1)^{p_1} f_1 (f_0 f_1)^{p_2} f_1 \dots (f_0 f_1)^{p_k} f_1 (f_0 f_1)^{p_{k+1}},$$

*where* $0 \leqslant k$, $0 \leqslant p_1 < p_2 < \cdots < p_k < n-1$, *and* $0 \leqslant p_{k+1} \leqslant n-1$. *Then*

$$\mathsf{s}\big(x_0^n\big) = x_0^{p_1+1} x_1^{p_2-p_1} x_0^{p_3-p_2} \dots x_{1-[\![k]\!]}^{p_k-p_{k-1}} x_{[\![k]\!]}^{n-p_k-1}.$$

**Proof.** Let us fix an integer $n \geqslant 1$. From (5a) for any $p \geqslant 0$ we have

$$(f_0 f_1)^p \big(x_0^n\big) = x_0^n.$$

Therefore, for $k = 0$ we have

$$\mathsf{s}\big(x_0^n\big) = f_0 f_1 (f_0 f_1)^{p_{k+1}} \big(x_0^n\big) = x_0^n,$$

and when $k > 0$ from Proposition 9 we have

$$\begin{aligned}\mathsf{s}\big(x_0^n\big) &= f_0 f_1 (f_0 f_1)^{p_1} f_1 (f_0 f_1)^{p_2} f_1 \dots (f_0 f_1)^{p_k} f_1 \big(x_0^n\big)\\ &= x_0^{p_1+1} x_1^{p_2-p_1} x_0^{p_3-p_2} \dots x_{1-[\![k]\!]}^{p_k-p_{k-1}} x_{[\![k]\!]}^{n-(p_1+1)-\sum_{i=2}^{k}(p_i-p_{i-1})}\\ &= x_0^{p_1+1} x_1^{p_2-p_1} x_0^{p_3-p_2} \dots x_{1-[\![k]\!]}^{p_k-p_{k-1}} x_{[\![k]\!]}^{n-p_k-1}. \qquad \square\end{aligned}$$

**Proposition 10.** *The infinite system of relations*

$$f_0^2 = 1, r_0, r_1, r_2, \dots,$$

*is minimal, that is none of the relations follows from the others.*

**Proof.** Let us show that the relation

$$f_0^2 = 1$$

does not follow from the relations $\{r_p, p \geqslant 0\}$. Indeed, each relation $r_p$, for $p \geqslant 0$, includes the symbol $f_1$ in both its left- and right-hand side, and therefore it cannot be applied to $f_0^2 = 1$.

Moreover, the relation

$$r_0: \quad f_1^3 = f_1$$

does not follow from the set of relations $f_0^2 = 1, \{r_p, p \geqslant 1\}$, either. Let us consider its right-hand side, the semigroup word $f_1$. The unique relation which may be applied to it is $f_0^2 = 1$; and the set of semigroup words equivalent to $f_1$ is described in the following way:

$$f_0^{2p_1} f_1 f_0^{2p_2}, \quad p_1, p_2 \geqslant 0.$$

Obviously, this set does not include the semigroup word $f_1^3$.

Let us denote the left- and right-hand sides of the relation $r_p$, for $p > 0$, by the symbols $w_p$ and $v_p$, respectively, that is

$$\begin{aligned}w_p &= f_1 (f_0 f_1)^p f_1 (f_0 f_1)^p f_1,\\ v_p &= f_1 (f_0 f_1)^p f_1 (f_0 f_1)^{p-1} f_0.\end{aligned}$$

Let us fix a positive integer $\ell \geqslant 1$ and prove that the set of semigroup words equivalent to $v_\ell$, obtained by applying the relations $f_0^2 = 1$, $\{r_p,\ p \geqslant 0,\ p \neq l\}$, does not include any semigroup words which end in the symbol $f_1$.

Let us consider the set of semigroup words

$$\Omega_i = \big\{ f_1 f_0^{1+2t_1} f_1 f_0^{1+2t_2} \dots f_1 f_0^{1+2t_{i-1}} f_1 f_0^{2t_i} \ \big|\ t_1, t_2, \dots, t_i \geqslant 0 \big\},$$

where $i > 0$. All words in the set $\Omega_i$, for $i > 0$, are pairwise equivalent, and let us choose the word of minimal length

$$\omega_i = f_1 f_0 f_1 f_0 \dots f_1 f_0 f_1 = f_1 (f_0 f_1)^{i-1}$$

as the representative of $\Omega_i$. For $i = 0$ let us consider the set of words

$$\Omega_0 = \{ f_0^{2t_1} \mid t_1 \geqslant 0 \},$$

with representative $\omega_0 = 1$.

Let $\mathsf{s} \in S_{I_2}$ be an arbitrary semigroup element. It can be unambiguously written in the following way

$$f_0^{\varepsilon_1} \nu_0 \nu_1 \dots \nu_k f_0^{\varepsilon_2},$$

where $k \geqslant 0$, $\varepsilon_1, \varepsilon_2 \in \{0, 1\}$, $\nu_j \in \Omega_{i_j}$, $j = 0, 1, \dots, k$, $i_0 = 0$, $i_j > 0$, $j = 1, 2, \dots, k$, and if $k = 0$ let $\varepsilon_1 \leqslant \varepsilon_2$, $\varepsilon_1 + \varepsilon_2 \leqslant 1$. Using only the relation $f_0^2 = 1$, the element $\mathsf{s}$ can be unambiguously reduced to the following product

$$\mathsf{s} = f_0^{\varepsilon_1} \omega_{i_0} \omega_{i_1} \dots \omega_{i_k} f_0^{\varepsilon_2},$$

where requirements on the parameters are listed above.

Let us consider the set

$$\Upsilon(\mathsf{s}) = \left\{ \sum_{j=0}^{l} (-1)^{j+1} i_j \;\middle|\; l = 0, 1, 2, \dots, k \right\}.$$

The *width* of the semigroup word $\mathsf{s}$ is the positive integer

$$w(\mathsf{s}) = \max \Upsilon(\mathsf{s}) - \min \Upsilon(\mathsf{s}).$$

Let us note that $\mathsf{s}$ has width 0 if and only if $\mathsf{s} = f_0^p$ for some $p \geqslant 0$.

The relations $r_p$, for $p = 0, 1, \dots$, have the following representations:

$$\begin{aligned} r_0\colon &\quad \omega_0 \omega_1 \omega_1 \omega_1 = \omega_0 \omega_1; \\ r_p\colon &\quad \omega_0 \omega_{p+1} \omega_{p+1} \omega_1 = \omega_0 \omega_{p+1} \omega_p f_0, \qquad p > 0. \end{aligned} \tag{11}$$

Obviously, the left- and right-hand sides of $r_p$ have the same width $(p + 1)$, for all $p > 0$. Moreover, both sides of the relation $f_0^2 = 1$ have the same width 0, too.

From (11) it follows that the application of relations $f_0^2 = 1$ or $r_p$, for $p = 0, 1, \dots$ does not change the width of $\mathsf{s}$, and the relation $r_p$ can be applied to $\mathsf{s}$ if and only if $0 \leqslant p \leqslant w(\mathsf{s}) - 1$. Hence, only the relations

$$f_0^2 = 1, r_0, r_1, \dots, r_{\ell-1} \tag{12}$$

can be applied to the word $v_\ell$ and the words equivalent to it.

Let us separate $v_\ell$ into two parts

$$v_\ell^{(1)} = \omega_0\omega_{\ell+1}, \qquad v_\ell^{(2)} = \omega_0\omega_l f_0,$$

where $v_\ell = v_\ell^{(1)} \cdot v_\ell^{(2)}$. In addition,

$$w(v_\ell) = w\big(v_\ell^{(1)}\big) = \ell + 1, \qquad w\big(v_\ell^{(2)}\big) = \ell.$$

From Proposition 7 it follows that all words $v_\ell$, $v_\ell^{(1)}$ and $v_\ell^{(2)}$ have normal form (8). If the relation $r_p$ from (12) is applied to $v_\ell$, then there are three possible cases:

- $w_p$ and $v_p$ belong to $v_\ell^{(1)}$;
- $w_p$ and $v_p$ belong to $v_\ell^{(2)}$;
- $\omega_0\omega_{p+1}$ belongs to $v_\ell^{(1)}$, and $\omega_0\omega_{p+1}\omega_1$ and $\omega_0\omega_p f_0$ belong to $v_\ell^{(2)}$.

As $p < \ell$, the application of a relation from (12) does not change the width of the parts $v_\ell^{(1)}$ and $v_\ell^{(2)}$. Hence, if $\mathsf{s}$ is an arbitrary word which is obtained from $v_\ell$ by relations (12), it can be separated into two parts $\mathsf{s}^{(1)}$ and $\mathsf{s}^{(2)}$, $\mathsf{s} = \mathsf{s}^{(1)} \cdot \mathsf{s}^{(2)}$, where $w(\mathsf{s}^{(i)}) = w(v_\ell^{(i)})$ for $i = 1, 2$. As $w(\mathsf{s}^{(2)}) = \ell$ and the parities of the number of occurrences of $f_0$ in $\mathsf{s}^{(2)}$ and $v_\ell^{(2)}$ coincide, $\mathsf{s}^{(2)}$ ends on the symbol $f_0$. Therefore the word $\mathsf{s} = \mathsf{s}^{(1)} \cdot \mathsf{s}^{(2)}$ ends in $f_0$ too, and the word $\omega_0\omega_{p+1}\omega_{p+1}\omega_1$, which ends on $f_1$, is not equivalent to $v_\ell$. $\square$

**Proof of Theorem 1.** From Lemmas 1 and 2 it follows that in the semigroup $S_{I_2}$ the relations $f_0^2 = 1$ and $r_p$, for $p \geqslant 0$, hold. In Proposition 7 it is proved that, using these relations, each element can be unambiguously reduced to normal form. It is enough to show that semigroup elements, which are written in different normal forms, define different automatic transformations over the set $X_2^\omega$.

Let $\mathsf{s}_1, \mathsf{s}_2$ be arbitrary semigroup elements, written in normal form. As $f_0$ is a bijection and $f_1$ is not a bijection, then any semigroup word which includes the symbol $f_1$ defines an automatic transformation which is not a bijection, and therefore differs from both transformations 1 and $f_0$. Due to this remark, it is enough to consider elements in normal form (8). Let us write

$$\mathsf{s}_1 = f_0^{\varepsilon_1} f_1 (f_0 f_1)^{p_1} f_1 (f_0 f_1)^{p_2} f_1 \ldots (f_0 f_1)^{p_k} f_1 (f_0 f_1)^{p_{k+1}} f_0^{\varepsilon_2},$$
$$\mathsf{s}_2 = f_0^{\mu_1} f_1 (f_0 f_1)^{t_1} f_1 (f_0 f_1)^{t_2} f_1 \ldots (f_0 f_1)^{t_\ell} f_1 (f_0 f_1)^{t_{\ell+1}} f_0^{\mu_2},$$

where $\varepsilon_1, \varepsilon_2, \mu_1, \mu_2 \in \{0, 1\}$, $k, \ell \geqslant 0$, $0 \leqslant p_1 < p_2 < \cdots < p_k$, $0 \leqslant t_1 < t_2 < \cdots < t_\ell$, and $p_{k+1} \geqslant 0$, $t_{\ell+1} \geqslant 0$.

Let us assume that the elements $\mathsf{s}_1$ and $\mathsf{s}_2$ define the same automatic transformation over $X_2^\omega$. Then for any $u \in X_2^\omega$ the equality holds

$$\mathsf{s}_1(u) = \mathsf{s}_2(u). \tag{13}$$

As $f_0$ is a bijection, the equalities

$$\mathsf{s}_1 = \mathsf{s}_2, \qquad f_0\mathsf{s}_1 = f_0\mathsf{s}_2, \qquad \mathsf{s}_1 f_0 = \mathsf{s}_2 f_0,$$

hold simultaneously. Moreover, from (13) for any element $\mathsf{s}_3 \in S_{I_2}$ it follows that

$$\mathsf{s}_1\mathsf{s}_3(u) = \mathsf{s}_2\mathsf{s}_3(u).$$

Let us consider possible values of $\varepsilon_1$ and $\mu_1$.

(1) $\varepsilon_1 = 0$ and $\mu_1 = 0$. Due to the remark above, this case is equivalent to the case $\varepsilon_1 = 1$ and $\mu_1 = 1$, which is described below.
(2) $\varepsilon_1 = 0$ and $\mu_1 = 1$. As $k, l \geqslant 0$, the semigroup words $\mathsf{s}_1$ and $\mathsf{s}_2$ start by the symbols $f_1$ and $f_0 f_1$, respectively. For the input word $u = x_1$ we have

$$\mathsf{s}_1(x_1) = f_1\big((f_0 f_1)^{p_1} f_1 \dots (f_0 f_1)^{p_k} f_1 (f_0 f_1)^{p_{k+1}} f_0^{\varepsilon_2}(x_1)\big) = x_1,$$

and

$$\mathsf{s}_2(x_1) = f_0 f_1\big((f_0 f_1)^{t_1} f_1 \dots (f_0 f_1)^{t_\ell} f_1 (f_0 f_1)^{t_{\ell+1}} f_0^{\mu_2}(x_1)\big) = f_0(x_1) = x_0.$$

Therefore, the elements $\mathsf{s}_1$ and $\mathsf{s}_2$ define different automatic transformations over the set $X_2^\omega$. The case $\varepsilon_1 = 1$ and $\mu_1 = 0$ is similar.
(3) $\varepsilon_1 = 1$ and $\mu_1 = 1$.

Let us assume that $\varepsilon_2 = 0$ and $\mu_2 = 0$. From (13) it follows that the elements

$$\mathsf{s}_1(f_0 f_1)^{(p_k + t_\ell + 1)} f_1 \quad \text{and} \quad \mathsf{s}_2(f_0 f_1)^{(p_k + t_\ell + 1)} f_1$$

define the same automatic transformation. Using Proposition 9, we have

$$\begin{aligned}
\mathsf{s}_1(f_0 f_1)^{(p_k+t_\ell+1)} f_1\big(x_0^*\big) &= f_0 f_1 (f_0 f_1)^{p_1} f_1 \dots (f_0 f_1)^{p_k} f_1 (f_0 f_1)^{p_{k+1}+p_k+t_\ell+1} f_1\big(x_0^*\big) \\
&= x_0^{p_1+1} x_1^{p_2-p_1} x_0^{p_3-p_2} \dots x_{1-[\![k]\!]}^{p_k-p_{k-1}} x_{[\![k]\!]}^{p_{k+1}+t_\ell+1} x_{1-[\![k]\!]}^*, \\
\mathsf{s}_2(f_0 f_1)^{(p_k+t_\ell+1)} f_1\big(x_0^*\big) &= f_0 f_1 (f_0 f_1)^{t_1} f_1 \dots (f_0 f_1)^{t_\ell} f_1 (f_0 f_1)^{t_{\ell+1}+p_k+t_\ell+1} f_1\big(x_0^*\big) \\
&= x_0^{t_1+1} x_1^{t_2-t_1} x_0^{t_3-t_2} \dots x_{1-[\![l]\!]}^{t_\ell-t_{\ell-1}} x_{[\![l]\!]}^{t_{\ell+1}+p_k+1} x_{1-[\![l]\!]}^*.
\end{aligned}$$

As the words in the right-hand sides coincide, we obtain the requirements $k = \ell$, $p_i = t_i$, $i = 1, 2, \dots, k+1$. This means that the elements $\mathsf{s}_1$ and $\mathsf{s}_2$ are written in the same normal form (8).

The case $\varepsilon_2 = 1$ and $\mu_2 = 1$ is considered similarly, because we may consider elements $\mathsf{s}_1 f_0$ and $\mathsf{s}_2 f_0$.

Next, let us assume $\varepsilon_2 = 0$ and $\mu_2 = 1$ (the case $\varepsilon_2 = 1$ and $\mu_2 = 0$ is considered similarly). If $k = 0$ or $p_{k+1} > p_k$, then elements

$$\mathsf{s}_1 f_1 = f_0 f_1 (f_0 f_1)^{p_1} f_1 (f_0 f_1)^{p_2} f_1 \dots (f_0 f_1)^{p_k} f_1 (f_0 f_1)^{p_{k+1}} f_1,$$
$$\mathsf{s}_2 f_1 = f_0 f_1 (f_0 f_1)^{t_1} f_1 (f_0 f_1)^{t_2} f_1 \dots (f_0 f_1)^{t_\ell} f_1 (f_0 f_1)^{t_{\ell+1}} f_0 f_1,$$

are written in normal form (8) and define the same transformation over $X_2^\omega$. From the proof above in case (3). it follows that $k+1=\ell$, $p_i = t_i$, $i = 1, 2, \dots, k+1$, and $t_{\ell+1}+1=0$; but this contradicts the condition $t_{\ell+1} \geqslant 0$.

Let $k > 0$ and $0 \leqslant p_{k+1} \leqslant p_k$. Let us assume $\mathsf{s}_3 = f_1(f_0 f_1)^{p_{k+1}}$, then the element $\mathsf{s}_2\mathsf{s}_3$ is already written in normal form (8):

$$\begin{aligned}\mathsf{s}_2 f_1 (f_0 f_1)^{p_{k+1}} &= f_0 f_1 (f_0 f_1)^{t_1} f_1 (f_0 f_1)^{t_2} f_1 \dots (f_0 f_1)^{t_\ell} f_1 (f_0 f_1)^{t_{\ell+1}} f_0 f_1 (f_0 f_1)^{p_{k+1}} \\ &= f_0 f_1 (f_0 f_1)^{t_1} f_1 (f_0 f_1)^{t_2} f_1 \dots (f_0 f_1)^{t_\ell} f_1 (f_0 f_1)^{t_{\ell+1}+p_{k+1}+1}.\end{aligned}$$

The element $\mathsf{s}_1\mathsf{s}_3$ is reduced, and its normal form is the following:

$$\begin{aligned}\mathsf{s}_1 f_1 (f_0 f_1)^{p_{k+1}} &= f_0 f_1 (f_0 f_1)^{p_1} f_1 (f_0 f_1)^{p_2} f_1 \dots (f_0 f_1)^{p_k} f_1 (f_0 f_1)^{p_{k+1}} f_1 (f_0 f_1)^{p_{k+1}} \\ &= f_0 f_1 (f_0 f_1)^{p_1} f_1 (f_0 f_1)^{p_2} f_1 \dots (f_0 f_1)^{p_{k-1}} f_1 (f_0 f_1)^{p_k}.\end{aligned}$$

For the elements $\mathsf{s}_1\mathsf{s}_3$ and $\mathsf{s}_2\mathsf{s}_3$, as proved above, the following requirements hold:

$$k \geqslant 1, \quad k-1=\ell, \quad p_1 = t_1, \quad p_2 = t_2, \dots, p_{k-1} = t_\ell, \quad p_k = t_{\ell+1} + p_{k+1} + 1. \tag{14}$$

As $f_0$ is a bijection, a similar reasoning can be carried out for the elements $\mathsf{s}_1 f_0$ and $\mathsf{s}_2 f_0$, where $\mathsf{s}_1$ and $\mathsf{s}_2$ are rearranged. For the case $t_{\ell+1} > t_\ell$ or $\ell = 0$ we obtain a contradiction with the requirement $p_{k+1} \geqslant 0$, and in the case $0 \leqslant t_{\ell+1} \leqslant t_\ell$ and $\ell > 0$ the requirement $\ell - 1 = k$ should be fulfilled, but it contradicts the requirements (14).

Thus, the relations $f_0^2 = 1, r_0, r_1, \dots$ form a system of defining relations. In Proposition 10 it is proved that this system is minimal, and therefore the semigroup $S_{I_2}$ is infinitely presented.

To solve the word problem in $S_{I_2}$, it is necessary to reduce semigroup words $\mathsf{s}_1$ and $\mathsf{s}_2$ to normal form (8), and then to check them for graphical equality. From Proposition 8 this can be done in no more than

$$\left[\frac{|\mathsf{s}_1|}{2}\right] + \left[\frac{|\mathsf{s}_2|}{2}\right]$$

steps, and the word problem is solved in polynomial time.

Let us prove the second part of Theorem 1 in a similar way as the first part. Let us fix the integer $n \geqslant 1$ and let $\mathsf{s}_1$ and $\mathsf{s}_2$ are arbitrary elements of the semigroup $W_n$. The elements 1, $f_0$, $f_1 \dots$ and $f_0 f_1 \dots$ define pairwise distinct transformations $(x_0, x_1)$, $(x_1, x_0)$, $(x_1, x_1)$, and $(x_0, x_0)$ over the set $X_2^1$, respectively. Therefore, using the proof above, it is enough to consider $n > 1$ and $\mathsf{s}_1, \mathsf{s}_2$ such that they are written in normal form (10):

$$\begin{aligned}
\mathsf{s}_1 &= f_0 f_1 (f_0 f_1)^{p_1} f_1 (f_0 f_1)^{p_2} f_1 \dots (f_0 f_1)^{p_k} f_1 (f_0 f_1)^{p_{k+1}} f_0^{\varepsilon_2}, \\
\mathsf{s}_2 &= f_0 f_1 (f_0 f_1)^{t_1} f_1 (f_0 f_1)^{t_2} f_1 \dots (f_0 f_1)^{t_\ell} f_1 (f_0 f_1)^{t_{\ell+1}} f_0^{\mu_2},
\end{aligned} \tag{15}$$

where $\varepsilon_2, \mu_2 \in \{0, 1\}$, $k, \ell \geqslant 0$, $0 \leqslant p_1 < p_2 < \dots < p_k < n-1$, $0 \leqslant t_1 < t_2 < \dots < t_\ell < n-1$, and $0 \leqslant p_{k+1} + \varepsilon_2 \leqslant n-1$, $0 \leqslant t_{\ell+1} + \mu_2 \leqslant n-1$. Let us consider these elements in the same way as it was done for elements of the semigroup $S_{I_2}$. Besides, it is enough to consider the cases $\varepsilon_2 = \mu_2 = 0$ and $\varepsilon_2 = 1$, $\mu_2 = 0$.

Let us assume that $\varepsilon_2 = \mu_2 = 0$. Then from Corollary 3 for the input word $u = x_0^n$ it follows that

$$\begin{aligned}
\mathsf{s}_1\big(x_0^n\big) &= x_0^{p_1+1} x_1^{p_2-p_1} x_0^{p_3-p_2} \dots x_{1-[\![k]\!]}^{p_k-p_{k-1}} x_{[\![k]\!]}^{n-p_k-1}, \\
\mathsf{s}_2\big(x_0^n\big) &= x_0^{t_1+1} x_1^{t_2-t_1} x_0^{t_3-t_2} \dots x_{1-[\![l]\!]}^{t_\ell-t_{\ell-1}} x_{[\![l]\!]}^{n-t_\ell-1},
\end{aligned}$$

and from assumption (13) we have the requirements

$$k = \ell, \quad p_i = t_i, \quad i = 1, 2, \dots, k.$$

With no loss of generality let us assume $0 \leqslant p_{k+1} < t_{k+1}$.

If $k = 0$ or $p_k < n - 1 - t_{k+1} + p_{k+1}$, let us consider the element $\mathsf{s}_3 = (f_0 f_1)^{n-1-t_{k+1}} f_1$. Then $\mathsf{s}_1\mathsf{s}_3$ does not reduce, because $p_{k+1} + n - 1 - t_{k+1} < n - 1$, and $\mathsf{s}_2\mathsf{s}_3$ is reduced to the following element:

$$\begin{aligned}
\mathsf{s}_2\mathsf{s}_3 &= f_0 f_1 (f_0 f_1)^{p_1} f_1 (f_0 f_1)^{p_2} f_1 \dots (f_0 f_1)^{p_k} f_1 (f_0 f_1)^{t_{k+1}} \cdot (f_0 f_1)^{n-1-t_{k+1}} f_1 \\
&= f_0 f_1 (f_0 f_1)^{p_1} f_1 (f_0 f_1)^{p_2} f_1 \dots (f_0 f_1)^{p_k} f_1 (f_0 f_1)^{n-1}.
\end{aligned}$$

For the input word $u = x_0^n$ we have

$$\begin{aligned}
\mathsf{s}_1\mathsf{s}_3\big(x_0^n\big) &= f_0 f_1 (f_0 f_1)^{p_1} f_1 (f_0 f_1)^{p_2} f_1 \dots (f_0 f_1)^{p_k} f_1 (f_0 f_1)^{n-1-t_{k+1}+p_{k+1}} f_1\big(x_0^n\big) \\
&= x_0^{p_1+1} x_1^{p_2-p_1} x_0^{p_3-p_2} \dots x_{[\![k+1]\!]}^{p_k-p_{k-1}} x_{1-[\![k+1]\!]}^{(n-1-t_{k+1}+p_{k+1})-p_k} x_{[\![k+1]\!]}^{n-1-(n-1-t_{k+1}+p_{k+1})}, \\
\mathsf{s}_2\mathsf{s}_3\big(x_0^n\big) &= f_0 f_1 (f_0 f_1)^{p_1} f_1 (f_0 f_1)^{p_2} f_1 \dots (f_0 f_1)^{p_k} f_1 (f_0 f_1)^{n-1} \\
&= x_0^{p_1+1} x_1^{p_2-p_1} x_0^{p_3-p_2} \dots x_{1-[\![k]\!]}^{p_k-p_{k-1}} x_{[\![k]\!]}^{n-1-p_k},
\end{aligned}$$

which contradicts assumption (13).

In the case $k > 0$ and $p_k \geqslant n - 1 - t_{k+1} + p_{k+1}$, let us consider the element $\mathsf{s}_4 = (f_0 f_1)^{n-1-t_{k+1}} f_1 (f_0 f_1)^{n-1-t_{k+1}+p_{k+1}}$. Then elements $\mathsf{s}_1\mathsf{s}_4$ and $\mathsf{s}_2\mathsf{s}_4$ are reduced to the following elements:

$$\begin{aligned}
\mathsf{s}_1\mathsf{s}_4 &= f_0 f_1 (f_0 f_1)^{p_1} f_1 \dots (f_0 f_1)^{p_k} f_1 (f_0 f_1)^{p_{k+1}} \cdot (f_0 f_1)^{n-1-t_{k+1}} f_1 (f_0 f_1)^{n-1-t_{k+1}+p_{k+1}} \\
&= f_0 f_1 (f_0 f_1)^{p_1} f_1 (f_0 f_1)^{p_2} f_1 \dots (f_0 f_1)^{p_{k-1}} f_1 (f_0 f_1)^{p_k}; \\
\mathsf{s}_2\mathsf{s}_4 &= f_0 f_1 (f_0 f_1)^{p_1} f_1 \dots (f_0 f_1)^{p_k} f_1 (f_0 f_1)^{t_{k+1}} \cdot (f_0 f_1)^{n-1-t_{k+1}} f_1 (f_0 f_1)^{n-1-t_{k+1}+p_{k+1}} \\
&= f_0 f_1 (f_0 f_1)^{p_1} f_1 (f_0 f_1)^{p_2} f_1 \dots (f_0 f_1)^{p_k} f_1 (f_0 f_1)^{n-1}.
\end{aligned}$$

Similarly, for the input word $u = x_0^n$ we have

$$\begin{aligned}
\mathsf{s}_1\mathsf{s}_4(x_0^n) &= f_0 f_1 (f_0 f_1)^{p_1} f_1 (f_0 f_1)^{p_2} f_1 \ldots (f_0 f_1)^{p_{k-1}} f_1 (f_0 f_1)^{p_k} (x_0^n) \\
&= x_0^{p_1+1} x_1^{p_2-p_1} x_0^{p_3-p_2} \ldots x_{1-[\![k-1]\!]}^{p_{k-1}-p_{k-2}} x_{[\![k-1]\!]}^{n-1-p_{k-1}}, \\
\mathsf{s}_2\mathsf{s}_4(x_0^n) &= f_0 f_1 (f_0 f_1)^{p_1} f_1 (f_0 f_1)^{p_2} f_1 \ldots (f_0 f_1)^{p_k} f_1 (f_0 f_1)^{n-1} \\
&= x_0^{p_1+1} x_1^{p_2-p_1} x_0^{p_3-p_2} \ldots x_{1-[\![k]\!]}^{p_k-p_{k-1}} x_{[\![k]\!]}^{n-1-p_k},
\end{aligned}$$

which contradicts assumption (13). Hence, the elements (15) with $\varepsilon_2 = \mu_2 = 0$ define the same transformation over $X_2^n$ if and only if $k = \ell$, $p_i = t_i$, $i = 1, 2, \ldots, k+1$.

Consider now the case $\varepsilon_2 = 1$, $\mu_2 = 0$. Let us assume that $\ell = 0$ or $t_\ell < t_{\ell+1}$. In this case the elements $\mathsf{s}_1 f_1$ and $\mathsf{s}_2 f_1$ are not reduced:

$$\begin{aligned}
\mathsf{s}_1 f_1 &= f_0 f_1 (f_0 f_1)^{p_1} f_1 (f_0 f_1)^{p_2} f_1 \ldots (f_0 f_1)^{p_k} f_1 (f_0 f_1)^{p_{k+1}+1}, \\
\mathsf{s}_2 f_1 &= f_0 f_1 (f_0 f_1)^{t_1} f_1 (f_0 f_1)^{t_2} f_1 \ldots (f_0 f_1)^{t_\ell} f_1 (f_0 f_1)^{t_{\ell+1}} f_1.
\end{aligned}$$

From assumption (13) and the proof above the requirements $k = \ell + 1$, $p_i = t_i$, $1 \leqslant i \leqslant k$, $p_{k+1} + 1 = 0$ follow. The last requirement contradicts the condition $p_{k+1} \geqslant 0$ of (15). A similar reasoning can be carried out for the elements $\mathsf{s}_1 f_0$ and $\mathsf{s}_2 f_0$, and we reach a contradiction in the case $k = 0$ or $p_k < p_{k+1}$.

Let us now consider the case $k, \ell > 0$, $t_\ell \geqslant t_{\ell+1}$ and $p_k \geqslant p_{k+1}$. The elements $\mathsf{s}_1 f_1 (f_0 f_1)^{t_{\ell+1}}$ and $\mathsf{s}_2 f_1 (f_0 f_1)^{t_{\ell+1}}$ are reduced to the following normal forms:

$$\begin{aligned}
\mathsf{s}_1 f_1 (f_0 f_1)^{t_{\ell+1}} &= f_0 f_1 (f_0 f_1)^{p_1} f_1 (f_0 f_1)^{p_2} f_1 \ldots (f_0 f_1)^{p_k} f_1 (f_0 f_1)^{\min(p_{k+1}+1+t_{\ell+1}, n-1)}, \\
\mathsf{s}_2 f_1 (f_0 f_1)^{t_{\ell+1}} &= f_0 f_1 (f_0 f_1)^{t_1} f_1 (f_0 f_1)^{t_2} f_1 \ldots (f_0 f_1)^{t_{\ell-1}} f_1 (f_0 f_1)^{t_\ell}.
\end{aligned}$$

From assumption (13) and the proof above the requirements

$$k = \ell - 1, \quad p_i = t_i, \quad 1 \leqslant i \leqslant k, \qquad \min(p_{k+1} + 1 + t_{\ell+1}, n-1) = t_\ell \tag{16}$$

follow. Similarly, from the equality

$$\mathsf{s}_1 f_0 f_1 (f_0 f_1)^{t_{\ell+1}} (x_0^n) = \mathsf{s}_2 f_0 f_1 (f_0 f_1)^{t_{\ell+1}} (x_0^n)$$

we get the requirement $k - 1 = \ell$, which contradicts the requirements (16).

The theorem is completely proved. □

**Proof of Corollary 1.** Let us fix a number $n \geqslant 1$, and prove that the cardinality of the semigroup $W_n$ is

$$|W_n| = 2 + (2n-1)2^n.$$

Any element of form (10) is defined by a set of $k$ parameters $\{p_1, p_2, \dots, p_k\}$, and by $\varepsilon_1$, $p_{k+1}$, $\varepsilon_2$. Parameter $\varepsilon_1$ has two possible values, "the tail" $(f_0 f_1)^{p_{k+1}} f_0^{\varepsilon_2}$ has length varying from 0 to $(2n-2)$, and the set $\{p_1, p_2, \dots, p_k\}$ is a $k$-element subset of $\{0, 1, \dots, n-2\}$, where $k$ is some integer in $\{0, 1, \dots, n-1\}$. Therefore,

$$|W_n| = \underbrace{2}_{1, f_0} + \underbrace{2}_{\varepsilon_1} \cdot \underbrace{2^{n-1}}_{p_1, \dots, p_k} \cdot \underbrace{(2n-1)}_{p_{k+1}+\varepsilon_2} = 2 + (2n-1)2^n.$$

As was shown in $(*)$, for all $n \geqslant 1$

$$\left|\operatorname{End}\left(X_2^n\right)\right| = 2^{2\frac{2^n-1}{2-1}} = 4^{2^n-1},$$

and the Hausdorff dimension of the semigroup $S_{I_2}$ is

$$\operatorname{Hdim} S_{I_2} = \liminf_{n\to\infty} \frac{\log(2 + (2n-1)2^n)}{(2^n-1)\log 4} = 0,$$

which proves the corollary. □

## 5. Growth functions

We derive, in this section, the growth series of the semigroup $S_{I_2}$, as well as the asymptotics of the growth functions $\gamma_{S_{I_2}}$ and $\gamma_{I_2}$.

### *5.1. Growth series*

**Lemma 5.** *Let $q(n)$ be the number of partitions of $n \in \mathbb{N}$ in distinct, odd parts, and form $\Psi(X) = \sum q(n)X^n$. Then*

$$\Psi(X) = \sum_{m=0}^{\infty} \frac{X^{m^2}}{(1-X^2)\dots(1-X^{2m})} = (1+X)\left(1+X^3\right)\left(1+X^5\right)\dots.$$

**Proof.** Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$ be such a partition of $n$. Then $\lambda_i \geqslant 2i-1$ for all $i = 1, 2, \dots, m$, and

$$\left(\lambda_1 - 1, \lambda_2 - 3, \dots, \lambda_m - (2m-1)\right)$$

is a partition of $n - m^2$ in at most $m$ even parts. By "flipping," this is the same as a partition of $n - m^2$ into even parts that are at most $2m$, whence the first equality.

The second equality is standard: an integer partition $(\lambda_1, \dots, \lambda_m)$ in distinct odd parts corresponds to a monomial $X^{\lambda_1} \dots X^{\lambda_m}$. □

It follows from Proposition 7 that the word growth series of $S_{I_2}$ is

$$\Delta_{S_{I_2}}(X) = \sum_{n\geqslant 0} \delta_{S_{I_2}}(n) X^n = \underbrace{(1+X)}_{1,f_0} + \underbrace{\underbrace{(1+X)}_{f_0^{\varepsilon_1}} \underbrace{X}_{f_1} \Psi(X) \underbrace{\frac{1}{1-X^2}}_{(f_0 f_1)^{p_{k+1}}} \underbrace{(1+X)}_{f_1^{\varepsilon_2}}}_{\text{form (8)}}.$$

Indeed all subwords $(f_0 f_1)^{p_1} f_1 (f_0 f_1)^{p_2} \dots (f_0 f_1)^{p_k} f_1$ of the second form (8) correspond uniquely to an integer partition $(2p_1+1, 2p_2+1, \dots, 2p_k+1)$ in distinct odd parts; and $(2p_1+1)+(2p_2+1)+\dots+(2p_k+1)$ is the length of this subword. We obtain:

$$\begin{aligned}\Delta_{S_{I_2}}(X) &= 1 + X + \frac{X+X^2}{1-X}\Psi(X) = (1+X)\left(1 + \frac{X}{1-X}\prod_{n\geqslant 0}\left(1+X^{2n+1}\right)\right) \\ &= (1+X)\left(1+\frac{X}{1-X}\left(1+\frac{X}{1-X^2}\left(1+\frac{X^3}{1-X^4}(1+\cdots)\right)\right)\right), \qquad (17)\end{aligned}$$

which proves the first part of Theorem 2.

As mentioned in Remark 1, the set of elements which can be represented as a product of $n$ generators, includes the sets of elements of length $n, n-2, \dots$. Therefore

$$\gamma_{I_2}(n) = \sum_{i=0}^{[\frac{n}{2}]} \delta_{S_{I_2}}\left(2i + [\![n]\!]\right)$$

whence

$$\Gamma_{I_2}(X) = \frac{1}{1-X^2}\Delta_{S_{I_2}}(X) = \frac{1}{1-X}\left(1+\frac{X}{1-X}\prod_{n\geqslant 0}\left(1+X^{2n+1}\right)\right).$$

As $\gamma_{S_{I_2}}(n) = \sum_{i=0}^{n} \delta_{S_{I_2}}(i)$, one has

$$\Gamma_{S_{I_2}}(X) = \frac{1}{1-X}\Delta_{S_{I_2}}(X) = \frac{1+X}{1-X}\left(1+\frac{X}{1-X}\prod_{n\geqslant 0}\left(1+X^{2n+1}\right)\right).$$

Last two equalities complete the proof of Theorem 2.

### *5.2. Asymptotics*

We quote the following result by Richmond [18]:

**Theorem 5.** *If* $\gcd(a_1, \dots, a_s, M) = 1$*, then the number of partitions of* $n$ *into distinct parts all congruent to some* $a_i \bmod M$ *has the asymptotic value*

$$2^{(\frac{s-3}{2}+\frac{1}{M}(\sum a_i))}3^{-1/4}n^{-3/4}\exp\left(\pi\sqrt{\frac{sn}{3M}}\right)\left(1+\mathcal{O}\left(n^{-1/2+\delta}\right)\right)$$

*for any* $\delta > 0$.

In particular, for $q(n)$, we take $M = 2$, $s = 1$ and $a_1 = 1$ to obtain the asymptotics

$$q(n) \sim 2^{-1/2}3^{-1/4}n^{-3/4}\exp\left(\pi\sqrt{\frac{n}{6}}\right),$$

where $f(n) \sim g(n)$ means $\lim f(n)/g(n) = 1$.

The following result appears as Lemma 3.4 in [10]. Its proof follows from the Euler-MacLaurin summation formula:

**Lemma 6** [10]. *Let* $f$ *be a series with* $f(n) \sim n^\alpha \exp(\beta\sqrt{n})$*, and define* $g(n) = \sum_{i=1}^n f(i)$*. Then*

$$g(n) \sim \frac{2}{\beta}n^{\alpha+1/2}\exp\left(\beta\sqrt{n}\right).$$

Let us return to the first expression in (17). The term $(X + X^2)/(1 - X)$ expands to $X + 2X^2 + 2X^3 + \cdots$. We deduce:

$$\delta_{S_{I_2}}(n) = q(n-1) + 2\sum_{i=0}^{n-2} q(i) \tag{18a}$$

for $n \geqslant 2$. Moreover,

$$\gamma_{I_2}(n) = 1 + \sum_{i=0}^{n-1}(n-i)q(i), \tag{18b}$$

$$\gamma_{S_{I_2}}(n) = 2 + \sum_{i=0}^{n-1}(2n-2i-1)q(i). \tag{18c}$$

**Proof of Theorem 3.** It follows from Lemma 6 and (18a) that

$$\delta_{S_{I_2}}(n) \sim 2\sum_{i=0}^{n} q(i) \sim \frac{4\sqrt{6}}{\pi}\sqrt{n}\cdot q(n) \sim \frac{2^2 3^{1/4}}{\pi}n^{-1/4}\exp\left(\pi\sqrt{\frac{n}{6}}\right).$$

Once more, applying Lemma 6 to the equation at line above, we have the sharp estimate

$$\gamma_{S_{I_2}}(n) = \sum_{i=0}^{n} \delta_{S_{I_2}}(i) \sim \frac{48}{\pi^2} n \cdot q(n) \sim \frac{2^{7/2} 3^{3/4}}{\pi^2} n^{1/4} \exp\left(\pi \sqrt{\frac{n}{6}}\right),$$

with the ratios of left- to right-hand side tending to 1 as $n \to \infty$.

Similarly, the growth function of the automaton $I_2$ admits the sharp estimate

$$\gamma_{I_2}(n) \sim \frac{24}{\pi^2} n \cdot q(n) \sim \frac{2^{5/2} 3^{3/4}}{\pi^2} n^{1/4} \exp\left(\pi \sqrt{\frac{n}{6}}\right),$$

which completes the proof of Theorem 3. □

**Proof of Corollary 2.** From Theorem 3 it follows that

$$[\gamma_{S_{I_2}}] = \left[\exp\left(\sqrt{n}\right)\right],$$

and by Proposition 4 the same asymptotics hold for $[\gamma_{I_2}]$. □

## References


[1] A.G. Abercrombie, Subgroups and subrings of profinite rings, Math. Proc. Cambridge Philos. Soc. 116 (1994) 209–222.

[2] Y. Barnea, A. Shalev, Hausdorff dimension, pro-$p$ groups, and Kac–Moody algebras, Trans. Amer. Math. Soc. 349 (1997) 5073–5091.

[3] F. Gécseg, Products of Automata, Monogr. Theoret. Comput. Sci. EATCS Ser., vol. 7, Springer-Verlag, Berlin, ISBN 3-540-13719-X, 1986.

[4] V.M. Gluškov, Abstract theory of automata, Uspekhi Mat. Nauk 16 (1961) 3–62.

[5] R.I. Grigorchuk, On the Milnor problem of group growth, Dokl. Akad. Nauk SSSR 271 (1983) 30–33.

[6] R.I. Grigorchuk, Semigroups with cancellations of degree growth, Mat. Zametki 43 (1988) 305–319, 428.

[7] R.I. Grigorchuk, V.V. Nekrashevich, V.I. Sushchansky, Automata, dynamical systems, and groups, Proc. Steklov Inst. Math. 231 (2000) 128–203.

[8] P. de la Harpe, Topics in Geometric Group Theory, Univ. of Chicago Press, Chicago, IL, ISBN 0-226-31719-6, 0-226-31721-8, 2000.

[9] G. Lallement, Semigroups and Combinatorial Applications, Wiley, New York, ISBN 0-471-04379-6, 1979.

[10] A.A. Lavrik-Männlin, On some semigroups of intermediate growth, Internat. J. Algebra Comput. 11 (2001) 565–580.

[11] G.H. Mealy, A method for synthesizing sequential circuits, Bell System Tech. J. 34 (1955) 1045–1079, MR 17,436b.

[12] M.B. Nathanson, Number theory and semigroups of intermediate growth, Amer. Math. Monthly 106 (1999) 666–669.

[13] M.B. Nathanson, Asymptotic density and the asymptotics of partition functions, Acta Math. Hungar. 87 (2000) 179–195.

[14] I.I. Reznykov, The growth functions of two-state mealy automata over a two-symbol alphabet and the semigroups, defined by them, PhD thesis, Kyiv Taras Schevchenko University, 2002.

[15] I.I. Reznykov, V.I. Sushchansky, The growth functions of 2-state automata over the 2-symbol alphabet, Reports of the NAS of Ukraine, 2002, 76–81.

[16] I.I. Reznykov, V.I. Sushchansky, 2-generated semigroup of automatic transformations whose growth is defined by Fibonacci series, Mat. Stud. 17 (2002) 81–92.

[17] I.I. Reznykov, V.I. Sushchansky, The two-state Mealy automata over the two-symbol alphabet of the intermediate growth, Mat. Zametki 72 (2002) 102–117.
[18] L.B. Richmond, On a conjecture of Andrews, Utilitas Math. 2 (1972) 3–8.
[19] S.N. Sidki, Automorphisms of one-rooted trees: Growth, circuit structure, and acyclicity, J. Math. Sci. (N.Y.) 100 (1) (2000) 1925–1943, Algebra, 12.

## Further reading

[20] M. Abért, B. Virág, Dimension and randomness in groups acting on rooted trees, math.GR/0212191.