A Structure Theorem for Noetherian P.I. Rings with Global Dimension Two

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Let $\mathcal{A}$ be the Artin radical of a Noetherian ring $R$ of global dimension two. We show that $\mathcal{A} = ReR$ where $e$ is an idempotent; $\mathcal{A}$ contains a heredity chain of ideals and the global dimensions of the rings $R/\mathcal{A}$ and $eRe$ cannot exceed two. Assume further that $R$ is a polynomial identity ring. Let $P$ be a minimal prime ideal of $R$. Then $P = P^2$ and the global dimension of $R/P$ is also bounded by two. In particular, if the Krull dimension of $R/P$ equals two for all minimal primes $P$ then $R$ is a semiprime ring. In general, every clique of prime ideals in $R$ is finite and in the affine case $R$ is a finite module over a commutative affine subring. Additionally, when $\mathcal{A} = 0$, the ring $R$ has an Artinian quotient ring and we provide a structure theorem which shows that $R$ is obtained by a certain subidealizing process carried out on rings involving Dedekind prime rings and other homologically homogeneous rings.

1. INTRODUCTION

The major purpose of the present paper is to provide a structure theorem for Noetherian polynomial identity (P.I.) rings with global dimension two. Along the way we discover several new properties of these rings. Apart from finding an application in our work, some are also of independent interest.

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For Noetherian rings, the global dimension one (hereditary) case is well determined. By Chatter's theorem such a ring is a direct sum of prime rings and an Artinian ring. Harada showed that each Artinian part is Morita equivalent to a generalized triangular matrix ring while Jacobinski and Robson (independently) described a procedure which explains how the prime component is built up from a maximal hereditary order, and vice versa. Moreover, Robson and Small have shown that a hereditary Noetherian prime P.I. ring is, in fact, a classical order over a Dedekind domain.

In order to explain our results we need to introduce some terminology, and in particular define the \( \ast \)-process of Robson which will play a crucial role.

Let \( R \subseteq S \) be rings and \( M \) a maximal two-sided ideal in \( R \) which is a left ideal in \( S \). Suppose also that \( M \) is generative, that is, \( MS = S \). Clearly, \( R \subseteq \Pi_S(M) \), the idealizer of \( M \) in \( S \). The ring \( R \) is called a \emph{tame subidealizer} of \( M \) in \( S \), and in fact \( M^\ast = \text{Hom}_R(H_R, R_R) = S \). Further, \( R \) is called a \emph{tame idealizer} if in addition \( R = \Pi_S(M) \). Moreover we say that \( R \) is a \emph{special tame subidealizer} provided that \( M \) is, in addition a projective \( S \)-module. In each of these cases we say that \( S \) is obtained from \( R \) by the \( \ast \)-process. Analogously one can deal with the \#-process where \( M^\# = \text{Hom}_R(H_R, R_R) \). This \#-process was used by Robson to proceed from \( R \), an hereditary Noetherian prime (HNP) ring, to a maximal order containing it.

Proceeding to the two dimensional case, we need to develop new techniques as, unlike for hereditary rings, the problem can no longer be split into separate, prime, and Artinian cases. In our results an important role is played by the \emph{Artin radical} \( A(R) \) of a ring \( R \). This is an ideal of \( R \) which contains all Artinian one-sided ideals and is itself Artinian as an \( R \)-module on either side. Moreover, we have \( A(R/A(R)) = 0 \). We shall show that \( A(R) \) and \( A(R)_R \) are projective modules, and that \( A(R) = A(R) \). It follows by standard results that \( \text{gl. dim } R/A(R) \leq 2 \). This enables us to split our investigation into two parts, one dealing with the structure of \( A(R) \) and the other with that of \( R/A(R) \) where the latter object is a ring of global dimension two (or less) but additionally, with zero Artin radical. To resolve these separate issues different methods are required.

The following facts concerning \( A(R) \) do not require the P.I. assumption and are applicable to any Noetherian ring of global dimension two. A key result here grants the existence of a hereditary chain of ideals \( 0 = I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots \subseteq I_k = A(R) \) where \( I_j/I_{j-1} \) is a hereditary ideal in \( R/I_{j-1} \) for \( j = 1, 2, \ldots, k \). We recall here that \( I \) is a \emph{hereditary ideal} if \( I_R \) is projective and \( \text{End}(I_R) \) is a semi-simple Artinian ring. This notion first appeared in the theory of quasi-hereditary finite dimensional algebras [CPR] and was later extended to the non-Artinian case (e.g., by S. König [K]). The gist of
the result here is that $A(R)$ is built from a flag of ideals where the endomorphism rings of the quotients of successive terms have a very simple structure. In proving the last statement we follow the quasi-hereditary technique rather closely and, in particular, the work of Dlab and Ringel [DR]. Two other key results on the Artin radical are that $A(R) = ReR$ for some $e = e^2 \in R$ and $\text{gl.dim} \ eRe \leq 2$. Now $eRe$ is an Artinian ring and so by [DR], $eRe$ is a quasi-hereditary ring. This means that a sequence as above exists in $eRe$. Therefore, the results described above may be viewed as a generalization of those in [DR] to the non-Artinian case.

We shall now describe the case when $\text{gl.dim} \ R = 2$ and $A(R) = 0$. Here the P.I. assumption is important for most of the results and is crucial for the main structure theorem. As a consequence of this assumption $R$ contains only a finitely many idempotent ideals [R51] allowing us to use induction on this number. We show that such a ring has an Artinian hereditary quotient ring $Q(R)$. Inside $Q(R)$ the $*$-process can be performed on maximal projective ideals of $R$. So if $M$ is a maximal ideal of $R$ with $M^2 = M$ and $M_R$ projective then $M^* = S$ is an overring of $R$, and $R_S$ is finitely generated and projective. The ring $S$ inherits the conditions that $R$ satisfies but has fewer idempotent ideals. Repeating this process yields a direct sum of rings $T$ where $A(T) = 0$ and $T$ has no maximal ideal as above. If $T$ is a semiprime ring then $T$ is a “homologically homogeneous” (hom-hom) ring [SZ], a notion which coincides, in our case, with that of a hereditary maximal order, if $k.\text{dim} \ R = 1$, and is reasonably well understood in the case $k.\text{dim} \ R = 2$. If $T$ is not semiprime we show that $T \cong (\mathbb{C} \times \mathbb{N}, D)$ where $D$ is a maximal hereditary order (i.e., a Dedekind prime ring), $C$ shares the assumptions on $R$ but has fewer idempotent ideals, and $\mathbb{C} \times \mathbb{N}$ and $\mathbb{N}_D$ are finitely generated and projective. Of course, the same procedure can now be applied to $C$.

Analogous to the properties of $A(R)$, in the P.I. case we also have $P = P^2$ and $\text{gl.dim} \ R/P \leq 2$ for minimal prime ideals $P$ of $R$. In addition, $R_P$ and $P_R$ are projective when $A(R) = 0$. Indeed the case $A(R) = 0$ seems to yield results which run parallel to the hereditary prime case.

We also obtain two somewhat unexpected results. A clique (i.e., a minimal localisable set of primes) is always finite. This is in spite of the fact that $R$ is usually far from being a finite module over a central subring. The other surprising fact is that when $R$ is affine, $R$ is a finite module over a commutative (not necessarily central) subring.

Finally, we note that attempts to extend our results to higher global dimension face formidable obstructions. For example, the ring $R[x]$ where $R = \mathbb{Z}/p^2 \mathbb{Z}/p^2$ ($p$ prime) shows that Theorem 3.6 is no longer true in dimension 3. It is also worth pointing out that our methods in Section 3 rely crucially on the P.I. assumption, as we need to apply [BW, Corollary 5]
to obtain that the nil radical and minimal primes are projective. It is likely, however, that these results hold in a more general setting.

Further properties of the rings studied and statements of results in greater detail can be found in the section below.

**Statements of the Main Results**

**Theorem A** (The Structure of the Artin Radical). *Let* $R$ *be a Noetherian ring with global dimension 2 (or less), and* $A(R)$ *its Artin radical. Then*

(i) $A(R) = A(R)^2$ *and* $A(R)$ *is projective as a left and right* $R$ *module. Consequently* $\text{gl.dim } R/A(R) \leq 2$;

(ii) $A(R) = ReR$ *for some idempotent* $e$;

(iii) There is an hereditary chain of ideals in $R$, $I_1 = Re_1 R \subseteq R(e_1 + e_2) R = \cdots I_k = Re_k R$, where $e = e_1 + e_2 + \cdots + e_k$ *is a decomposition of* $e$ *into primitive, mutually orthogonal idempotents*;

(vi) $\text{gl.dim } eRe \leq 2$.

**Remark.** Observe that this theorem is true without the P.I. assumption on $R$.

**Theorem B** (The Case of $A(R) = 0$). *Let* $R$ *be a Noetherian P.I. ring with global dimension 2 (or less) and* $A(R) = 0$. *Then* $R$ *has an Artinian quotient ring and* $R$ *is obtained by finitely many iterations of tame idealizers and/or special tame subidealizer, from* $\bigoplus_{i=1}^d A_i$, *where each* $A_i$ *satisfies:*

(i) $\text{gl.dim } A_i \leq 2$, *and the minimal primes of* $A_i$ *consist of a single clique.*

(ii) If $A_i$ *is semiprime, then* $A_i$ *is a homologically homogeneous (hom-hom) ring (along each maximal clique) of k.dimension 2 or a Dedekind prime ring.*

(iii) If $A_i$ *is not semi-prime then* $k \text{dim } A_i = 1$ *and* $A_i = (c, D)$, *where* $D$ *is a Dedekind prime ring,* $cV$ *and* $V_D$ *are finitely generated and projective, and* $C$ *satisfies all the conditions on* $R$ *but has fewer minimal primes.*

(iv) Each of the tame idealizer and special tame subidealizer is performed inside $Q(R)$, *the Artinian quotient ring of* $R$.

Conversely, given $R$, *a Noetherian P.I. ring of global dimension 2 with* $A(R) = 0$, *then the* $*$-process, *applied to* $R$, *yields, after finitely many steps, $\bigoplus_{i=1}^d A_i$, as above.*

**Remarks.** (1) Iterating the $*$-process on $C$, if it is not Dedekind, yields at the end (after finitely many steps), a finite direct-sum of hom-hom rings
(along their maximal cliques) or generalized upper triangular matrices with Dedekind prime rings in their corners.

2 The hom-hom rings appear in Theorem B(ii), as a final object. They have a fairly well understood structure (e.g., [BoH]) and are considered by some as a good substitute to maximal orders. Moreover, in [RV] a complete classification of this family of rings is obtained in case the center of $R$ is normal, local, and of characteristic 0.

3 Suppose $R$ is obtained as a tame idealizer from $S$ where gl.dim $S = 2$. It is not clear (to us) what is a necessary and sufficient condition for gl.dim $R \leq 2$. This problem, surprisingly, does not occur if $R$ is a tame special subidealizer, as shown in Corollary 5.11, and gl.dim $R \leq 2$ is secured in that case.

Many important properties and results are proved along the way as well as corollaries. We summarize some of them into one theorem.

**Theorem C.** Let $P$ be a Noetherian P.I. ring with global dimension 2 (or less). Then

1. $R$ has an Artinian quotient ring $Q(R)$, if and only if $A(R)$ is left principal as well as right principal. In particular $Q(R)$ exists if $A(R) = 0$ (Theorem 3.6).

2. $R$ is semiprime, provided $k.dim R/P = 2$, for each minimal prime ideal $P$ (Theorem 3.11).

3. $P^2 = P$ for each minimal prime ideal $P$ and therefore gl.dim $R/P \leq 2$. In particular if $N(R)$ is prime then $R$ is prime (Theorem 3.4.)

4. $R$ is a finite module over a commutative affine subring provided $R$ is in addition affine (Theorem 4.12).

5. Every clique in $R$ is finite and the cardinality of the clique exceeds (or equals) the number of distinct minimal primes which are contained in members of the clique (Theorem 5.13, Proposition 5.14).

6. Suppose also $A(R) = \{0\}$. Then there exists a Noetherian P.I. semi-prime ring $S$, gl.dim $S \leq 2$, $R \subset S$, and $S$ is a finitely generated projective left $R$-module (Theorem 4.16).

We next consider the question of how is $R$ built from $R/A(R)$ and $eRe$. It is proved in Theorem 2.18 that $R$ is obtained from $R/A(R)$ and $eRe$ via a (finite) succession of the “not so trivial extension” construction. This construction is due to Parshall-Scott (e.g., [DR1, p. 172]). However, given $S(= R/A(R))$ a P.I. Noetherian ring with gl.dim $S \leq 2$ and $A(S) = 0$ and $T(= eRe)$ an Artinian P.I. ring with gl.dim $T \leq 2$, it is not clear how to grant that $R$, which is built via the previous construction, will have gl.dim $R \leq 2$. 
We offer two applications of our methods and results.

**Theorem 6.1.** Let $R$ be a Noetherian P.I. ring with $\text{gl.dim } R \leq 2$. Suppose that $R$ has exactly two maximal (two-sided) ideals. Then $R$ is one of the following:

(i) $R$ is Artinian,

(ii) $R$ is prime,

(iii) $R \cong (R/P_1 \cap P_2^{(2)})$, and $R/P_i$ is a maximal order with $\text{gl.dim } R/P_i \leq 2$. Here, $(P_1, P_2)$ is the set of minimal prime ideals of $R$.

Our final application is a generalization of a theorem of König and Wiedemann [KW].

**Theorem 6.3.** Let $R$ be a semi-prime Noetherian P.I. ring with $\text{gl.dim } R \leq 2$. Then $R$ is quasi-hereditary.

The paper is organized as follows. In Section 2 we provide the proof of Theorem A as well as of Theorem 2.18. In Section 3 we prove an essential part of Theorem C, however, items (4), (5), and (6) are deferred to later sections. We also consider here a question raised by Vasconcelos. In Section 4 we essentially start with the proof of Theorem B. We provide here, as well, proofs of Theorems C(4) and C(6). Section 5 is mainly concerned with the completion of the proof of Theorem B. We also prove here Theorem C(5). In Section 6 we prove Theorems 6.1 and 6.3.

**Conventions and Notation**

We follow the standard usage and terminology of ring theory and refer the reader to [McR, GW] for the background material. Some of the notation used is explained below.

$A \supset B$ means that $B$ is properly contained in $A$.

$\mathcal{Q}(I)$ is the set of all elements of $R$ which are regular mod. the ideal $I$.

$k\text{.dim } R$ stands for the classical Krull dimension of $R$.

$\text{pr.dim } M$ denotes the projective dimension of the module $M$.

$P \cap Q$ stands for a link between the prime ideals $P$ and $Q$ and means that there exists an ideal $A$, $PQ \subseteq A \subseteq P \cap Q$ such that $P \cap Q/A$ is left torsion free $R/P$-module as well as right torsion free $R/Q$-module.

$r(R)$ is the number of distinct proper idempotent ideals in $R$.

$Q(R)$ is the full quotient ring of $R$ (whenever it exists).

$N(R)$ stands for the nil radical of $R$, and $N(I)$ is the intersection of all prime ideals which contain the ideal $I$. 
\[ Z(R) \text{ is the center of } R. \]

\[ R(x_1, \ldots, x_k) \text{ stands for the ring generated by } R \text{ and } x_1, \ldots, x_k. \]

Two standard results we need, on several occasions. First, every Noetherian hereditary ring is a direct sum of prime rings and Artinian rings [MCR, p. 151]. Second, if \( I \) is a one-sided projective, idempotent ideal, then \( \text{gl.dim} R/I \leq \text{gl.dim} R [\text{MCR}, \text{p. 242}]. \)

### 2. THE STRUCTURE OF THE ARTIN RADICAL \( \mathcal{A}(R) \)—(THE PROOF OF THEOREM A)

We deal in this section with the structure of \( \mathcal{A}(R) \), the Artin radical of \( R \). It is a two-sided ideal which contains every one-sided Artinian ideal of \( R \) and \( \mathcal{A}(R)/\mathcal{A}(R) = 0 \). For further properties of this ideal, the reader should consult [CH], or the Introduction. We assume here that \( R \) is Noetherian with \( \text{gl.dim} R \leq 2 \) (but not necessarily P.I.). We prove, at the beginning, some general results, and we end the section with results which resemble similar ones for quasi-hereditary Artinian algebras. In fact, some of the quasi-hereditary technique is used here. In proving Theorem A we follow the proof of [DR] rather closely, and then we use Theorem 2.14 to show that \( \text{gl.dim} eRe \leq 2 \). There is an alternative approach. Indeed one can start with \( \text{gl.dim} eRe \leq 2 \), use [DR] results to exhibit an hereditary chain inside \( eRe \), and then inflate it to an hereditary chain inside \( R \) which terminates with \( ReR \).

**Lemma 2.1.** Let \( R \) be a Noetherian ring and \( \mathcal{A}(R) \) its Artin radical. Then

\[ \text{l-ann}_R(\text{r-ann}_R(\mathcal{A}(R))) = \mathcal{A}(R). \]

**Proof.** \( \mathcal{A}(R) \) is a faithful, Artinian, right \( R/\text{r-ann}_R(\mathcal{A}(R)) \)-module, and therefore \( R/\text{r-ann}_R(\mathcal{A}(R)) \) is right (and left) Artinian. Now \( \text{l-ann}_R(\text{r-ann}_R(\mathcal{A}(R))) \) is a finitely generated right \( R/\text{r-ann}(\mathcal{A}(R)) \)-module. Consequently, \( \text{l-ann}_R(\text{r-ann}_R(\mathcal{A}(R))) \) is a right Artinian ideal and therefore \( \text{l-ann}_R(\text{r-ann}_R(\mathcal{A}(R))) \subseteq \mathcal{A}(R) \). The converse inclusion is trivial. Q.E.D.

**Corollary 2.2.** Suppose \( R \) is Noetherian with \( \text{gl.dim} R \leq 2 \). Then \( \mathcal{A}(R), \mathcal{A}(R)_R \) are projective \( R \)-modules.

**Proof.** Denote \( X \equiv \text{r-ann}_R(\mathcal{A}(R)) \) and hence, by Lemma 2.1, \( \text{l-ann}_R(X) = \mathcal{A}(R) \). Consequently, there is a left \( R \)-module inclusion; \( R/\mathcal{A}(R) \rightarrow X \oplus \cdots \oplus X \). Consequently, since \( X \subseteq R \) we have that \( \text{pr.dim}_R[R/\mathcal{A}(R)] \leq \text{pr.dim}_R X \leq 1 \). Therefore \( R \mathcal{A}(R) \) is projective. The other statement follows by symmetry. Q.E.D.
**Theorem 2.3.** Let \( R \) be a Noetherian ring with \( \text{gl.dim } R \leq 2 \). Then \( A(R)^2 = A(R) \), \( A(R)^2 \), \( \text{gl.dim } R \leq 2 \). Consequently

\[ A(R)^2 = A(R) \]

**Proof.** \( A(R) \) is finitely generated and projective. Let \( \{ x_i, f_i \} \) be a dual basis. That is, \( a = \sum x_i f_i(a) \) for each \( a \in A(R) \), where \( x_i \in A(R) \) and \( f_i \in \text{Hom}_R(A(R), R) \). Clearly, \( f_i(A(R)) \) is a right ideal in \( R \) which is a homomorphic image of \( A(R) \). Consequently, \( f_i(A(R)) \subseteq A(R) \), and therefore \( a \in A(R)^2 \), that is, \( A(R) = A(R)^2 \). The result \( \text{gl.dim } R \leq 2 \) now follows from a result of Fields e.g., [McR, p. 240]. Q.E.D.

**Remark.** Theorem 2.3 clearly separates our discussion into two disjoint cases: \( A(R) \) and \( R/A(R) \). The latter has global dimension 2 or less and \( A(R/A(R)) = 0 \). This will be the subject of a later section.

**Lemma 2.4.** Let \( R \) be a Noetherian ring with \( A(R) = A(R)^2 \). Then \( A(R) = ReR \), for some idempotent \( e \) in \( A(R) \).

**Proof.** Let \( N(R) \) be the nilpotent radical of \( R \). By [McR, 4.1.8], \( A(R/N) \) is a direct summand of \( R/N \) and is a semi-simple Artinian ring. Now \( A(R) + N/N \) is an idempotent ideal in \( A(R/N) \) and therefore a direct summand of it. Consequently \( A(R) + N/N = (R/N)e \), for some central \( e \) in \( R/N \). Equivalently \( A(R) = Re + A(R) \cap N(R) \), where \( e \) is an idempotent in \( A(R) \) which projects onto \( e \). Let \( k \) be chosen so that \( N(R)^k = 0 \). Then

\[
A(R) = A(R)^k \subseteq ReR + [A(R) \cap N(R)]^k = ReR \subseteq A(R). \quad \text{Q.E.D.}
\]

**Lemma 2.5.** Let \( R \) be a Noetherian ring with \( \text{gl.dim } R \leq 2 \). Then \( R A(R) \equiv \bigoplus_{i=1}^k Re_i^{(n_i)} \), where \( e = e_1 + \cdots + e_k \), is a decomposition of \( e \) into primitive, mutually orthogonal, idempotents.

**Proof.** By Lemma 2.4 and Theorem 2.3, \( A(R) = ReR \). Consequently there exists a left \( R \)-module onto map \( \nu, \nu': Re_i^{(n_i)} \to A(R) \). By the left projectivity of \( R A(R) \) we have \( A(R) \oplus v' \subseteq Re_i^{(n_i)} \). Now \( Re = Re_1 \oplus \cdots \oplus Re_k \), and the result follows by the Krull-Schmidt theorem. Q.E.D.

**Lemma 2.6.** Let \( P \) be a finitely generated projective left \( R \)-module. Then \( A(P) = A(R) \cdot P \), where \( A(P) \) is the Artin radical of \( P \).

**Proof.** The result is obvious for a free left finitely generated \( R \)-module \( F \). Let \( P' \) be a projective module so that \( P \oplus P' = F \) where \( RF \) free. Clearly \( A(P) \oplus A(P') = A(F) = A(R) \cdot F = A(R) \cdot P \oplus A(R) \cdot P' \). Consequently \( A(P) = A(R) \cdot P \). Q.E.D.
**Proposition 2.7.** Let $R$ be a Noetherian ring with $\text{gl.dim } R \leq 2$. Let $\tilde{R}P$ be a finitely generated, projective, Artinian $R$-module. Then $\tilde{R}P \cong \bigoplus_{i=1}^{k} \tilde{R}e_i^{(n_i)}$, where the $\{e_i\}$ are chosen as in Lemma 2.5.

**Proof.** Let $\tilde{R}U$ be a finitely generated projective $R$-module satisfying $\tilde{R}P \oplus_{\tilde{R}} U = R^{(n)}$. Multiplying by $\tilde{A}(R)$ on the left yields $\tilde{A}(R) \cdot P \oplus \tilde{A}(R)U = \tilde{A}(R)^{(n)}$. By assumption $P = \tilde{A}(P)$ and by Lemma 2.6, $\tilde{A}(P) = \tilde{A}(R) \cdot P$. Hence $P \oplus \tilde{A}(R) \cdot U = \tilde{A}(R)^{(n)}$. The result now follows from Lemma 2.5 and the Krull–Schmidt theorem. Q.E.D.

**Proposition 2.8.** Let $\tilde{R}M$ be a finitely generated module over the Noetherian ring $R$ and $\tilde{A}(R)M = M$. Then, there exists an onto $R$-module map $\beta: \tilde{R}P \to \tilde{R}M$, where $P$ is a projective Artinian $R$-module and $\ker \beta \subseteq N(R)P$. Conversely, the existence of $\beta$ and $P$ implies that $\tilde{A}(R)M = M$.

**Remark.** This is really a statement about the existence of a projective cover, and the proof resembles known results.

**Proof.** The condition $\tilde{A}(R)M = M$ implies that $\tilde{R}M$ is an Artinian $R$-module. Let $N = N(R)$. Now $\tilde{A}(R/N)$ splits inside $R/N$, that is, $\tilde{A}(R/N) = (R/N)^\tilde{f}$, where $\tilde{f}$ is a central idempotent in $R/N$. Let $A = \tilde{A}(R)$. Then since $A = M$ we get $(A + N/N) \cdot M/NM = M/NM$. In particular, $M/NM$ is a unital finitely generated module over the semi-simple Artinian ring $\tilde{A}(R/N) = \tilde{(R/N)^\tilde{f}}$. Hence $M/NM \cong (R/N)^{\tilde{f}^{(n_i)}} \oplus \cdots \oplus (R/N)^{\tilde{f}^{(n_i)}}$, where $\{\tilde{f}_i\}$ are primitive, orthogonal idempotents which decompose $\tilde{f}$. Let $P = \bigoplus_{i=1}^{k} (R\tilde{e}_i)^{(n_i)}$, where $\{\tilde{e}_i\}$ is a set of mutually orthogonal idempotents which project onto $\{\tilde{f}_i\}$. Then $\tilde{R}P$ is a finitely generated projective $R$-module. Moreover $P/NP \cong \bigoplus_{i=1}^{k} (R\tilde{e}_i/N\tilde{e}_i)^{(n_i)} = \bigoplus_{i=1}^{k} (R/N)^{\tilde{f}^{(n_i)}} \cong M/NM$. We denote by $\alpha$ the isomorphism $P/NP \cong M/NM$. Consider $\nu = \alpha \circ \pi'$, where $\pi': P \to P/NP$ is the natural projection. Also let $\pi': M \to M/NM$ be the natural projection. Then the projectivity of $\tilde{R}P$ implies the existence of $\beta: P \to M$, with $\nu = \pi \circ \beta$. Clearly $\beta(P) + NM/NM = \nu(P) = M/NM$, which implies that $\beta(P) + NM = M$. Therefore, by Nakayama’s lemma, $M = \beta(P)$. Also $\ker \beta \subseteq \ker \nu = NP$, as needed. Conversely, by Lemma 2.6, $\tilde{A}(R)P = P$, hence $\tilde{A}(R)(P/\ker \beta) = (\tilde{A}(R)P + \ker \beta)/\ker \beta = P/\ker \beta = M$. Q.E.D.

**Corollary 2.9.** Let $\tilde{R}M$ be a finitely generated $R$-module over a Noetherian ring $R$. Suppose $\tilde{A}(R)M = M$ and let $\tilde{R}P$ be the “projective cover” of $M$ (as in Proposition 2.8). Let $\tilde{R}Q$ be a finitely generated projective $R$-module which is mapped onto $M$. Then $P$ is isomorphic to a direct summand of $Q$.

**Proof.** This is standard (e.g., [Rw, Vol. 1, p. 233]).
**Definition.** (i) An ideal $I$ in $R$ is called an hereditary ideal if $I^2 = I$, $RI$, $IR$ are projective and $IN(R)I = (0)$.

(ii) The sequence of ideals $0 = I_0 \subset I_1 \subset \cdots \subset I_k$ is called an hereditary chain if $I_i/I_{i-1}$ is an hereditary ideal in $R/I_{i-1}$, for each $i = 1, \ldots, k$.

The next result proves Theorem A (iii).

**Theorem 2.10.** Let $R$ be a Noetherian ring with $\text{gl.dim } R \leq 2$. Let $A(R) = ReR$ and $e = e_1 + \cdots + e_s$ be a decomposition of $e$ into mutual orthogonal, primitive idempotents. Then, there exists an hereditary chain of ideals $0 = I_0 \subset I_1 \subset \cdots \subset I_k = A(R)$, and (after a possible renumbering of $e_1, e_2, \ldots$) $I_i = R(e_1 + \cdots + e_i)R$, for $i = 1, \ldots, k$.

**Proof.** The proof follows [DR, Theorem 2] rather closely. Let $f \in A(R)$ be a primitive idempotent with minimal Loewy length $L(Rf)$. By Proposition 2.7, $Rf \cong Re_f$ for some $f$. We first show that $fN(R)f = 0$. Indeed let $0 \neq x \in fNf$. Then, right multiplication by $x$ yields an endomorphism of $Rf$ with non-zero kernel $K$, $K \subseteq N(R)f$. Consequently, by $\text{gl.dim } R \leq 2$, $K$ is projective (and obviously Artinian). So, by Proposition 2.7, $K = Re_{f^{(i)}} \oplus \cdots \oplus Re_{f^{(m)}}$, which implies, by the minimality of $L(Rf)$, that $L(K) \geq L(Rf)$. However, $K \subseteq N(R)f$ implies that $L(K) \leq L(Rf) - 1$, which contradicts the previous inequality. Therefore $fN(R)f = 0$; hence $fRf$ is simple Artinian. We next show that $I = RfR$ is projective as left and right $R$-modules. Clearly, $I = \sum_{j=1}^s Rf_{ij}$, for some $y_{ij}$ in $R$, hence, there is an $R$-module epimorphism from $Q = (Rf)^{(s)}$ onto $I$. Now $I^2 = I \subseteq A(R)$ implies that $A(R)f = I$ and, by Proposition 2.8, $I$ has a projective cover $P$. By Corollary 2.9 and the Krull–Schmidt theorem we have that $P \cong (Rf)^{(s)}$, with $s \leq n$. Let $\beta: (Rf)^{(s)} \rightarrow I$, be the projective cover map. We shall show that $\beta$ is injective. Indeed if $\text{Ker } \beta \neq 0$, then $\beta'$, the restriction of $\beta$ to one of the coordinates, is not injective. So $\beta': Rf \rightarrow I$ has non-zero kernel. However, ker $\beta \subseteq (Nf)^{(i)}$ implies that ker $\beta' \subseteq Nf$. Now ker $\beta'$ being projective provides us with the inequality $L(\text{Ker } \beta') \geq L(Rf)$ which is a contradiction to $L(\text{Ker } \beta') \leq L(Nf) \leq L(Rf) - 1$. So $(Rf)^{(i)} = I$ as needed. A similar argument can be given for the right projectivity of $I$. Recall that $Rf \cong Re_j$ for some $j$. We renumber $(e_j)$ so that $Rf \cong Re_1$, and therefore $I_1 = Re_1R$ is a left and right projective $R$-module and $e_1N(R)e_1 = 0$. Therefore $I_1N(R)I_1 = Re_1RN(R)Re_1R = 0$, that is, $I_1$ is an hereditary ideal. Now, $\text{gl.dim } R/I_1 \leq 2$. Also $A(R/I_1) = A(R)/I_1$, and $e_2, \ldots, e_s$ is a decomposition of $\bar{e}$ into primitive orthogonal idempotents. The result now follows by induction on $k$. Finally $e_j = e_j(e_1 + \cdots + e_i)$ for each $j \leq i$, shows that $I_i = Re_1R + \cdots + Re_iR = R(e_1 + \cdots + e_i)R$, for each $1 \leq i \leq k$.

Q.E.D.
Remark. Theorem 2.10 can be regarded as a generalization of Theorem 2 in [DR].

We shall proceed now to show that \( \text{gl.dim } eRe \leq 2 \). To this end we need the following several results.

**Lemma 2.11.** Let \( e^2 = e \in R \) such that \( ReR \) is finitely generated and projective. Then \( Re \) is a finitely generated projective right \( eRe \)-module.

**Proof.** \( Re \oplus V \cong (eR)^{(r)} \), as right \( R \)-modules. Multiplication by \( e \) on the right yields \( ReRe \oplus Ve \cong (eRe)^{(r)} \), as right \( eRe \)-module. Q.E.D.

**Corollary 2.12.** Let \( P_R \) be a projective \( R \)-module then \( Pe_{eRe} \) is a projective \( eRe \)-module.

**Proof.** \( P \oplus W \cong R^{(n)} \) implies that \( Pe \oplus We \cong (Re)^{(n)} \), as right \( eRe \)-modules. Now use the previous lemma. Q.E.D.

**Lemma 2.13.** Let \( 0 \to K_R \to P_R \to M_R \to 0 \) be an exact sequence with \( P_R \) projective and \( ReR \) projective. Then the induced sequence of right \( eRe \)-modules, \( 0 \to Ke \to Pe \to Me \to 0 \), is exact.

**Proof.** Let \( \alpha : P_R \to M_R \) be the given surjection and define \( \beta \) to be the restriction of \( \alpha \) to \( Pe \). Then \( \beta : Pe \to Me \) is onto and a right \( eRe \)-module map. Also \( \text{Ker } \beta = Pe \cap \text{Ker } \alpha = Pe \cap K = Ke \).

The following result should have been known (special cases do appear in the literature).

**Theorem 2.14.** Let \( R \) be a ring with \( e^2 = e \in R \) and \( ReR \) is finitely generated projective. Then \( r.\text{gl.dim } eRe \leq r.\text{gl.dim } R \).

**Proof.** Let \( n = r.\text{gl.dim } R \). We clearly may assume that \( n \) is finite. Let \( M \) be a right ideal in \( eRe \). Then \( r.\text{pr.dim}_R (MeR) \leq n - 1 \). Given the projective resolution

\[
0 \to P_{n-1} \to P_{n-2} \to \cdots P_0 \to MeR \to 0,
\]

and the sequence

\[
0 \to P_{n-1}e \to P_{n-2}e \to \cdots P_0e \to (MeR)e \to 0.
\]

Then by Corollary 2.12 and Lemma 2.13, the latter sequence is an \( eRe \)-projective resolution of \( M = (MeR)e \), and the result follows. Q.E.D.

**Remarks.** (1) A more elaborate proof shows the same without \( ReR \) being finitely generated.

(2) A proof of Theorem 2.14 can also be given by using Lemma 2.11 and [KK, Proposition 2.2].
(3) In fact, we only need $ReRe$ finitely generated and projective, to obtain the inequality $r.gl.dim eRe \leq r.gl.dim R$. Using this observation, with $P_R$ being a finitely generated projective right $R$-module, we obtain the following, somewhat stronger result. We leave the details to the interested reader.

**Theorem 2.15.** Let $R$ be a ring and $P_R$ a finitely generated projective $R$-module. Suppose that $\text{Hom}_{R}(P, R)$ is a finitely generated projective $\text{End}_{R}(P)$-module. Then $r.gl.dim \text{End}_{R}(P) \leq r.gl.dim R$.

We now establish Theorem A (iv).

**Corollary 2.16.** Let $R$ be a Noetherian ring with $\text{gl.dim } R \leq 2$ and $A(R) = ReR$ with $e^2 = e$. Then $\text{gl.dim } eRe \leq 2$.

**Proof.** This is straightforward by using Theorem 2.3 and Theorem 2.14. Q.E.D.

Our next result is an application of Theorem 2.10.

**Proposition 2.17.** Let $R$ be a Noetherian ring with $\text{gl.dim } R \leq 2$. Let $M$ be a maximal two-sided ideal in $R$ which is a minimal prime, as well. Then $M^2 = M$.

**Proof.** Recall that $A(R/A(R)) = 0$. Consequently if $M \supseteq A(R)$ we must get that $M/\text{Ann}(R)$ is a maximal ideal as well as a minimal prime in a Noetherian ring with zero Artin radical. This contradicts [St, Lemma 2.2]. Therefore $M \nsubseteq A(R)$ and in particular $A(R) \neq 0$. The proof will proceed by induction on the length ($A(R)$). We retain the notation of Theorem 2.10. If $M \supseteq I_1$, then, since $\text{gl.dim } R/I_1 \leq 2$ and $\text{length}[R/A(R/I_1)] = \text{length}[R/A(R)/I_1] < \text{length } A(R)$, we get that $[M/I_1]^2 = M/I_1$. Consequently, since $I_1^2 = I_1$, we obtain $M^2 = M$, as needed. Suppose therefore that $I_1 \nsubseteq M$, that is, $e_1 \notin M$. Recall that $\text{N}(e_1 Re_1) = e_1 \text{N}(R)e_1 = 0$ and that $e_1 Re_1$ is a simple Artinian ring. Therefore $e_1 \notin e_1 Me_1$ implies that $e_1 Me_1 = 0$. Let $\overline{R} = R/M^2$, which is a local ring. Then $(Re_1 R)M(Re_1 R) = \overline{R} e_1 Me_1 \overline{R} = (0)$. Hence, since $Re_1 R$ is a two-sided ideal in $\overline{R}$ which is not contained in $M$, we get that $\overline{R} e_1 \overline{R} = \overline{R}$. Consequently $\overline{R} M \overline{R} = (0)$, or $M = M^2$.

**Remarks.** (1) This result can be considered as a generalization of [BF, Proposition 1.4], in the Artinian case, since we did not really use the $\text{gl.dim } R \leq 2$ assumption but only the existence of an hereditary chain inside $A(R)$. (It is rather straightforward to translate the statement $\text{Ext}^1(S, S) = 0$ for $S$ simple, into the statement $M^2 = M$ where $M = \text{l-ann } R S$).
The next result shows how to build $R$ from its “simple” parts: $R/A(R)$ and $eRe$. This method is called “not so trivial extension” by Parshall and Scott (e.g., [D R 1, pp. 172–173]). So in principle knowledge of $R/A(R)$ by Theorem B and that of $eRe$ (via the Artinian theory) should determine $R$. This is not quite so because there is no mechanism (to the best of our knowledge) to decide when $\text{gl.dim} R \leq 2$ (given, to begin with, that $\text{gl.dim} R/A(R) \leq 2$ and $\text{gl.dim} eRe \leq 2$).

Theorem 2.18. Let $R$ be a Noetherian ring with $\text{gl.dim} R \leq 2$ and $A(R) = \text{Re} R$ its Artinian radical. Then $R$ is obtained from $R/A(R)$ and $eRe$ via a succession of the “not so trivial extension” construction.

Proof. We follow the method of [D R 1]. By Theorem 2.10 we have the hereditary chain $I_1 = Re_1 R \subset R(e_1 + e_2) R \subset \cdots \subset ReR = I_k$, in particular $e_1 R e_1$ is semi-simple. Also, $e_1$ can be chosen to satisfy $e_1 R (1 - e_1) \subseteq N(R)$ as well as $(1 - e_1) Re_1 \subseteq N(R)$. Clearly $U \equiv (1 - e_1) Re_1 (1 - e_1)$ is an ideal in $D = (1 - e_1) R (1 - e_1)$ and, since $e_1 Re_1$ is semi-simple, $U^2 \subseteq N(R) \cap e_1 Re_1 = 0$. Moreover, if $D = R/Re_1 R$ then $D/U \cong D$ and therefore $0 \to U \to D \to D \to 0$ is a “Hochschild extension.” Hence $D$ is uniquely determined by $U$ and $D$. Let $X = (1 - e_1) Re_1$ and $Y = e_1 R (1 - e_1)$. Observe that $Re_1 R \cdot X \subseteq N(R) \cap e_1 Re_1 = (0)$ and $Y \cdot Re_1 R \subseteq N(R) \cap e_1 Re_1 = (0)$ which shows that $X$ is a $D - e_1 Re_1$ bimodule and $Y$ is a $e_1 Re_1 - D$ bimodule. Also, since $Re_1 R$ is an hereditary ideal, then $Re_1 \otimes_{e_1 Re_1} e_1 R \cong Re_1 R$, via the multiplication map. Consequently (e.g., [D R 1, p. 159]) $X \otimes_{e_1 Re_1} Y \cong U$ by the multiplication map and $Y \otimes X = 0$. Consequently, since $R = e_1 Re_1 \oplus (1 - e_1) Re_1 \oplus e_1 R(1 - e_1) \oplus (1 - e_1) R(1 - e_1) = e_1 Re_1 \oplus X \oplus Y \oplus D$, then $R$ is uniquely determined by $e_1 Re_1$, $X$, $Y$, $X \otimes Y = U$, and $R/Re_1 R = D$. Now $D = R/Re_2 R$ has a shorter hereditary chain,

$$\tilde{I}_2 \subset D e_2 D \subset I_3 \subset \cdots \subset \tilde{I}_k = D e D = ReR/Re_1 R,$$

and $\text{gl.dim} D \leq 2$, so we get the result by induction on the length of the hereditary chain. Recall that $e_1 Re_1 \subseteq eRe$ so $R$ is built from $eRe$, $R/Re_1 R$, and the bimodules $X, Y$ as claimed. Q.E.D.

3. THE PROOF OF (PART OF) THEOREM C

As stated in the title we assemble here several results, of independent interest, which will be used in Section 4 as well. The issues with which we are concerned here will include the behavior of minimal primes and the
existence of (an Artinian) quotient ring as well as a question of Vasconcelos. However, the proofs of items (4), (5), and (6) of Theorem C are deferred to Sections 4 and 5.

**Lemma 3.1.** Let $R$ be a Noetherian P.I. ring with $\text{gl.dim } R \leq 2$, $I = P_1 \cap \cdots \cap P_k$ where $P_i$ are non-maximal prime ideals. Then $I$ and $P_i$ are projective left and right $R$-modules, for each $i$.

**Proof.** By assumption $R/I$ and $R/P_i$ has no Artinian submodules. Therefore, by BW, Corollary 5, pr.dim$_R(R/I) \leq 1$ and pr.dim$_R(R/P_i) \leq 1$. Hence, by [BW, Corollary 5], pr.dim$_R(R/I) \leq 1$ and pr.dim$_R(R/P_i) \leq 1$. Q.E.D.

**Proposition 3.2.** Let $R$ be a Noetherian ring and $I$ a semi-prime ideal.

Suppose that

(i) $\bigcap_i I^i = (0)$, and

(ii) $I_R$ and $R/I$ is a projective $R$-module.

Then $R$ has an Artinian quotient ring.

**Proof.** This is really Theorem 2.2 of [CH1]. For the sake of completeness we reproduce the argument. Let $c \in \mathcal{R}(R)$. Then $l(c) = \text{l-ann}_R(c) \subseteq N(R)$. Therefore, by the semi-primeness of $I$, we get $I(c) \subseteq I$. Let $f \in \text{Hom}_R(I_R, R_R)$. Then $l(c) \cdot c = 0$ implies that $f(l(c))c = 0$, that is, $f(l(c)) \subseteq I(c)$. Now, $I_R$ is projective so let $(a, f_a)$ be a dual basis of $I_R, a \in I, f_a \in \text{Hom}_R(I_R, R_R)$. Let $x \in l(c)$; then $x = \sum a f_a(x) \subseteq l(c)$, that is, $l(c) = \text{II}(c)$. By iterations $l(c) \subseteq \bigcap_i I^i = (0)$. Similarly, using the right projectivity of $I$, we get $r-\text{ann}_R(c) = 0$. We now apply Small’s theorem. Q.E.D.

**Proposition 3.3.** Let $R$ be a Noetherian P.I. ring with $\text{gl.dim } R \leq 2$. Suppose that $A(R) = 0$. Then $R$ has an hereditary, Artinian, quotient ring.

**Proof.** We have by [St, Lemma 2.2] (since minimal primes are affiliated), that no minimal prime ideal in $R$ is co-Artinian. Consequently, by Lemma 3.1, $N(R)$ is a left and right projective $R$-module. The existence of $Q(R)$ now follows by Proposition 3.2. Let $S = \mathcal{C}(0)$. Then $N(R)_S$ is left and right projective $R_S = Q(R)$-module. This shows that $Q(R)$ is hereditary. Q.E.D.

The following result is the promised generalization of Proposition 2.17.

**Theorem 3.4.** Let $R$ be a Noetherian P.I. ring with $\text{gl.dim } R \leq 2$. Then $P^2 = P$ for every minimal prime ideal $P$ in $R$.

**Proof.** If $P$ is in addition co-Artinian then the result follows from Proposition 2.17. Consequently we may assume that $P$ is not maximal.
Therefore \( A(R) \subseteq P \). Using Theorem 2.3 we may translate the problem to \( R/A(R) \) and \( P/A(R) \). Hence, we may assume that \( A(R) = 0 \). By Proposition 3.3 the ring \( R \) has an Artinian, hereditary, quotient ring. Let \( S = \mathfrak{S}(0) \). Then \( P_2 \) being a minimal prime in the hereditary Artinian ring \( R_2 = Q(R) \) is an idempotent ideal, that is, \( P_2^2 = P_2 \). Therefore there exists \( t \in \mathfrak{S}(0) = S \), so that \( tP \subseteq P^2 \). Now \( P/P^2 \) is a left projective \( R/P \)-module (since \( P \) is projective) and hence it is \( R/P \)-torsion free. However, \( t \cdot P/P^2 = (0) \), with \( t \in \mathfrak{S}(0) \subseteq \mathfrak{S}(P) \). This contradicts the torsion freeness, unless \( P = P^2 \). Q.E.D.

We now proceed to generalize Proposition 3.3. In order to do this we need the following possibly well-known result.

**Lemma 3.5.** Let \( R \) be a ring satisfying the A.C.C. on right annihilators. Let \( x \in R \) with \( x = x^2u \). Then \( x = xu \), the element \( xu \) is an idempotent in \( R \), and \( xR = (xu)R \).

**Proof.** Denote \( r \text{-ann}_R x = r(x) \). We have \( xR = x^2R \). Hence \( xR = x^kR \) for each \( k \). Let \( m \) be chosen such that \( r(x^m) = r(x^{m+1}) \). Then, by Fitting’s Lemma one has \( x^mR \cap r(x^m) = (0) \). Now \( r(x) \cap xR \subseteq r(x^m) \cap xR = (0) \). Also, \( x = x^2u \) implies that \( x(1-xu) = 0 \), that is, \( 1-xu \in r(x) \).

Therefore, \( x = xu = (1-xu)x \in r(x) \cap xR = (0) \), that is, \( x = xu \). This shows that \( xu^2 = xu \) and \( xR = (xu)R \). Q.E.D.

**Theorem 3.6.** Let \( R \) be a Noetherian P.I. ring with \( \text{gl.dim} R \leq 2 \). Then the following are equivalent

(i) \( A(R) \) is a principal left as well as right ideal,

(ii) \( R \) has an Artinian quotient ring.

**Proof.** That (ii) implies (i) follows from a result by Ginn and Moss (e.g., [CH, Theorem 4.14; MCR, 4.1.8]) and does not require the \( \text{gl.dim} R \leq 2 \) or the P.I. assumption.

Suppose now that \( A(R) = Rx \) and \( A(R) = yR \). Then \( A(R)^2 = A(R) \) implies that \( y = y^2u \) and \( x = ux^2 \). Consequently, by Lemma 3.5, \( A(R) = eR, A(R) = fR \) and \( e^2 = e, f^2 = f \). An easy exercise shows that \( e = f \) and that \( e \) is central. Hence \( R = eR \oplus (1-e)R \) a direct sum of ideals. Clearly \( 1-e)R \approx R/A(R) \) which has by Proposition 3.3 an Artinian quotient ring. It follows that \( Q(R) \approx A(R) \oplus Q((1-e)R) \). Q.E.D.

**Example 3.7.** Let \( R = (\begin{smallmatrix} x & \alpha \varepsilon \\ 0 & \varepsilon \end{smallmatrix} : \varepsilon \in \mathfrak{S}, \alpha \in \mathfrak{S}(0) \) \), where \( e_{11}x \) acts trivially on \( Fe_{12} \) and \( Fe_{22} \). Then, by [MCR, p. 246], \( \text{gl.dim} R = 2 \). However, \( A(R) = Fe_{12} \oplus Fe_{22} = Re_{22} \), but \( A(R) \) is not right principal and \( R \) does not have an Artinian quotient ring.

We now prove a useful generalization of Theorem 3.4.
PROPOSITION 3.8. Let R be a Noetherian P.I. ring with gl.dim $R \leq 2$. Let $I = \bigcap_{i=1}^{k} P_i$, be an intersection of minimal primes where no $P_i$, $i = 1, \ldots, k$, is co-Artinian. Then $I' = I^{t+1}$ for some $t$, and consequently gl.dim $R/I' \leq 2$.

Proof. We first show how to reduce the case $\mathcal{A}(R) = 0$. Indeed, since no $P_i$ is maximal, $\mathcal{A}(R) \subseteq P_i$ for each $i$. So $\mathcal{A}(R) \subseteq I$ and $\mathcal{A}(R) = \mathcal{A}(R)^2$ combined with $I' = I^{t+1}$ in $R/\mathcal{A}(R) = \bar{R}$ will furnish the result. So, without loss of generality, we may assume (since gl.dim $R \leq 2$) that $\mathcal{A}(R) = 0$. Now, by Proposition 3.3, $Q(R)$ exists and is Artinian. Let $S = \mathcal{S}(0)$. Then $S \subseteq \mathcal{S}(I)$ and $R_S = Q(R)$. Also $I_S' = I_S^{t+1}$ for some $t$ (since $Q(R)$ is Artinian). Consequently there exists $x \in S$ so that $xt' \subseteq I^{t+1}$. Hence, $I'/I^{t+1}$ is not left $R/I$-torsion free. This leads to a contradiction, since $R/I$ being projective, implies that $I'/I^{t+1}$ is a left projective $R/I$-module. The conclusion is, therefore, $I' = I^{t+1}$. Q.E.D.

We next prove one of the statements of Theorem C. In order to do so we need the following results.

LEMMA 3.9. Let $R$ be a Noetherian P.I. ring with gl.dim $R \leq 2$. Let $Q$ be a non-maximal prime ideal in $R$ and $x$ the clique of $Q$. Then $x$ is finite and therefore localizable.

Proof. By [GW, p. 205] every $P \in x$ is non-maximal. Then, by Lemma 3.1, $RP$ and $(P, x)$ is projective. Therefore $P[x]$ is a left projective ideal in $R[x]$. Let $x' = \{P[x] | P \in x\}$. Then, by the Goodearl–Stafford Lemma (e.g., [B, Lemma 4.2]), $x'$ is localizable. Say $T = \bigcap_{P \in x'} \mathcal{S}(P[x])$. Then $P[x]_T$ is a left and right projective $R[x]_T$ module. Now this implies that $R[x]_T$ is hereditary. Therefore $x'_T$ is finite and consequently $x$ is finite. Finally, the localizability of $x$ follows from [GW, p. 214]. Q.E.D.

COROLLARY 3.10. Let $R$ be a Noetherian P.I. ring with gl.dim $R \leq 2$, $P$ a height one non-maximal prime ideal, and $x$ the clique of $P$. Then $x$ is localizable, $R_x$ is hereditary, and prime, and in particular no prime ideal in $x$ is minimal.

Proof. By Lemma 3.9, $x$ is localizable and $R_x$ has a single clique of maximal ideals. In particular no maximal ideal contains a non-trivial central idempotent. Also by Lemma 3.1 each $Q \in x$ is projective, implying that $Q_x$ is projective as well. Consequently $R_x$ is hereditary. Now, $R_x$ being void of non-trivial central idempotents is either prime or Artinian. The latter opinion is impossible since $P_x$ is not minimal. Q.E.D.

The next result settles Theorem C(2).

THEOREM 3.11. Let $R$ be a Noetherian P.I. ring with gl.dim $R \leq 2$. Suppose that $k.dim R/P = 2$ for each minimal prime $P$ in $R$. Then $R$ is semi-prime.
Proof. By assumption $A(R) \subseteq P$ for each minimal prime $P$. Hence, since $A(R) = A(R)^2$, we get that $A(R) = 0$, which implies, by Proposition 3.3, that $R$ has an Artinian quotient ring $Q(R)$. Let $P_1, \ldots, P_k$ be the complete set of minimal prime ideals in $R$. By assumption we can choose $Q_i$, a prime ideal in $R$, height $(Q_i/P_i) = 1$, and $Q_i$ not maximal. Let $x_i = \text{clique}(Q_i)$ for $i = 1, \ldots, k$. By Lemma 3.9, $x_i$ is finite and localizable. Hence, by [GW, 12.21], $x = \bigcup_i x_i$ is localizable. Also $\mathcal{O}(x_i)$ is an Ore set and hence its image, in the prime ring $R/P_i$, is an Ore set. This implies that $\mathcal{O}(x_i) \subseteq \mathcal{O}(P_i)$ for each $i$. Therefore $\mathcal{O}(x) = \bigcap_i \mathcal{O}(x_i) \subseteq \bigcap_i \mathcal{O}(P_i) = \mathcal{O}(N(R)) = \mathcal{O}(0)$. Hence $Q(R_x) = Q(R)$. Now, by Corollary 3.10 we have no maximal ideal in $R_x$ is minimal, which implies that in the hereditary ring $R_x$, no maximal ideal is minimal. Consequently $R_x$ is a direct sum of prime rings. This implies that $Q(R_x) = Q(R)$ is semi-simple Artinian. Q.E.D.

The following example shows that the minimal prime ideals in Theorem 3.11 need not split.

Example 3.12. As in [BH] we find $S$, a prime affine Noetherian P.I. ring over the field $F$ with $\text{gl.dim } S = k.\dim S = 2$ and $M = M^2$ a maximal (two-sided) ideal in $S$ which is right projective and $S/M \cong M_2(F)$. Let $W$ be a maximal right ideal containing $M$. Then $\text{pr.dim } S/W = \text{pr.dim } S/M$, shows that $W$ is a right projective $S$-module. Define $R = F + (W \oplus W)$. We claim that $R$ satisfies the required properties. Indeed, $(S \oplus S)(W \oplus W) = S \oplus S$ shows, by [R,S, Theorem 5(i)], that

$$
2 = \text{gl.dim } (S \oplus S) \leq \text{gl.dim } R \\
\leq \sup\{\text{gl.dim } (S \oplus S), 1 + \text{gl.dim } R/W \oplus W \\
+ \text{pr.dim } [S \oplus S/W \oplus W]_{S,S}\} \\
= \sup[2, 1 + \text{pr.dim } [S/W]_S].
$$

Now $W$ being a right projective $S$-module implies that $\text{gl.dim } R = 2$. Also $R$ has only two minimal primes and no non-trivial central idempotent.

Corollary 3.13. Let $R$ be a Noetherian P.I. Ring with $\text{gl.dim } R \leq 2$ and $I = P_1 \cap \cdots \cap P_k$ a semi-prime ideal. Then $I = I^2$ provided $\text{k.dim } R/P_i = 2$, for each $i$.

Proof. We have, by Proposition 3.8, that $I' = I^{t+1}$ for some $t$. Moreover, $I$ being projective implies that $I'$ is projective and consequently $\text{gl.dim } R/I' \leq 2$. We now apply Theorem 3.11 to the ring $R/I'$. Q.E.D.
We shall now use some of the previous results to consider the following question of Vasconcelos.

**Question [V, p. 21, Question 6.3(ii)].** Let $R$ be a finite module over its Noetherian local center. Suppose $\text{gl.dim } R = \text{k.dim } R = 2$. Is $R$ a prime ring?

We shall first provide a counterexample to this question and then give a positive answer to a variation of it.

**Example 3.14.** Let $C = \mathbb{k}[x, y]_{(x, y)}$ be the polynomial ring in two variables localized at $(x, y)$. Let $m = (x, y)_{(x, y)}$, Consider $A = (\frac{C}{m}, \frac{C}{m})$, and $M = (\frac{m}{m}, \frac{C}{m})$, a maximal two-sided ideal in $A$. Now $(\frac{m}{m}, \frac{C}{m}) = (\frac{C}{m}, \frac{C}{m})$ shows that $M_A$ is projective. Define $R = (\frac{A}{M}, \frac{A}{M})$. Then, by [McR, p. 246], we have that

$$2 = \text{gl.dim } A \leq \text{gl.dim } R$$

$$\leq \sup \{\text{gl.dim } A, \text{gl.dim } A/M + \text{pr.dim}[A/M]_A + 1\} = 2.$$

Moreover, $2 = \text{k.dim } A = \text{k.dim } R$, and $Z(R) = ((\frac{b}{0}, 0)|b \in Z(A)) \equiv C$. Therefore $Z(R)$ is a local regular domain and $R$ is a non-semi-prime ring which satisfies all the required properties.

The next theorem is a positive result in this direction.

**Theorem 3.15.** Let $R$ be a finite module over its Noetherian center. Suppose, in addition, that $\text{k.dim } R = \text{gl.dim } R = 2$, $Z(R)$ is a domain and $A(R) = (0)$. Then $R$ is a prime ring.

**Proof.** We know, by Proposition 3.3, that $R$ has an Artinian quotient ring $Q(R)$. In particular the clique of every minimal prime consists of minimal primes. Let $P$ be a minimal prime and $p = P \cap Z(R)$. We claim that $p$ is a minimal prime in $Z(R)$. Indeed if $q \subseteq p$, where $q$ is a minimal prime, then we can find, by “Going up,” prime ideals $Q \subseteq P$, satisfying $Q \subseteq Z(R) = q$ and $P \supseteq Z(R) = p$. Hence, by a result of Müller [GW, p. 296] $P \subseteq \text{clique } (P)$ and therefore it is a minimal prime, which is a contradiction. Consequently, given that $Z(R)$ is a domain, we have that $P \cap Z(R) = (0)$, for every minimal prime $P$. Let $\lambda = Z(R) \setminus (0)$. Then $\lambda \subseteq \cap \{\text{clique } (P)|P \text{ is a minimal prime}\} = \varnothing(N(R)) = \varnothing(0)$.

It is standard that $\text{k.dim } Z(R) = \text{k.dim } R = 2$. So, let $q$ be a height one non-maximal prime ideal in $Z(R)$. Then $\delta = Z(R) \setminus q \subseteq \varnothing(0)$ and therefore $R_\delta \subseteq Q(R)$. Now, by Lemma 3.1, $\text{gl.dim } R_\delta \leq 1$ and since $Z(R_\delta) = Z(R)_\delta$ is local, $R_\delta$ has no central idempotents. Moreover, since $R_\delta$ is not Artinian, we must conclude, by the structure theorem for Noetherian hereditary rings, that $R_\delta$ is prime. Equivalently, $R$ is prime. Q.E.D.
We end this section with the following result.

**Proposition 3.16.** Let $R$ be a Noetherian P.I. ring with $\text{gl.dim } R \leq 2$ and $P$ a non-maximal prime ideal. Then, either $P = P^2$ or $P$ is localizable, and $\cap_i P^i$ is a minimal prime ideal.

**Proof.** Let $Q \subseteq P$ be a minimal prime. Then $Q^2 = Q$ shows that $\cap_i P^i \supseteq Q$. Let $x$ be the clique of $P$. If $Q = P$ the result is trivial, so assume that $Q \subset P$. Then, as in Corollary 3.10, $R$, is hereditary and therefore either $P_2 = P_1$ or $P_1$ is localizable. Now $P_2 = P_1$, implies that $tP \subseteq P_2$, for some $t \in \mathfrak{c}(x) \subseteq \mathfrak{c}(P)$. This shows that $P/P^2$ is not a torsion free left $R/P$-module, but this contradicts the left projectivity of $P/P^2$ over $R/P$, unless $P = P^2$.

Suppose now that $P_1$ is localizable. Consequently $P$ is localizable. Therefore $P/Q$ is localizable in $R/Q$. In particular $\cap_i P^i \subseteq Q$. This together with $Q = Q^2 \subseteq \cap_i P^i$ yields $\cap_i P^i = Q$. Q.E.D.

**Example 3.17.** Let $k[x]$ be the commutative polynomial ring over the field $k$, and let $R = \begin{pmatrix} k[x] & k & k[x] \\ 0 & k & k \\ 0 & 0 & k[x] \end{pmatrix}$.

Then by [McR, p. 246], $\text{gl.dim } R = 2$ and $R$ is a Noetherian affine P.I. ring. Observe that $R_{xy} = \mathcal{A}(R)$ and $R/\mathcal{A}(R) \cong \begin{pmatrix} \mathbb{A}^1 & \mathbb{A}^1 \\ \mathbb{A}^1 & \mathbb{A}^1 \end{pmatrix}$. In fact, $R$ exhibits all the “pathologies” of a Noetherian affine P.I. ring with global dimension 2.

**4. Preliminaries Toward the Proof of Theorem B**

The running assumptions in this section on $R$ are $\text{gl.dim } R \leq 2$, $\mathcal{A}(R) = 0$, and $R$ is a Noetherian P.I. ring. As we know from Proposition 3.3, the ring $R$ has an hereditary and Artinian quotient ring. We will perform the $\ast$-process inside $Q(R)$ producing in each step a ring $S$ of a similar nature but with a smaller number of idempotent ideals. The proof will proceed by induction on this number and therefore the critical case will be the case where one cannot perform the $\ast$-process to begin with. This case is dealt with in Theorem 5.8. We also prove here that every affine Noetherian P.I. ring of $\text{gl.dim } R \leq 2$ is a finite module over a commutative affine subring. This should be compared with Theorem 13 of [BH], where it is shown, in the semi-prime case, that the commutative affine subring can be taken to
be central. We end the section with the proof of Theorem C(6) where we show that for $R$ as above, there exists a semi-prime over ring $S$ so that $\rho S$ is finitely generated projective and $\text{gl.dim} S \leq 2$. This result "explains" the rather surprising, well behaved, nature of $R$.

We start with a result which resembles a similar one in the (semi)-prime case. In fact the proof is really the same. So we only outline it.

**Proposition 4.1.** Suppose $R$ has an Artinian quotient ring $Q(R)$, and $M$ is a non-minimal prime ideal. Then $M^* = \text{Hom}_R(M_R, R_R)$ can be identified with $\{q \in Q(R) | qM \subseteq R\}$, and $M^* = \text{Hom}_R(\rho^* M_R, R_R)$ with $\{q \in Q(R) | qM \subseteq R\}$.

**Proof.** Let $f \in M^*$ and $q \in Q(R)$. We can find, using $M_{q'(0)} = Q(R)$, an element $s \in q'(0)$ such that $qs = m \in M$. We extend the definition of $f$ to $Q(R)$ by $f(q) = f(m)s^{-1}$. One uses the Ore condition to show that this is well defined. Next, using again the Ore condition, one shows that $f \in \text{End}(Q(R)_{Q(R)})$. Finally, clearly every $g \in \text{End}(Q(R)_{Q(R)})$ can be identified with left multiplication by an element of $Q(R)$. Q.E.D.

**Notation.** $t(R)$ is the number of proper idempotent ideals in $R$.

**Corollary 4.2.** Suppose that $R$ is a Noetherian P.I. ring with an Artinian quotient ring. Let $M$ be a maximal ideal in $R$ satisfying $M^2 = M$ and $M$ is a projective right $R$-module. Then $t(M^*) < t(R) < \infty (t(M^*) < t(R)$, respectively).

**Proof.** It is standard, using $M^2 = M$, to see that $M^*$ is a ring. The rest is exactly the same as in [BH, p. 594]. Q.E.D.

**Proposition 4.3.** Let $R$ be a Noetherian P.I. ring having an Artinian quotient ring. Let $V$ be a maximal ideal in $R$, which is not a minimal prime. Suppose $V$ is a right projective $R$-module and $V^*V = R$. Then

1. $V$ contains a unique minimal prime $P$ of $R$,
2. $\cap_i V^i = P$,
3. $\mathcal{C}(V)$ and $\mathcal{C}(P)$ are left Ore sets,
4. $V^*N(R) = N(R)$ and $VN(R) = N(R)$ (we do not need $V^*V = R$ here).

A similar, right-handed version statement holds if $\rho V$ is left projective and $\rho V^{*\rho} = R$.

**Proof.** Let $P \subseteq V$ be a minimal prime. $V^*P \subseteq V^*V = R$ and $V^*P \subseteq Q(R)P$ imply that $V^*P \subseteq Q(R)P \cap R = P$. That is, $V^*P = P$. Now $V$ being projective is equivalent to $1 \in VV^*$. Hence $P = 1 \cdot P \subseteq (VV^*)P = V(V^*P) = VP$. Iterations yield $P \subseteq \cap_i V^iP \subseteq \cap_i V^i$. Also $V^*P = P$ shows
Let $V^* + P/P \subseteq (V/P)^*$ and therefore $V^* V = R$ implies that $(V/P)^* \cdot V/P = R/P$. Consequently, by [H, Theorem 4.3], $V/P$ is a localizable maximal ideal in the prime Noetherian P.I. ring $R/P$, and therefore, by Jategaonkar’s theorem, $\cap_i (V/P)^i = (0)$. Hence, $\cap_i V^i \subseteq P$. Combining with the reverse inclusion, we get $\cap_i V^i = P$ which settles (1) and (2) simultaneously. Next, to prove (4), we observe that $V^* N(R) \subseteq R$ and hence $N(R) \leq (Q(R) N(R) \cap R) = N(R)$, that is, $V^* N(R) = N(R)$. Also, $N(R) = 1. N(R) \subseteq V^* N(R) = VN(R)$. These together with $V^* V = R$ imply that $(V/N(R))^* \cdot V/N(R) = R/N(R)$. Again, by [H, Theorem 4.3], this shows that $\bar{V} = V/N(R)$ is localizable in $\bar{R} \equiv R/N(R)$. Let $W \neq V$ be a maximal ideal. This implies that $\bar{V} \cap \bar{W} = \bar{VW}$, that is, $V \cap W = VW + N(R)$. Therefore $V^* (V \cap W) = V^* (VW + N(R)) \subseteq V^* VW + V^* N(R) = W + N(R) = W$. Also $V^* (V \cap W) \supseteq V^* (VW) = W$ implies that $V^* (V \cap W) = W$. Consequently $V \cap W = 1. V \cap W \subseteq V^* (V \cap W) = V^* (V \cap W) = VW$. This shows, by standard results (e.g., [GW, 12.21]), that $\mathfrak{e}(V)$ is a left Ore set. Finally, to show that $\mathfrak{e}(P)$ is left Ore, let $Q$ be a prime ideal with $P \cap Q$. Then, by [GW, p. 211], since $\mathfrak{e}(V) \subseteq \mathfrak{e}(P)$, we have $\mathfrak{e}(V) \subseteq \mathfrak{e}(Q)$. Therefore $Q \subseteq V$. By (1) we must have that $P \subseteq Q$ and hence $P = Q$. Q.E.D.

**Notation 4.4.** We now return to our main subject matter namely, a ring $R$ which is a Noetherian P.I. ring with $\text{gl.dim } R \leq 2$ and $\mathcal{A}(R) = (0)$. By Proposition 3.3, the ring $R$ has an Artinian quotient ring. We shall keep, for the rest of the section, the following convention. Let $x_i, i = 1, \ldots, d$, be the complete list of cliques of minimal primes in $R$. Set $I_i = \cap_i (Q \in x_i), i = 1, \ldots, d$.

The following lemma is well known (e.g., [M, p. 602, Lemma 2.1; GW, Corollary 11.9]).

**Lemma 4.5.** Let $R$ be an Artinian ring with cliques of minimal primes $x_i, i = 1, \ldots, r$, $I_i = \cap_i (Q \in x_i)$ and $n$ an integer satisfying $I_i^n = I_i^{n+1}$ for each $i$. Then $I_i^n \cap I_j^n = I_j^n I_i^n = I_j^n I_i^{n+1}$ for each $i$ and consequently $R \equiv R/I_i^n \oplus \cdots \oplus R/I_j^n$.

**Proof.** We only give a sketch. If $I_i^n \cap I_j^n/I_j^n I_i^n \neq 0$ it will lead to maximal ideals which are bonded but the first contains $I_j$ and the second contains $I_i$. Thus, they are in different cliques. This contradicts the fact that they are bonded. The remaining assertion follows from the Chinese remainder theorem. Q.E.D.

**Lemma 4.6.** $\cap_{i=1}^d I_i^n = (0)$ for some $n$, where the notation is as in Notation 4.4.

**Proof.** Let $S \equiv \mathfrak{e}(0)$ and $Q(R) = R_S$. We can find, by Proposition 3.8, a number $n$ so that $I_i^n = I_i^{n+1}$ for each $i = 1, \ldots, d$. Hence $I_i^n = I_i^{n+1}$ for
each $i$. Now $Q(R)$ being Artinian implies, by Lemma 4.5, that $\cap_i I_i^n = (0)$. Consequently $\cap_{i=1}^d I_i^n = 0$. Q.E.D.

**Lemma 4.7.** \( \text{gl.dim } R/(I_i^n + \cap_{j\neq i} I_j^n) = 0 \), for each $i = 1, \ldots, d$, where $d > 1$ and the notation is as in Lemma 4.6.

**Proof.** Observe that the result trivially follows if $\text{gl.dim } R/I_i = 1$. Suppose, to the contrary, that $(I_i^n + \cap_{j\neq i} I_j^n) \subseteq P$, where $P$ is a height one prime ideal in $R$ which is not maximal. Then $I_i^n \subseteq P$ and $I_j^n \subseteq P$ for some $j \neq i$ imply that there are minimal prime ideals $V, W$ in $R$, $V$ minimal over $I_i$, and $W$ minimal over $I_j$, such that $V \subset P$ and $W \subset P$. Also $V \neq W$ since $i \neq j$. Let $x = \text{clique } (P)$. Then we have, by Corollary 3.10, that $R_x$ is prime. Consequently $V_x = 0 = W_x$, that is, $\forall t \in \mathcal{C}(x)$. Now $x = \text{clique } (P)$ is localizable in $R$, hence its image in $R/V$ is localizable, implying that $\mathcal{C}(x) \subseteq \mathcal{C}(V)$. Now this, together with $W_x = 0$, implies by the primeness of $V$, that $W \subseteq V$. Therefore $W = V$ which contradicts $W \neq V$. Q.E.D.

**Lemma 4.8.** Keeping the notation of Lemma 4.6, we have

$$\bigcap_{i=1}^k I_i^n = I_1^n \cdots I_k^n,$$

for every $k \leq d$. Q.E.D.

**Proof.** It is easily seen, by the dual basis lemma, that if $X, Y$ are ideals which are right (left) projective $R$-modules then so is $XY$. Consequently, $I_i^n$ is a right and left projective $R$-module for each $i$. Also, by Proposition 3.2, $R/I_i^n$ has an Artinian quotient ring, for each $i$ (this can be deduced from Proposition 3.3 as well, since $\text{gl.dim } R/I_i^n \leq 2$ and $A(R/I_i^n) = (0)$). We assume, by induction, that $\bigcap_{i=1}^{k-1} I_i^n = I_1^n \cdots I_{k-1}^n = I$, say. By the above $I$ is a right projective $R$-module and therefore $I/I \cdot I_k^n$ is a right projective $R/I_k^n$-module. In particular $I/I \cdot I_k^n$ is torsion free with respect to the regular elements of $R/I_k^n$. Now let $S = \mathcal{C}_R(0)$. Clearly, by Lemma 4.5, one has $I_k \cap (I_k^n) = I_k(I_k^n) = I_k(I_k^n) = I_k$, which implies $(I \cap I_k^n) \subseteq I \cdot I_k^n$ for some $I \in \mathcal{C}_R(0) \subseteq \mathcal{C}(I_k^n)$. This leads to a torsion element in $I/I \cdot I_k^n$ (which was excluded), unless $I \cap I_k^n = I \cdot I_k^n$, as desired. Q.E.D.

**Proposition 4.9.** Let $R$ be a Noetherian P.I. ring with $\text{gl.dim } R \leq 2$ and $A(R) = 0$. Suppose, in addition, that $M^n M = R$ for every maximal ideal $M$ in $R$ which is a projective right $R$-module. Then there is an integer $n$ such that

$$R \equiv R/I_1^n \oplus \cdots \oplus R/I_d^n,$$

where, we keep the notation of 4.4.

**Proof.** Choose $n$ as in Lemma 4.6. It suffices to show, by the Chinese remainder theorem, that $I_i^n + (\cap_{j\neq i} I_j^n) = R$, for each $i = 1, \ldots, d$. By
Lemma 4.8 we know that $I^n_i$ and $\bigcap_{j \neq i} I^n_i$ are right projective $R$-modules. Therefore, by Lemma 4.6, $I^n_i + \bigcap_{j \neq i} I^n_i = I^n_i \oplus \bigcap_{j \neq i} I^n_i$. Consequently $K = I^n_i + \bigcap_{j \neq i} I^n_i$ is a right (and left) projective $R$-module. Suppose, to the contrary, that $K \neq R$. Then, by Lemma 4.7, $R/K$ is right (and left) Artinian. Let $T_R \subset R/K$ be a simple right $R$-submodule, and set $M = r \text{-ann}_R(T)$. Then $\text{pr.dim}[R/M]_R = \text{pr.dim} T_R \leq \text{pr.dim}[R/K]_R = 1$. That is, $M$ is a maximal ideal which is a right projective $R$-module. By assumption we have $M^*M = R$. Now $M \supset I^n_i$ and $M \supset I^n_j$ for some $j \neq i$. Consequently, $M$ contains two different minimal primes, in contradiction to Proposition 4.3(1). Q.E.D.

We shall disgress now, from the proof of Theorem B, and prove that an affine Noetherian P.I. ring $R$ with gl.dim $R \leq 2$ is a finite module over a commutative affine subring. To this end we need two preliminary results.

**Lemma 4.10.** Let $R$ be a Noetherian affine P.I. ring over the field $F$. Suppose that $R/A(R)$ is a finite module over an affine commutative subring. Then $R$ is a finite module over a commutative affine subring.

**Proof.** Clearly $A(R)$ is a finitely generated faithful module over $R/\text{lan}_R(A(R)) = S$. Hence $S$ is affine, Artinian, and therefore finite dimensional over $F$. Consequently $A(R)$ is finite dimensional over $F$. Every $x \in R$ acts, via the left regular representation, as a linear transformation on the finite dimensional $F$-vector space $A(R)$. Let $h_i(t)$ be its characteristic polynomial. So $h_i(x)A(R) = 0$. Likewise, using the right regular representation, we have $A(R)k_i(x) = 0$. Hence $p_i(t) = h_i(t)k_i(t)$ satisfies $p_i(x)A(R) = A(R)p_i(x) = 0$. By assumption, $R/A(R)$ is a finite module over an affine commutative subalgebra $F[\overline{a}_1, \ldots, \overline{a}_s]$. We use now the notation $[x, y] = xy - yx$. Now $[a^n_i, a^m_j] \in A(R)$ implies that $[a^n_i, a^m_j]p_n(a_i) = 0 = p_n(a_i)[a^n_i, a^m_j]$, for each $n$ and $m$. Consequently $[a^n_i, p_n(a_i)] = [a^n_i, p_n(a_i)]p_n(a_i) + p_n(a_i)[a^n_i, p_n(a_i)] = 0$, for each $n$. Therefore, $C = F[p_n^2(a_i), \ldots, p_n(a_i)]$ is a commutative subring of $F[a_1, \ldots, a_s]$. Moreover, $F[a_1, \ldots, a_s] + A(R)/A(R) = F[\overline{a}_1, \ldots, \overline{a}_s]$ is a finite module over $C$. Therefore $R/A(R)$ is a finite module over $C$. Finally, since $A(R)$ is finite dimensional over $F$, we get that $R$ is a finite module over $C$. Q.E.D.

**Lemma 4.11.** Let $R$ be a Noetherian affine P.I. ring with gl.dim $R \leq 2$ and $A(R) = \{0\}$. Suppose $M$ is a maximal ideal in $R$ which is right projective as a right $R$-module and $M^2 = M$. Then $R$ is a finite module over a commutative affine subring, provided $M^*$ is also such a ring.

**Proof.** Suppose $M^*$ is a finite module over its commutative affine subring $B$. Clearly (e.g., by Proposition 4.17) $M^*/M$ is a finitely generated right $R/M$-module. The latter, being finite dimensional over $F$ (the field over which $R$ is affine), implies that $M^*/M$ is finite dimensional over $F$. 
Consequently $B/M \cap B \cong B + M/M$ is finite dimensional over $F$. Observe that $M \cap B \subseteq R \cap B$ and is an ideal in $R \cap B$. Therefore $B/M \cap B$ is a finite module over $B \cap R/M \cap B$. This implies that $B$ is a finite module over $B \cap R$. Now, by Eakin’s theorem, $B \cap R$ is Noetherian and, by the Artin–Tate Lemma, it is also affine. Therefore $M^*$ is a finite module over the affine commutative ring $B \cap R$ and consequently $R \supset M^*$ has the same property.

**Theorem 4.12.** Let $R$ be an affine Noetherian P.I. ring with $\text{gl.dim } R \leq 2$. Then $R$ is a finite module over a commutative affine subring.

**Proof.** Say $R = F\{x_1, \ldots, x_m\}$, where $F$ is a central subfield. By Theorem 2.3 and Lemma 4.10 we may assume that $A(R) = 0$. The theorem is now proved by induction on $t(R) = \text{the number of proper idempotents}$ ideal in $R$, which is finite by Corollary 4.2. Suppose there exists a maximal ideal $M$ in $R$, $M^*M = M$, and $M$ is a projective right $R$-module. Then, by Corollary 4.2 $t(M^*) < t(R)$, and, by [RS], $\text{gl.dim } M^* \leq 2$. By the induction hypothesis $M^*$ is a finite module over a commutative affine subring. Now Lemma 4.11 will furnish the desired conclusion. The only remaining case to consider is $M^*M = R$ whenever $M$ is a maximal ideal in $R$ and $M$ is a projective right $R$-module. Observe that this will settle the step $t(R) = 0$, as well, since $M^*M = M$ implies, if $M_R$ is projective, that $M^2 = M$. Now, by Proposition 4.9, $R = R/I^n_1 \oplus \cdots \oplus R/I^n_r$. We need to distinguish between two cases, as follows. Recall that $k \dim R \leq \text{gl.dim } R \leq 2$, by [GS]. Suppose $k \dim R/I^n_1 = 2$. Then, by Theorem 3.11, $R/I^n_1$ is semi-prime. Now by [BH], $R/I^n_1$ is a finite module over a commutative affine (central) subring. If $k \dim R/I^n_1 = 1$, then, by [B1], $R/N(I^n)$ is a finite module over $R[a]$, hence it is a finite module over $F[a]$ for some $a \in R$. Now $N(I^n)/N^{j+1}(I^n)$ is a finite $R/N(I^n)$-module for each $j$, hence $N(I^n)/N^{j+1}(I^n)$ is a finite module over $F[a]$ for each $j$. Consequently $R$ is a finite module over a commutative affine subring since each $R/I^n$ has this property.

Q.E.D.

The purpose of the following example is to show that the affine commutative subring of Theorem 4.12 need not be central.

**Example 4.13.** Let $k[x]$ be the polynomial ring in one variable and $V = (k\{x\}, k\{x\})$. Let $a = (1, 0)$, $b = (0, 1)$. Then $V$ is a finitely generated torsion free left $k[a]$-module and $V$ is a finitely generated torsion free right $k[b]$-module (with generators $(1, 0)$, $(0, 1)$). Consequently, since $k[a]$ is a free $k[a]$, $k[a]V$ and $V_{k[a]}$ are free. Define $R = (k\{x\}, k\{x\})$. Then, by [McR, p. 246], $\text{gl.dim } R \leq 2$ and $R$ is clearly affine Noetherian P.I. with $A(R) = \{0\}$. Finally, the center $Z(R) = \{(a, b) | ab = ba\}$, for all $a \in V$. This applied to $u = (1, 0)$ and to $v = (0, 1)$ shows for $e = (1, 0)$, $d = (0, 1)$,
\( (q(x), 0, 0) \), that \( p(x) = q(x) \) and \( p(x) = q(x^2) \). Consequently \( p(x) = q(x) \in k \). Hence

\[
Z(R) = \left\{ \begin{pmatrix} \alpha & 0 \\ \alpha & \alpha \\ 0 & \alpha \end{pmatrix} \mid \alpha \in k \right\}.
\]

Now, \( R \) is not a finite \( Z(R) \)-module since it is not finite dimensional over \( k \). However, \( R \) is finite over \( k[a] \oplus k[b] \).

The following result can be considered as an application of the \( \ast \)-process as well as of Theorem 2.14. It explains, at least heuristically, why a Noetherian P.I. ring \( R \) with \( gl.dim \leq 2 \) and \( A(R) = (0) \) is well behaved. The “reason” is that there exists a semi-prime Noetherian P.I. ring \( S \), with \( gl.dim S \leq 2 \), which dominates \( R \). We need some preliminary results.

**Proposition 4.14** [CPS, Remark 1.6]. Let \( I = ReR \) be a finitely generated projective right \( R \)-module and \( e = e^2 \). Then \( eRe \) is Morita equivalent to \( \text{End}(I_R) \).

**Proof.** Consider the projection \( eR \oplus \cdots \oplus eR \equiv eR^{(n)} \rightarrow ReR \). This projection splits, since \( I_R \) is projective, by a \( R \)-module map \( \varphi \). So \( eR^{(n)} = \varphi(I) \oplus F \), for some projective \( R \)-module \( F \). Let \( f \) be the projection of \( eR^{(n)} \) onto \( \varphi(I) \), that is, \( f^2 = f \) and \( f \) acts as zero on \( F \). Let

\[
g = \begin{pmatrix} e & 0 \\ \cdots & \cdots \\ 0 & e \end{pmatrix}.
\]

Then \( g \) is the canonical projection \( g: R^{(n)} \rightarrow eR^{(n)} \). Let \( R^{(n)} = eR^{(n)} \oplus G \) for some projective \( R \)-module \( G \) (in fact \( G = (1-e)R^{(n)} \)). We extend \( f \) to \( \tilde{f} \) where \( \tilde{f} \) acts as zero on \( G \) and as \( f \) on \( eR^{(n)} \). Clearly \( \tilde{f} = gf = fg = f^2 \). Therefore \( f: R^{(n)} \rightarrow \varphi(I) \) is a projection and by [CPS, Remark 1.4(b)], \( M_n(I) = M_n(R)\tilde{f}M_n(R) \). Now by applying \( g \) from both sides, and using \( \tilde{f} = g\tilde{f} = \tilde{f}g \), we get \( M_n(eRe) = gM_n(I)g = gM_n(R)\tilde{f}M_n(R)g = (gM_n(R)g)\tilde{f}(gM_n(R)g) = M_n(eRe)\tilde{f}M_n(eRe) \). Also for every \( a \in M_n(eRe) \) we obviously have \( fa = fa \). Consequently \( M_n(eRe) = M_n(eRe)\tilde{f}M_n(eRe) \) which shows that \( M_n(eRe) \) is Morita equivalent to \( fM_n(eRe)f \). Now \( M_n(eRe) = \text{End}(eR^{(n)}) \) so, as in [MCR, Lemma 3.5.6], we have that \( fM_n(eRe)f = \text{End}(I_R) \). Hence we have that \( M_n(eRe) \) is Morita equivalent to \( \text{End}(I_R) \). The result now follows since \( eRe \) is Morita equivalent to \( M_n(eRe) \). Q.E.D.

We next need the following Lemma.
Lemma 4.15. Let $R$ be a ring with a finite number of minimal prime ideals and $P = ReR$, $e = e^2$, is a minimal prime. Then $eRe$ has fewer minimal primes than $R$.

Proof. Let $\{P = P_1, \ldots, P_n\}$ be the set of minimal primes in $R$. Hence $P_1 \cap \cdots \cap P_n = N(R)$, implies $(eP_1e) \cap \cdots \cap (eP_ne) \subseteq N(R) \cap eRe = eN(R)e = N(eRe)$. The result will now follow, using $eP_2e = eRe$, once we show that $eVe$ is a prime ideal of $eRe$, for every prime ideal $V$ in $R$. So let $A, B$ be ideals in $eRe$ and $AB = eVe$. Then $A(eRe)B \subseteq eVe$ and therefore $(ReeR)(ReeR) \subseteq ReVeR \subseteq V$. Consequently $ReeR \subseteq V$ (or $ReeR \subseteq V$) and therefore $A = eReeRe \subseteq eVe$ (or $B \subseteq eVe$). Q.E.D.

Theorem 4.16. Let $R$ be a Noetherian P.I. ring with $\text{gl.dim } R \leq 2$ and $A(R) = (0)$. Then there exists a semi-prime ring, $S$, $S \supset R$ satisfying $\text{gl.dim } S \leq 2$ and $S$ is finitely generated and projective.

Proof. One gets, by the $\ast$-process, after finitely many steps, a ring $T, T \supset R$, where $T$ is finitely generated and projective, $\text{gl.dim } T \leq 2$ and $M^*M = T$ for each maximal ideal $M$ which is a projective right $t$-module. Consequently, by Proposition 4.9, $T = T_1 \oplus \cdots \oplus T_d$, where in each $T_i$, the minimal primes consist of a single clique. Moreover by Lemma 2.6, $A(T) = A(R)T = 0 \cdot T = 0$. So without loss of generality, working separately with each $T_i$, we may assume that in $R$ the minimal primes consist of a single clique. Recall that $Q(R)$ is hereditary and therefore by Harada’s theorem [H] we can find a minimal prime $P'$ in $Q(R)$ so that $\text{l-ann}_{Q(R)}(P') = (0)$. Consequently, if $P = P' \cap R$, then $\text{l-ann}_R(P) = (0)$. Also, $P^2 = P$ by Theorem 3.4 and $P_R$ being projective by Lemma 3.1, imply, by [CPS, Remark 1.4(b)], that there exists an integer $n$ so that $M_n(P) = M_n(R)eM_n(R)$, where $e^2 = e$. Switching from $R$ to $M_n(R)$ causes no harm since $M_n(R)$ is a finitely generated projective left $R$-module, where $R \subseteq M_n(R)$ by the diagonal embedding. Therefore we may also assume that $P = ReR$ where $e^2 = e$ and $\text{l-ann}_R(P) = (0)$. Let $S \equiv \text{End}(P_R)$. Then $R$ is embedded in $S$ via the left multiplication. Moreover, by Proposition 4.14, $S$ is Morita equivalent to $eRe$ and consequently, by Theorem 2.14, $\text{gl.dim } S = \text{gl.dim } eRe \leq 2$. Also $P_R$ being projective implies that $1 \in P \text{ End}(P_R) = PS$. This together with $SP = P$ shows that $P_S$ is finitely generated and projective. Now, by Lemma 2.6, $A(S) = A(R)S = 0, S = 0$, so $S$ satisfies all the conditions that $R$ does. The result now follows by induction on the number of minimal primes since $S$ has fewer minimal primes than $R$ does by Lemma 4.15. Q.E.D.

Remark. The previous theorem can be considerably generalized, using a related proof, as follows.
**Theorem 4.16.** Let $R$ be a Noetherian ring with $\text{gl.dim } R \leq 2$, $A(R) = \{0\}$, and $R$ has an Artinian quotient ring. Then there exists a semi-prime ring $S$, $S \supset R$, $R S$ is finitely generated and projective, and $\text{gl.dim } S \leq 2$.

We need, for later use, the following (possibly known) result.

**Proposition 4.17.** Let $M$ be a left ideal in $S$ and $R$ so that $S M S$ is finitely generated and projective. Suppose that $R$ is Noetherian. Then $S$ is Noetherian.

**Proof.** Let $V$ be a right-ideal in $S$. Then $V M \subseteq V \cap (S M) = V \cap M$. Hence $V = (V \cap M) S$. Now $V \cap M$ is a finitely generated right ideal in $R$, that is, $V \cap M = \sum_{i=1}^k v_i R$. Therefore $V = (V \cap M) S = (\sum_{i=1}^k v_i R) S = \sum_{i=1}^k v_i S$. This shows that $S$ is right Noetherian.

Remark. We would like to recall that condition $(\#)$, but not the terminology, already appeared in Proposition 4.9.

5. THE PROOF OF THEOREM B

In the present section we complete the proof of Theorem B. Here $R$ will denote a Noetherian P.I. ring with $\text{gl.dim } R = 2$ and $A(R) = 0$. As follows from Proposition 3.3, the ring $R$ has an Artinian hereditary quotient ring. We now define the following condition.

**Definition 5.1.** Let $R$ be a Noetherian ring with an Artinian quotient ring. We say that $R$ satisfies condition $(\star)$, if $M^* M = R$ holds for every maximal ideal $M$ in $R$ which is right projective. Similarly, $R$ satisfies condition $(\#)$, if $M M^* = R$ for every maximal ideal $M$ in $R$ which is left projective.

Remark. We would like to recall that condition $(\star)$, but not the terminology, already appeared in Proposition 4.9.

In order to use the facts that $Q(R)$ is an hereditary Artinian quotient ring we need to recall Harada’s theorem.

**Theorem 5.2 (Harada [Ha]).** Let $D_i = M_{i i}$ be a division ring, and $M_{i j}$ a $D_i - D_j$ bimodule, for every $r \geq i > j \geq 1$. Suppose there exists a $D_i - D_j$
bimodule injection $\bigoplus_k M_{ik} \otimes M_{kj} \to M_{ij}$ for each $r \geq i > j \geq 1$. Let

$$R = \begin{pmatrix}
M_{11} & 0 & \cdots & 0 \\
M_{21} & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
M_{r1} & \cdots & \cdots & M_{rr}
\end{pmatrix}.$$ 

Then each hereditary Artinian ring is Morita equivalent to a ring of this form. Here the matrix multiplication is the obvious one, making use of $\otimes$.

**Definition 5.3.** Let $R$ be a Noetherian ring and $P$ a prime ideal in $R$. We say that $P$ is a sink if $Q \supset P$ for some $Q$, but no linkage arrow $\rightarrow$ is going out of $P$. Analogously $P$ is called a source if $P \supset Q$, for some $Q$ but no arrow $\rightarrow$ is going into $P$.

**Proposition 5.4.** Let $R$ be an hereditary Artinian ring.

(i) $PN = N, P \cap P' = PP'$, for all sinks $P$ and $P'$,

(ii) $NP = N, P \cap P' = PP'$, for all sources $P$ and $P'$,

(iii) $\text{l-ann}(P) =$ product of all other primes (in a certain order, each occurring only once), where $P$ is a sink.

**Proof.** Recall that two Morita equivalent rings have a semi-group isomorphism $\varphi$, between their ideals, which preserve inclusion. Consequently $\varphi(I \cap J) \supset \varphi(I) \varphi(J)$ if and only if $I \cap J \supset IJ$ and $\varphi(I \cap J) = \varphi(I) \cap \varphi(J)$. Therefore, (i), (ii), and (iii) will follow once we prove them for the lower triangular matrices which appear in Theorem 5.2. So let

$$R = \begin{pmatrix}
M_{11} & 0 \\
\vdots & \ddots \\
M_{r1} & M_{rr}
\end{pmatrix}$$

be as above.

Let $P_i = \{x \in R | e_i x e_i = 0\}$. Clearly $P_1, \ldots, P_r$ are all the maximal ideals in $R$. Moreover $(P_a P_b)_{(i,j)} = \sum_{k=1}^r M_{ik}^a M_{jk}^b$. Consequently, for $(i, j) \neq (a, b)$, we have that $M_{ij} M_{ij}$ (or $M_{ij} M_{ij}$) appears in the sum, namely $(P_a P_b)_{(i,j)} = M_{ij} = (P_a \cap P_b)_{(i,j)}$ for $(i, j) \neq (a, b)$. Now if $a < b$ then $(P_a P_b)_{(a,b)} = 0 = (P_a \cap P_b)_{(a,b)}$. Consequently $P_a \nmid P_b$, if $a < b$. Consider the case $a > b$. Then $(P_a \cap P_b)_{(a,b)} = M_{ab}$. Also $(P_a P_b)_{(a,b)} = \sum_{k=b+1}^{a-1} M_{ak} M_{kb}$, since $M_{ab}^{(a)} = 0$ and $M_{ab}^{(b)} = 0$. Therefore $P_a \nmid P_b$, if and only if $a > b$ and $M_{ab} \supset \sum_{k=b+1}^{a-1} M_{ak} M_{kb}$. As a consequence we see that

$P_a \nmid P_b$, for $a > b$, implies that $M_{ab} = \sum_{k=b+1}^{a-1} M_{ak} M_{kb}$. 


Suppose now that \( P_a \) is a sink. Then \( P_a \neq P_i \) for every \( i < a \). Taking \( i = a - 1 \) then \( P_a \nsubseteq P_i \) translates to \( M_{aa-1} = 0 \). One continues by induction, using \( M_{ai} = \sum_{k=a-1}^{a-1} M_{ak} M_{ki} \), and deduces \( M_{ai} = 0 \) for each \( i < a \). Consequently in \( N(R) \) the \( a \)th row consists of zeros. This implies that \( P_a \cdot N = N \) for every sink \( P_a \) in \( R \). Also, by their very definition, one has that \( P_a \cap P_b = P_a \cdot P_b = P_1 \cdot P_a \) for any two sinks \( P_a, P_b \). The proof for (ii) is similar and therefore omitted.

To prove (iii) we can, as in step (i), consider \( R \) as the lower triangular matrices and \( P = P_a \) is a sink. One sees, by inspection, that \( P_r \cdot \cdots \cdot P_1 = 0 \). Now the argument of (i) shows that \( P_r \cdot \cdots \cdot P_{a-1} \cdot P_a = P_i \cap P_a \), for every \( i < a \). Consequently \( P_r \cdot \cdots \cdot P_{a+1} \cdot P_{a-1} \cdot \cdots \cdot P_a = 0 \), which shows that \( l\text{-ann}_R P_a \supseteq P_r \cdot \cdots \cdot P_{a+1} \cdot P_{a-1} \cdot \cdots \cdot P_a \). To see the converse inclusion one observes, as in (i), that in \( P_a \) the \( a \)th row consists of zeros, \( 1 - e_{aa} \in P_a \) and \( (1 - e_{aa})P_a = P_a \). Therefore \( P_a = (1 - e_{aa})R \). Consequently \( l\text{-ann}_R(P_a) = Re_{aa} \). Now \( e_{aa} \in P_i \) for every \( i \neq a \), which shows that \( e_{aa} \in P_r \cdot \cdots \cdot P_{a+1} \cdot P_{a-1} \cdot \cdots \cdot P_a \). Hence \( l\text{-ann}_R(P_a) = Re_{aa} \supseteq P_r \cdot \cdots \cdot P_{a+1} \cdot P_{a-1} \cdot \cdots \cdot P_a \). Q.E.D.

Corollary 5.5. Let \( R \) be a Noetherian \( P.I \). Ring with \( \text{gl.dim} \, R = 2 \), \( A(R) = 0 \), and \( N(R) \neq 0 \). Then

(i) \( P \cdot N = N, P \cap P' = PP' = PP' \), for every minimal prime \( P, P' \) in \( R \) which are sinks.

(ii) Any product of minimal primes which are sinks (or sources) contains \( N \) and is an idempotent.

(iii) \( N \cdot P \cap P' = PP' = PP' \), for every minimal prime \( P, P' \) in \( R \) which are sources.

(iv) \( l\text{-ann}_R(P) = \) product of all other minimal primes (in certain order), for any sink \( P \).

Proof. Let \( \mathcal{E}_a(0) = S \). Then it is clear that \( P \) is a sink in \( R \) if and only if \( S \) is a sink in \( R \). Consequently, by Proposition 5.4, \( P_a \cdot N_s = N_s \). Therefore \( t \cdot N \subseteq P \cdot N \) for some \( t = \mathcal{E}_a(0) = S \). Now \( N/PN = \) a left projective \( R/P \)-module (since \( R \cdot N \) is projective) and \( t \cdot \mathcal{E}_a(0) \subseteq \mathcal{E}(P) \) implies that \( t \cdot N \) acts regularly on \( N/PN \). This contradicts \( t \cdot N \subseteq P \cdot N \). A similar argument applies to \( P \cap P' = PP' \), etc. This establishes (i). Now (ii) is a trivial consequence of (i) and Theorem 3.4. Now (iii) holds by similar arguments (using Proposition 5.4).

Finally to prove (iv) it \( x \in l\text{-ann}(P) \) and let \( Q_1, \ldots, Q_d \) be the other minimal primes which satisfy, by Proposition 5.4(iii), \( l\text{-ann}_{R/R}(P) = Q_{1s} \cdots Q_{ds} \). In particular \( x \in Q_1 \cdots Q_d, \) for some \( t \in S \). Clearly \( x \in Q_i \), for each \( i \). Suppose \( x \in Q_1 \cdots Q_k = I \) but \( x \notin Q_1 \cdots Q_{k+1} \). Now \( I \) being projective implies that \( I/|Q_{k+1} \) is a right projective \( R/Q_{k+1} \)-module, and in particular \( I/\overline{Q}_{k+1} \) is a right \( R/Q_{k+1} \) torsion
free. Consequently $xt \in IQ_{k+1}$ implies, since $t \in \mathfrak{g}(Q_{k+1})$, that $x \in IQ_{k+1}$.

Hence $\text{l-ann}_R(P) \subseteq Q_1 \cdots Q_d$. The reverse inclusion is easy, since $\text{l-ann}_R(P) = \text{l-ann}_R(P_x) \cap R = Q_1 \cdots Q_d \cap R \supseteq Q_1 \cdots Q_d$. Q.E.D.

**Proposition 5.6.** Let $R$ be a Noetherian $P$-ring with $\text{gl.dim} R \leq 2$ and $A(R) = 0$. Suppose that $R$ satisfies condition (*). Let $I \supset N(R)$ be a co-Artinian ideal in $R$ which is right projective and $P = I$ a prime ideal. Then $P$ is right projective in $R$ and $P/N(R)$ is localizable and left projective in $R/N(R)$.

**Proof.** By Lemma 4.5 there exists ideals $A_i$, such that $R/I = R/A_1 \oplus \cdots \oplus R/A_d$, and the set of minimal primes over $A_i/I$ consists of a single clique, for each $i$. Clearly $\text{pr.dim}[R/A_i]_R \leq \text{pr.dim}[R/I]_R \leq 1$ shows that $A_i$ is a right projective, for $1 \leq i \leq d$. Given $i$, choose $T \subseteq R/A_i$ a simple right $R$-submodule and set $K = r$-$\text{ann}_R T$. Hence $K$ is a maximal ideal in $R$ and $\text{pr.dim}[R/K]_R = \text{pr.dim} T_R \leq \text{pr.dim}[R/A_i]_R \leq 1$, where the first inequality holds since $\text{gl.dim} R \leq 2$. Consequently $K_R$ is right projective, which implies, by condition (*) that $K^*K = R$. Now, by Proposition 4.3(4), one gets $K^*N = N$, $KN = N$, and therefore $(K/N)^*(K/N) = R/N$. Consequently, by [H, Theorems 3.5 and 4.3] we have that $K/N$ is invertible and localizable. Now, since $K/N = K/KN$ is a right projective $R/N$-module, we get, since $(K/N)^* = (K/N)^*$, that $K/N$ is a left projective $R/N$-module.

Finally, $K/N$ being localizable in $R/N$ implies that $A_i$ is localizable in $R/A_i$, for $1 \leq i \leq d$. Now, since $R/A_i$ has a single clique of prime ideals, we have that $K/A_i$ is the unique minimal (= maximal) prime in $R/A_i$. The final statement follows since every prime ideal $P$ containing $I$ must contain $A_j$ for some $j$. Q.E.D.

**Lemma 5.7.** Let $R$ be a left and right Noetherian ring, $N(R)$ left and right projective, and $s$ its index of nilpotency. Let $J$ be an ideal in $R$ such that $J \supset N(R)$, $J/N$ is a right projective $R/N$-module. Then $[J/N \otimes_{R/N} N^{s-1}]_R \equiv [JN^{s-1}]_R$, and consequently $JN^{s-1}$ is a right projective $R$-module.

**Proof.** The left-hand side is projective by standard results (e.g., [M, p. 145]). We therefore need to prove the isomorphism. We have, by standard results (e.g., [F, Lemma 2]), that $N^{s-1}$ is projective, and therefore $I \otimes_R N^{s-1} \equiv IN^{s-1}$ for every ideal $I$ in $R$. This implies that $J \otimes_R N^{s-1} \equiv JN^{s-1}$ as well as $N \otimes_R N^{s-1} = NN^{s-1} = 0$. Consequently the exactness of $0 \rightarrow N \rightarrow J \rightarrow J/N \rightarrow 0$ leads, by tensoring on the right with $N^{s-1}$, to $J \otimes_R N^{s-1} \equiv J/N \otimes_R N^{s-1}$. The result now follows since $J/\otimes_{R/N} N^{s-1} = J/N \otimes_R N^{s-1}$. Q.E.D.

The next result is crucial. It handles one type of obstacle, and together with Theorem 5.9 shows that $R$ is a generalized triangular matrix ring.
Theorem 5.8. Let $R$ be a Noetherian P.I. ring with $\text{gl.dim } R = 2$, $A(R) = \{0\}$, $N(R) \neq 0$ and suppose that the minimal primes in $R$ consist of a single clique. Suppose in addition that $R$ satisfies condition $(\ast)$. Then, there exists a minimal prime ideal $Q$ in $R$ which is a sink, such that every maximal ideal containing $Q$ is right projective.

Remark. One gets a similar, left-handed, conclusion for some minimal prime which is a source, if one assumes condition $(\#)$.

Proof. $N(R) \neq \{0\}$ implies, by Theorem 3.11, that $k.dim R = 1$. Denote by $(Q_1, \ldots, Q_r)$ the set of all sinks among the minimal primes. Suppose, by negation, the existence of a maximal ideal $M_i \supset Q_i$ with $M_i$ not projective for $i = 1, \ldots, r$. We also have, by Proposition 4.3(3) and condition $(\ast)$, that every minimal prime which is not a sink, is contained in a maximal ideal which is not right projective. Then, we can find $\bar{a} \neq 0$ in $Z(R/N(R))$, such that $\bar{a}$ is regular and $a$ belongs to all the non-projective maximal ideals mentioned above. Observe that, by the regularity of $\bar{a}$, $a \in \mathfrak{Z}(R)$. Let $I = N(R) + aR = N(R) + Ra$. Then, by the choice of $a$, $I$ is a co-Artinian ideal in $R$. Let $(V_1, \ldots, V_t)$ be the set of all maximal ideals in $R$ which contain $I$ and are projective right $R$-modules. The proof will proceed (from now on) in various steps.

Step 1. There exists a maximal integer $k$ so that

$$V_{i_1}^* \cdots V_{i_k}^* I \subseteq R.$$ 

Indeed, by the Noetherian condition, the chain $(V_{i_1}^* \cdots V_{i_n}^* I | n \geq 1)$ stabilizes for some $m$. Suppose, by negation, that $V_{i_1}^* \cdots V_{i_k}^* I = V_{i_1}^* \cdots V_{i_k}^* I$, for each $n \geq m$. Let $I_0 = V_{i_1}^* \cdots V_{i_k}^* I$; hence $V_{i_1}^* \cdots V_{i_k}^* I_0 = I_0$. Now, $1 \in V_{i_{m+1}} \cdots V_{i_{s_j}}^* \cdots V_{i_{s_j+1}}^*$ implies that $I_0 \subseteq V_{i_{m+1}} \cdots V_{i_{s_j}}^* \cdots V_{i_{s_j+1}}^* I_0 = V_{i_{m+1}} \cdots V_{i_{s_j}}^* I_0$. Therefore, $I_0 \subseteq \cap_j V_{i_j}^n$, for some $j \leq t$. Now, by Proposition 4.3(2), $\cap_j V_{i_j}^n$ is a minimal prime, which contradicts the Artinian property of $R/I$.

Step 2. No maximal ideal containing $I_0$ is right projective. Indeed if $M$ is a right projective, maximal ideal, with $M \supseteq I_0$, then $M \supseteq I$, which implies that $M = V_j$ for some $j \leq t$. Hence $V_j^* V_j^* \cdots V_{i_k}^* I = V_j^* I_0 = M^* I_0 \subseteq M^* M \subseteq R$, which contradicts the maximality of $k$.

Step 3. $V_{i_1}^* \cdots V_{i_k}^* = (V_{i_1} \cdots V_{i_k})^*$ and is a left projective $R$-module. Indeed $V_{i_1} \cdots V_{i_k}$ is right projective, being the product of right projective ideals. Hence, by the dual basis lemma, $(V_{i_1} \cdots V_{i_k})^*$ is left projective. To show the equality one first observes that by its very definition $V_{i_1}^* \cdots V_{i_k}^* \subseteq (V_{i_1} \cdots V_{i_k})^*$. For the converse we use $(V_{i_1} \cdots V_{i_k})^* V_{i_1} \cdots V_{i_n} \subseteq R$ and hence $(V_{i_1}^* \cdots V_{i_m}^*) V_{i_1} \cdots V_{i_{n-1}}^* (V_{i_m}^* V_{i_n}^*) \subseteq V_{i_m}^*$. Now, by the projectivity of
Consider the inclusions Step 2 since, in view of an ideal containing $r$-ann $N$. Now one continues by iteration.

Step 4. $V_i^* \cdots V_i^* I = N(R) + V_i^* \cdots V_i^* a_i$ for each $m$. Indeed, by Proposition 4.3, $V_i^* N = N$ for each $1 \leq j \leq t$. Now $V_i^* I = V_i^*(N + Ra) = V_i^* N + V_i^* a = N + V_i^* a$. One continues by iteration.

Let $s$ be the nilpotency index of $N(R) = N$.

Step 5. $N^{s-1}I_0$ is a projective left $R$-module. Indeed, $N^{s-1}I_0 = N^{s-2}(V_i^* \cdots V_i^* I) = N^{s-1}(N + V_i^* \cdots V_i^* a) = N^{s-1}V_i^* \cdots V_i^* a$, where the second equality is due to Step 4. Now, by Step 3, and the regularity of $a, V_i^* \cdots V_i^* a$ is a left projective $R$-module. Therefore, by standard results (e.g., [F, Lemma 2]) and the left projectivity of $N^{s-1}$, we get the left projectivity of $N^{s-1}I_0$.

We now have to treat the two disjoint possibilities, $N^{s-1}I_0 \subset N^{s-1}$ and $N^{s-1}I_0 = N^{s-1}$.

Step 6. $N^{s-1}I_0 \subset N^{s-1}$.

Recall that $R/I$ is Artinian which implies the same for $R/I_0$. Consequently $N^{s-1}/N^{s-1}I_0$ is right Artinian as well as left Artinian. Let $V = \text{l-ann}_R[N^{s-1}/N^{s-1}I_0]$. Then $R/V$ is Artinian and $V \supset N$. Therefore, $V \supset N$ and we can find a regular element $b \in z(R/N)$ with $b \in V$. Observe that $b \in \mathfrak{p}_0(0)$. Consequently $bN^{s-1}$ is a right projective $R$-module. Let $I_1 = bR + N = Rb + N$. Then $I_1$ is co-Artinian, implying that $N^{s-1}/I_1N^{s-1}$ is a left Artinian $R$-module. So, by Lenagan’s result, $N^{s-1}/I_1N^{s-1}$ is a right Artinian $R$-module. Let $W = \text{r-ann}_R[N^{s-1}/I_1N^{s-1}]$. Then $W$ is co-Artinian and $W \supset N$. Moreover, since $I_1N^{s-1} = bN^{s-1}$ is right projective, then $\text{pr.dim}_R[R/W] \leq \text{pr.dim}_R[N^{s-1}/I_1N^{s-1}] \leq 1$, implying that $W$ is a right projective $R$-module. Therefore, by Proposition 5.6, every maximal ideal containing $W$ is right Artinian. Now $I_1N^{s-1} = bN^{s-1} \subset VN^{s-1} \subset N^{s-1}I_0$ imply that $W = \text{r-ann}_R[N^{s-1}/I_1N^{s-1}] \subset \text{r-ann}_R[N^{s-1}/N^{s-1}I_0]$. Therefore, every maximal ideal containing $\text{r-ann}_R[N^{s-1}/N^{s-1}I_0]$ is right projective. This contradicts Step 2 since, in view of $I_0 \subset \text{r-ann}_R[N^{s-1}/N^{s-1}I_0]$, every such maximal contains $I_0$.

Step 7. $N^{s-1}I_0 = N^{s-1}$.

Let $K = \text{r-ann}_R N^{s-1}$ and $K = \text{r-ann}_{Q(R)}(N^{s-1}(Q(R))$. Now $K = K' \cap R$ shows that $K$ is a finite intersection of minimal prime ideals. $N^{s-1}I_0 = N^{s-1}$ is equivalent, by Step 4, to $N^{s-1}V_i^* \cdots V_i^* a = N^{s-1}$. Now, by the regularity of $a$ and $V_i^* V_j = R$, for each $i$, we get that $N^{s-1} = N^{s-1}(a^{-1})V_i^* \cdots V_i^*$. Let $y \in V_i \cdots V_i$, be an arbitrary element and $x = a^{-1}y$. Consider the inclusions $R \subset R(x) \subset Q(R)$ as well as $R/K \subset$
\( R(x)/K' \cap R(x) \subset Q(R)/K' \). Observe that \( \kappa \dim R/k \leq 1 \) and \( R/K \) is Noetherian. Hence, by Schelter's version of the Krull–Akizuki Theorem [S, Theorem 1], one deduces that \( R(x)/K' \cap R(x) = D \) is Noetherian (and semi-prime). Observe that \( N^{r-1} \) is a faithful f.g. right \( D \)-module. Consequently, by the \( H \)-condition (e.g., [GW, Proposition 8.9]), we have the right \( D \)-module inclusion \( D_D \to N^{r-1}_D \oplus \cdots \oplus N^{r-1}_D \), where the number of copies is finite. However, this embedding is also an inclusion of right \( R(K) \subseteq D \) modules which shows, by the Noetherian property of \( R/K \), that \( D \) is a finite \( R/K \) module. In particular \( \bar{x} \) is integral over \( R/K \). Let \( \bar{x}^m + \bar{x}^{m-1} \bar{r}_1 + \cdots + \bar{r}_m = 0 \) be the integral equation of \( \bar{x} \) over \( R/K \equiv \bar{R} \). Then, since \( a \) is central modulo \( R/N \), we get that \( (\bar{a})^m = \bar{a}^m = -\bar{a} \bar{y}^{m-1} \bar{r}_1 \cdots \bar{a} \bar{y} \bar{r}_m \).

Consequently, \( \bar{y}^m \in a \bar{R} \) for each \( \bar{y} \in \bar{V}_{i_1} \cdots \bar{V}_{i_s} = (V_{i_1} \cdots V_{i_s} + K) / K \).

Consequently there exists an integer \( d \) so that \( \bar{V}_{i_1} \cdots \bar{V}_{i_s} \subseteq a \bar{R} \). Therefore \( X = (V_{i_1} \cdots V_{i_s})^d \subseteq aR + K \). Now \( X \) is right projective, co-Artinian, and contains a product of sinks. Consequently, by Corollary 5.5(ii), \( X \supseteq N \) and, by Proposition 5.6, every maximal ideal containing \( X \) is right projective. This contradicts the fact that \( aR + K \subseteq M \) for some maximal ideal \( M \), which is, by choice, not right projective.

Q.E.D.

**Theorem 5.9.** Let \( R \) be a Noetherian P.I. ring with \( \text{gl.dim} R \leq 2 \), \( A(R) = (0) \), \( N(R) \neq (0) \), and suppose that the set of minimal primes consists of a single clique. Suppose also that \( R \) satisfies condition \( (\ast) \). Then there exists an idempotent \( e \) so that \( R = (e_R \cdot e_R) \), where \( f = 1 - e \) and \( fRf \) is a Dedekind prime ring. Also \( eRe \cdot eRf \) and \( eRf_{Rf} \) are projective.

**Proof.** There exists, by Theorem 5.8, a minimal prime ideal \( Q \), which is a sink, and such that \( M \) is right projective for every maximal ideal \( M \), with \( M \supseteq Q \). Consequently, by condition \( (\ast) \), \( M^*M = R \), for every maximal ideal \( M \) which contains \( Q \). Let \( I \) be the intersection of all the minimal primes which differ from \( Q \). If \( I + Q \subseteq M \), with \( M \) maximal, then \( M^*M = R \) leads to a contradiction via Proposition 4.3(1). Therefore \( I + Q = R \) and \( I \cap Q = N \) implies that \( R/N \cong I/N \oplus Q/N \). Let \( \bar{e} \) be an idempotent such that \( \bar{e}(R/N) = Q/N \). Let \( e^2 = e \in Q \), be a preimage of \( e \). Now, by Corollary 5.5. we have \( QN = N \), which implies \( Q/QN = e(R/N) \). This shows, by standard results, that \( Q = eR \). Let \( f = 1 - e \). Then \( I - \text{ann}_R(Q) = I - \text{ann}_R(eR) = Rf \). Now, by Corollary 5.5, \( I - \text{ann}_R(Q) \) is the product of all other minimal primes (in certain order) \( = J \). Therefore \( fRf \subseteq IQ = 0 \). Hence \( R = eRe + fRf + eRf \) and, by an easy computation, one gets that \( R = (e_R \cdot e_R) \), \( fRf \cong R/Q \), and \( eRe \cong R/J \). Finally \( M \supseteq Q \), with \( M \) maximal implies \( Q = 1 \cdot Q \subseteq MM^*Q = MQ \). Therefore, \( M/Q = M/MQ \) is right projective in \( R/Q \), which shows that \( R/Q \) is hereditary. That \( R/Q \) is a Dedekind ring follows from \( (M/Q)^* \cdot (M/Q) = R/Q \). Also \( eRf = Qf \), being a product of minimal prime ideals, is a left and right
We would like to check now when a tame subidealizer of $M$ in $S$, is of
global dimension 2 (or less). Here $\text{gl.dim } S \leq 2$, $A(S) = \{0\}$, $SM = M$, $MS = S$, and $S/M$ is an Artinian left $S$-module. This is relevant to
Theorem B because this is the converse of the $\star$-process and it is applied,
finitely many times, to the known objects $A_i$, appearing in Theorem B.

**Proposition 5.10.** Let $R$ be a ring with $\text{l.gl.dim } R = n$. Suppose that
$T \supset R$ is a ring which is finitely generated right $R$-module and $Ta \subset R$
for some $a \in \mathcal{C}_i(0)$. Then $R$ is projective or $\text{l.pr.dim } T \leq n - 2$, where $I =$
$I = \text{l-ann}_R T/R$.

**Proof.** The case $n = 1$ is obvious since $R$ is projective. We therefore
may assume that $n \geq 2$. The condition $Ta \subset R$ implies that $\text{l.pr.dim}_R T =$
$n - 1$. Therefore, if $R$ is not projective, $\text{l.pr.dim}_R T \leq n - 1$. Also if $R$ is projective, then $\text{l.pr.dim}_R T \leq 1 \leq n - 1$, as well.
Now $R$ being a finitely generated right $R$-module, implies, by standard
results, that there exists an injective left $R$-module map $R/I \rightarrow T/R(k)$,
for some $k$. Consequently, using $\text{l.gl.dim } R = n$, we have $\text{l.pr.dim}_R[I/R] \leq$
$n - 1$. That is, $\text{l.pr.dim}_R[I \leq n - 2]$.

**Corollary 5.11.** Let $S$ be a Noetherian P.I. ring with $A(S) = \{0\}$ and
$\text{gl.dim } S \leq 2$. Let $M$ be a generative co-Artinian left ideal in $S$ and $R$ a tame
Noetherian subidealizer of $M$ inside $S$ with $\text{gl.dim } R \leq 2$. Then either $R$ is
projective (equivalently $M$ is projective) or $\text{II}_S(M) = R$.

**Proof.** By Proposition 3.3, $S$ has an Artinian quotient ring. Let
$\{P_i, \ldots, P_i\}$ be the set of minimal primes. Clearly $A(S) = \{0\}$ implies that
no $P_i$ is co-Artinian. Therefore $M \nsubseteq P_i$, for each $i$ and consequently, by
[St, Lemma 2.5], $M \cap \mathcal{C}_i(P) \neq \emptyset$. Therefore, by [St, Proposition 2.4],
$M \cap \mathcal{C}_i(0) \neq \emptyset$. Let $a \in M \cap \mathcal{C}_i(0)$. Then $Sa \subset SM = M \subset R$. Moreover,
$S = MS$ shows that $1 = \sum_{i=1}^k m_i t_i$, $m_i \in M$, $t_i \in S$, and consequently $R$ is
finally generated. Therefore, by [CS], $S/\text{N}(S)$ is a finally generated
right $R/N(S)$ $R$-module and by a standard induction argument (on $\text{N}(S)/N(S)$) one shows that $S$ is
finally generated. Let $T = \text{II}_S(M)$. If $T \supset R$, then, by using the maximality of $M$, $\text{l-ann}_R T/R = M$ and,
by Proposition 5.10, $M$ is projective. It is easy to check (e.g., [RS]) that
$R$ being projective implies that $S$ is projective.

**Remark.** Observe that by [RS] the converse of the $\star$-process provides us
with rings of global dimension 2, in case we take a tame special
subidealizer, namely if $M$ is generative and left projective in $S$. Otherwise
we must take the full idealizer as our new object. This leaves us with the

projective $R$-module. Therefore it is a projective left $R/J$-module as well
as a projective right $R/Q$-module. Q.E.D.
The problem of deciding when a tame idealizer of a generative co-Artinian left ideal in a Noetherian P.I. ring $S$ with $\text{gl.dim} S \leq 2$ is of global dimension 2 as well.

The Conclusion of the Proof of Theorem B

The proof is carried by induction on $t(R)$, the number of proper idempotent ideals in $R$. Suppose first that $t(R) = 0$. Then if $M_R$ (respectively $R/M$) is projective and maximal ideal, then $M^* R = R$ (respectively $M^* R = R$). This shows that $R$ satisfies conditions $(\ast)$ and $(\#)$. Consequently, by Proposition 4.9, $R \cong R/I_1^{s} \oplus \cdots \oplus R/I_n^{s}$ and clearly $t(R/I_i^{s}) = 0$, for each $i \leq d$. Therefore $R/I_i^{s}$ satisfies both condition $(\ast)$ and $(\#)$, for $i \leq d$. Moreover, in $R/I_i^{s}$ the minimal primes consist of a single clique.

Now $k \dim R/I_i^{s} \leq \text{gl.dim} R/I_i^{s} \leq 2$. Suppose first that $R/I_i^{s}$ is semi-prime. Then $(M/I_i^{s})^*(M/I_i^{s}) = R/I_i^{s}$ shows by [H] that $M/I_i^{s}$ is localizable whenever it is maximal and right projective. A similar statement holds for left projective maximal ideals. Therefore $R/I_i^{s}$ is a hom-hom ring (along cliques) and in particular it is integral over its center by [SZ].

Suppose, therefore, that $R/I_i^{s}$ is not semi-prime and, hence by Theorem 3.11, $k \dim R/I_i^{s} = 1$ (it cannot be 0 since $\mathcal{A}(R) = \{0\}$). We change notation to $R = R/I_i^{s}$. Then by Theorem 5.9, $R \cong \left(\begin{array}{c} c \cr 0 \end{array}\right)$ where $\mathcal{R} = R/I_i^{s}$ for some minimal prime $Q$ which is a sink, and $R/Q$ is a Dedekind prime ring. Moreover $t(eR e) = 0$, since otherwise, we could construct a non-trivial idempotent ideal in $R$. Consequently the process can be repeated on $eR e$ (having a smaller number of minimal primes) and we get in this way that $R$ is isomorphic to upper triangular matrices with Dedekind prime rings along the diagonal. We may, therefore, assume that $t(R) > 0$.

If no maximal ideal in $R$ is right projective then $\text{pr.dim}(R/M)_{R} = 2$, for every maximal ideal $M$ in $R$. Therefore, $R$ is a hom-hom ring and consequently, by [SZ], $R$ is a direct sum of prime ring and is integral over its center. Similarly, the same happens if $R$ has no maximal left projective ideals. Also if for all maximal ideals $M$, $M_R$ being projective implies that $M^* M = R$, then $R$ satisfies conditions $(\ast)$. Now Theorems 5.9, 5.8, and Proposition 4.9 furnish, as in the case $t(R) = 0$, the desired result. Therefore we may assume the existence of a maximal ideal $M$ in $R$ which is right projective and $M^* M = M$. Then $M^*$ is a ring and $\text{gl.dim} M^* \leq \text{gl.dim} R \leq 2$ by standard results (e.g., [MC, p. 252]). Now, by Proposition 4.17, $M^*$ is left and right Noetherian and by Corollary 4.2, $t(M^*) < t(R)$. Moreover $M^*$ is clearly obtained from $R$ by the $\ast$-process. Therefore, we get, by induction (on $t(M^*)$), the desired result.

Q.E.D.

Proposition 5.12. Let $R$ be a Noetherian P.I. ring with an Artinian quotient ring. Suppose $M = M^2$ is a maximal ideal which is not a minimal
prime and is right projective. Let \( P \) be a maximal ideal in \( R \) with \( P \nsubseteq V \equiv \text{l-ann}_M S^* \). Then

(i) \( P' \cap R = P \), for a unique maximal ideal \( P' \) in \( M^* \),

(ii) \( P_1 \) \( \nsubset \) \( P_2 \) if and only if \( P_1 \cap P_2 = P_i \), where \( P_i \nsubseteq V \), \( i = 1,2 \).

**Proof.** Recall that \( S = M^* \) is a ring and \( SM = M \). Now \( V \) is a non-zero two-sided ideal in \( S \) which is contained in \( R \). We can find, by the Going Up property between \( R \) and \( S \), a maximal ideal \( P' \) in \( S \), such that \( P \) is minimal over \( P' \cap R \). Now \( V \subseteq P \) implies that \( V \nsubseteq P' \) and therefore \( V + P' = S \). Consequently, by \( R/P' \cap R \equiv R/P' \) \( \equiv V + P' \), \( P' \cap R \) is maximal and therefore \( P = P' \cap R \). Similarly \( S/PSPS = S/SPS + V/PSPS = S/SPS + R/SPS \equiv R/P \) shows that \( P = SPS \) and (i) follows. Suppose that \( P_1 \cap P_2 \). In particular \( \text{l-ann}_R[1, P_1 \cap P_2/P_1 P_2] \)

\( = P_1 \), and \( \text{r-ann}_R[1, P_2 \cap P_1/P_1 P_2] = P_2 \). Assume, to the contrary, that \( P_1 \nsubset P_2 \). Hence \( P_1 \cap P_2 \neq P_1 P_2 \) and therefore \( V(P_1 \cap P_2) \subseteq \text{l-ann}_R[1, P_1 \cap P_2/P_1 P_2] = P_1 \). \( P_2 \). Now, \( V \nsubset P_2 \) forces \( P_1 \cap P_2 \) \( \subseteq \) \( P_1 \). Consequently, since \( P_1 \cap P_2 \) \( \subseteq \) \( P_1 \cap P_2 \), \( P_1 \cap P_2 \) \( \subseteq \) \( P_1 \cap P_2 \) implies that \( V \subseteq P_1 \), which is a contradiction.

Conversely, say \( P_1 \nsubset P_2 \) but \( P_1 \nsubset P_2 \). Then \( V(P_1 \cap P_2) \subseteq P_1 \cap P_2 = P_1 \cap P_2 \subseteq P_1 \cap P_2 \). Therefore \( V \subseteq \text{l-ann}_R[1, P_1 \cap P_2/P_1 P_2] = P_1 \), that is, \( V \subseteq P_1 \cap R = P \), which was excluded. Q.E.D.

Our next result is rather surprising and exhibits the usefulness of Theorems A and B.

**Theorem 5.13.** Let \( R \) be a Noetherian \( P.I. \) ring with \( \text{gl.dim} R \leq 2 \). Then every clique \( x \) in \( R \) is finite.

**Proof.** We may assume, by Lemma 3.9, that \( x \) consists of maximal ideals. We first assume that \( \mathcal{A}(R) = \{0\} \). The proof is carried by induction on the ordered pair \( \langle p(R), \text{r}(R) \rangle \), where \( p(R) \) denotes the number of minimal primes in \( R \). The \( \langle 0,0 \rangle \) case is handled by Case 2(i). We deal with two separate cases.

**Case 1.** \( M = M^2 \), for some maximal right (or left) projective ideal in \( R \). Let \( S = M^* \) and \( V = \text{l-ann}_S S/M \). Observe that \( V \) is co-Artinian and therefore is contained in only finitely many maximal ideals. Now \( p(S) = p(R) \) but \( \text{r}(S) < \text{r}(R) \) implies, by induction, that in \( S = M^* \), every clique is finite. The result now follows from Proposition 5.12 and [GW, Theorem 11.18] observing that \( R/V \) is Artinian. A similar argument, using \( M^* \), works for \( R/M \) being left projective.

**Case 2.** \( R \) satisfies conditions (\( \ast \)) and (\( \# \)). Observe that Case 2 is the complement of Case 1. Now we have by Proposition 4.9 that \( R = R_1 \oplus \cdots \oplus R_d \), where in each \( R_i \), the set of minimal primes consists of a single clique. Clearly \( p(R) \geq p(R_i) \) as well as \( \text{r}(R) \geq \text{r}(R_i) \) and each \( R_i \).
satisfies conditions (*) and (#). Therefore we shall assume that the set of minimal primes in \( R \) consists of a single clique. Once more the discussion splits into two separate cases.

**Case (i).** \( R \) is semi-prime. Then by [H, SZ], we have that \( R = A_1 \oplus \cdots \oplus A_k \), where each \( A_i \) is a prime ring which is equal to its own trace ring \( T(A_i) \). Consequently, by [B2, Theorem 8], each clique in \( A_i \) is finite and the result follows.

**Case (ii).** \( R \) is not semi-prime. Therefore, by Theorem B, we have that \( R = C \oplus V \). Consequently, by B2, Theorem 8, each clique in \( C \) is finite and the result follows.

**Proposition 5.14.** Let \( R \) be a Noetherian P.I. ring with \( \text{gl.dim} \, R \leq 2 \), and \( x = \{P_1, \ldots, P_m\} \) be a clique in \( R \). Let \( \{Q_1, \ldots, Q_r\} \) be the set of all minimal primes which are contained in the members of \( x \). Then \( m \geq r \).

**Proof.** By Theorem 5.13, we may assume, after localizing at \( x \) that \( \{P_1, \ldots, P_m\} \) is the complete set of maximal ideals in \( R \). Also say that \( \{P_1 = Q_1, \ldots, P_d = Q_d\} \) is the set of maximal ideals which are minimal primes. It suffices to show that \( m - d \geq r - d \). Let \( I = Q_{d+1} \cap \cdots \cap Q_r \). Then \( I' = I^{i+1} \) for some \( i \), \( A(R/I') = 0 \) and \( \text{gl.dim} \, R/I' \leq 2 \). Moreover \( \{P_{d+1}/I', \ldots, P_m/I'\} \) is the set of all maximal ideals in \( R/I' \) and \( \{Q_{d+1}/I', \ldots, Q_r/I'\} \) is the set of all minimal primes, none of which is maximal. We may therefore prove the result with the additional assumption \( A(R) = 0 \) (and keeping the original notation). Let \( S = \Omega(0) = \Omega(N(R)) \). Then by standard results (e.g., [McR, p. 418]) there is a surjec-
6. APPLICATIONS

In the present section we provide two applications of the methods and results of the previous sections. In the first application we handle the structure of a Noetherian P.I. ring \( R \) with \( \text{gl.dim} R \leq 2 \) and with only two maximal ideals. It is shown that \( R \) must be of a “standard form.” The second application deals with a Theorem of König-Wiedemann [KW] which shows that a classical order of global dimension at most two over a complete Dedekind domain is quasi-hereditary. We generalize this result by using the \( \ast \)-process, to any (semi-)prime Noetherian P.I. ring \( R \) with \( \text{gl.dim} R \leq 2 \).

Recall that by [R, Proposition 7] a local Noetherian ring \( R \) with \( \text{gl.dim} \leq 2 \) is a maximal order. This was further generalized by [SZ]. Our first application deals with the case where \( R \) has exactly two maximal ideals.

**Theorem 6.1.** Let \( R \) be a Noetherian P.I. ring with \( \text{gl.dim} R \leq 2 \) and with only two maximal (two-sided) ideals. Then \( R \) has (at most) two minimal prime ideals \( P_1, P_2 \) and \( R \) is one of the following

(i) \( R \) is Artinian,

(ii) \( R \) is prime,

(iii) \( R \cong \left( \frac{R}{P_1}, \frac{P_1 \cap P_2}{P_1} \right) \), where \( R/P_1 \) is a maximal order with \( \text{gl.dim} R/P_1 \leq 2 \).

**Proof.** Suppose first that \( A(R) \neq 0 \) but \( R \) is not Artinian. Let \( M_1, M_2 \) be the two maximal ideals of \( R \). We may assume that \( M_2 \) is a minimal prime as well. Let \( P \) be any minimal prime so that \( P \neq M_2 \). Consequently, the only maximal ideal containing \( P \) is \( M_1 \). Moreover, \( P \neq M_1 \), since \( R \) is not Artinian. Therefore \( R/P \) is local with \( \text{gl.dim} R/P \leq 2 \), and \( P = P^2 \subseteq \bigcap \mathfrak{m} M_1 \subseteq P \). Consequently \( R \) has exactly two minimal primes \( P \) and \( M_2 \). Now \( A(R) \subseteq P \), but \( A(R) \nsubseteq M_2 \), imply, by Theorem 2.3 and Theorem 3.4, that \( P = A(R) \). Let \( K = \text{l-ann}_R(P) \). Then \( R/K \) is Artinian. Also \( KP = 0 \) and \( P \nsubseteq M_2 \) imply that \( K \subseteq M_2 \). We need to separate the argument into two cases.

**Case 1.** \( M_1 \) or \( R/M_1 \) is projective.

If \( K \nsubseteq M_1 \) then \( M_2 \) being the unique maximal above \( K \), implies, by Theorem 3.4 that \( K = M_2 \). Consequently \( M_2 P = 0 \). Also \( N(R) = M_2 \cap P \)
and $M_2 + P = R$. Let $e = e^2$ be an idempotent so that $M_2/N(R) = eR = Re$ and $P/N(R) = fR = fRe$, where $f = 1 − e$ and $R = R/N(R)$. This is possible since $P/N(R)$ is Artinian. Therefore $M_2 = eR + N(R)$, $P = fR + N(R)$, and $eRf \subseteq M_2, P = \{0\}$. Consequently $R = fRf + fRe + eRe$ and it is standard to check that $R \equiv (fRf, fRe)$. This implies (e.g., [M cR, p. 246]) that $\text{gl.dim } eRe \leq 2$ as well as $\text{gl.dim } fRf \leq 2$. Now, $eRe$ and $fRf$ are both local, so by [R, Proposition 7], we get that $eRe$ is a prime maximal order and $fRf$ is a simple ring. Consequently, $N(eRe) = eN(R)e = \{0\}$ and $N(fRf) = fN(R)f = \{0\}$. Let $x \in N(R)$. Then the previous equalities imply $x = exe + fx + exe = fxe$. Therefore $N(R) \subseteq fRe$ and, since $fRe \subseteq N(R)$ is trivial, we get $N(R) = fRe$. Consequently $R/M_2 \cong fRf$ and $R/P \equiv eRe$, as needed.

We may therefore suppose that $K \subseteq M_1$. We now use the right projectivity assumption on $M_1$. Let $f \in \text{Hom}_R(M_1, R_R)$. Then $KA(R) = \{0\}$ implies $f(K) A(R) = \{0\}$ and therefore $f(K) \subseteq K$. Let $\{m_1, f_1\}$ be a dual basis for $M_1R$. Then $K = \Sigma m_1 f_1 (K) \subseteq M_1K$, shows that $K \subseteq \cap_i M_i = P$. This is a contradiction since $R/K$ is Artinian and $R/P$ is not Artinian. This excludes the possibility $K \subseteq M_1$. A similar argument with $k = r$-ann$_R(A(R))$ and $R M_1$ being projective leads to a similar conclusion.

**Case 2.** Neither $M_1R$ nor $R M_1$ is projective.

Now $A(R) = P$ contains a simple left $R$-module $T_1$ and a simple right $R$-module $T_2$. Clearly $\text{l-ann}_R T_1 = M$ is a maximal ideal and $\text{l.pr.dim } R/M = \text{pr.dim } T_1 = 1$, that is, $R M$ is projective. Now since $R M_1$ is not projective we conclude that $M = M_2$. Similarly, working with $T_2$ we obtain that $M_2$ is also projective. Recall that, by Lemma 3.1, $P_R$ and $R P$ are projective and so the exact sequence

$$0 \rightarrow P \cap M_2 \rightarrow P \oplus M_2 \rightarrow P + M_2 \rightarrow 0$$

shows that $N(R) = P \cap M_2$ is left and right projective $R$-modules. Therefore, by Proposition 3.2, $R$ has an Artinian quotient ring and $A(R) = eR = Re$ splits. That is, $R = (1 − e)R(1 − e) \oplus eRe$, and, by a standard verification, $M_2 = (1 − e)R = R(1 − e)$ as well as $R \equiv R/P \oplus R/M_2$.

We assume now that $A(R) = \{0\}$. Let $M_1, M_2$ be two maximal ideals of $R$. So $A(R) = \{0\}$ implies that $M_i$ is not a minimal prime for each $i = 1, 2$. We may assume that $R$ is not prime and therefore, by Theorem 3.4, $R$ has at least two minimal primes. Let $P$ be a minimal prime. If $P \subseteq M_1 \cap M_2 = \text{Jac}(R)$, then $P = P^2 \subseteq \cap_i \text{Jac}(R) = \{0\}$, which was excluded. Therefore in $R/P$ there is a unique maximal ideal $M_1/P$ and $\cap_i M_i^n = P$. This shows that $R$ has exactly two minimal prime ideals $P_1, P_2$ and, since $\text{gl.dim } R/P \leq 2$, we have that $R/P$ is a maximal order. Moreover, since for each $M_i$, $\cap_i M_i^n$ is a minimal prime, then $M_i$ is not an idempotent.
ideal and in particular $R$ satisfies conditions (*) and (#). Therefore, by
Theorem 4.9, we have $R \cong R_1 \oplus \cdots \oplus R_k$, where the minimal primes in
each $R_i$ consist of a single clique. Now $k \leq 2$, since $R$ has only two
maximal ideals. Suppose $k = 2$. Then $R_1$ and $R_2$ are local maximal orders
and $R \cong R_1 \oplus R_2$. If $k = 1$ we invoke Theorem 5.9 and get $R \cong
\left( \frac{R}{R_1} \cap \frac{R}{R_2} \right)_{k/2}$.

Q.E.D.

In our next application we consider a semi-prime Noetherian P.I. ring $R$
with $\text{gl.dim } R \leq 2$ and show that it is quasi-hereditary. As mentioned
above this generalizes the same result in the classical order case due to
[KW]. The next definition generalizes the one in [KW].

**Definition 6.2.** Let $R$ be a semi-prime Noetherian P.I. ring. We say
that $R$ is quasi-hereditary if there exists a hom-hom ring $S$ with no
projective idempotent ideal, $Q(R) \supset S \supset R$, $R S$ is finitely
generated and projective, $V$ an ideal such that $V_R$ is projective, $V^2 = V$, $V$ is a left ideal
in $S$, and $R/V$ is quasi-hereditary and Artinian.

**Remark.** If $k \text{.dim } R = 1$ then $k \text{.dim } S = 1$ and therefore $S$ is a maximal
order and the previous definition agrees with [KW, Definition 2].

**Theorem 6.3.** Let $R$ be a semi-prime Noetherian P.I. ring with $\text{gl.dim } R \leq 2$. Then $R$ is quasi-hereditary.

**Proof.** Since $A(R)$ is a direct summand of $R$ we may assume that
$A(R) = \{0\}$. Suppose that $R$ is not a hom-hom ring: then there exists a
maximal ideal $M$ in $R$ so that $M_R$ is projective, $M^2 = M$ and $M$ is not a
minimal prime. By the $*$ process we have a sequence $M^* = R_1$, $(M =
M_1, R = R_0)$. $M_i$ is a right projective maximal ideal in $R_{i-1}$, $M_i^2 = M_i$, and
$M_i^* = R_i$. Clearly $R_i \subset R_{i+1}$, for each $i$ and we have, by induction on $i(R)$,
that this process must stop, after finitely many steps, at $S = M^*_k$ and no
right projective maximal ideal in $S$ is an idempotent. This is equivalent to
$S$ being right injectively smooth and so, by [SZ, Theorem 1.3], $S$ is also left
injectively smooth, that is, a right hom-hom ring (along each clique). Let
$V = M_1 M_2 \cdots M_k$. Clearly $M_{i+1} M_i \subseteq M_i^* M_i = M_i$ and therefore
$V = M_1 M_2 \cdots M_k (M_1 \cdots M_i) \subseteq M_1 M_2 \cdots M_k R_i \subseteq M_i$, for each $i$. Also
$V^* = \{q \in Q(R) \mid q V \subseteq R\} = \{q \in Q(R) \mid q M_k \cdots M_i R_i \subseteq R\}$
$= \{q \in Q(R) \mid q M_k \cdots M_i \} = M^*_k = S$.

Next, since $M^*_k$ is projective, we get that $1 \in M_i M^*_i$ for each $i$.
Consequently $V V^* = M_i \cdots M_1 M^*_i = M_i \cdots M_1 M^*_k M^*_i = M_i \cdots M_k (M_i M^*_i) M^*_i = M_i \cdots M_k (M_i M^*_i) M^*_k = M_i \cdots M_k (M_i M^*_i) M^*_k = \cdots = M_i M^*_k$.
Now $1 \in M_i M^*_i$ shows that $1 \in V V^*$ and therefore $V_R$ is projective.
Consequently $V^* = S$ is finitely generated and projective. Now observe
that $S V = M^*_k (M_k \cdots M_1) = (M^*_k M_k) M_{k-1} \cdots M_1 = M_k \cdots M_1 = V$.
Consequently $V = 1 \cdot V \subseteq (V V^*)^2 = V V^2$ and $V = V^2$. Finally it is easy
to check (since each $M_i$ is maximal) that $R/V$ is Artinian and by [McR., Theorem 7.3.10 (ii)], $\text{gl.dim } R/V \leq 2$. We now use [DR] to deduce that $R/V$ is quasi-hereditary.

**Remarks.** (1) Using some more elaborate arguments one can actually drop the semi-prime assumption in Theorem 6.3.

(2) As stated in the Introduction, the hom-hom ring appearing here is a fairly good substitute for a maximal order and if $k\text{.dim } R = 1$ the two notions coincide.

(3) One can give a different formulation, using idempotent elements, to the previous theorem as follows.

**THEOREM 6.4.** Let $R$ be a Noetherian semi-prime P.I. ring with $\text{gl.dim } R \leq 2$. Then there exists an integer $n$ and an idempotent $e = e^2 \in M_n(R) = A$ satisfying

(i) $eAe$ is a hom-hom ring with $\text{gl.dim } eAe \leq 2$

(ii) $AeA$ is projective

(iii) $A/AeA$ is Artinian with $\text{gl.dim } A/AeA \leq 2$. Moreover if $R$ is an order over a complete D.V.R. (in a separable finite dimensional algebra) then one can take $n = 1$ and so recover the version appearing in [KW].

**Proof.** We retain the notation of Theorem 6.3. [By CPS, Remark 1.4(b)] there exists an integer $n$ so that $M_n(V) = M_n(R)eM_n(R)$ where $e = e^2$. Let $A = M_n(R)$. Then, by [McR., Lemma 3.5.7], $eAe \cong \text{End}(V_n) = S$. Consequently $eAe$ is a hom-home ring with $\text{gl.dim } eAe \leq 2$. Now $AeA = M_n(V)$ is a right projective $A$-module since $V_n$ has the same property. Consequently, by [McR., Theorem 7.3.10 (ii)], $\text{gl.dim } A/AeA \leq 2$. Finally $A/AeA \cong M_n(R/V)$ is Artinian.

Q.E.D.

**REFERENCES**


