Available online at www.sciencedirect.com

SCIENCE DIRECT

Journal of Multivariate Analysis 97 (2006) 148 – 178

Journal of Multivariate Analysis

www.elsevier.com/locate/jmva

Asymptotics for testing hypothesis in some multivariate variance components model under non-normality

Arjun K. Gupta[a,*], Solomon W. Harrar[b], Yasunori Fujikoshi[c]

[a] *Department of Mathematics and Statistics, Bowling Green State University, Bowling Green, OH 43403-0221, USA*

[b] *Department of Mathematics and Statistics, South Dakota State University, Brookings, SD 57007, USA*

[c] *Department of Mathematics, Hiroshima University, Higashi-Hiroshima, Hiroshima 739-8526, Japan*

Received 10 June 2003
Available online 26 January 2005

Abstract

We consider the problem of deriving the asymptotic distribution of the three commonly used multivariate test statistics, namely likelihood ratio, Lawley–Hotelling and Bartlett–Nanda–Pillai statistics, for testing hypotheses on the various effects (main, nested or interaction) in multivariate mixed models. We derive the distributions of these statistics, both in the null as well as non-null cases, as the number of levels of one of the main effects (random or fixed) goes to infinity. The robustness of these statistics against departure from normality will be assessed.

Essentially, in the asymptotic spirit of this paper, both the hypothesis and error degrees of freedom tend to infinity at a fixed rate. It is intuitively appealing to consider asymptotics of this type because, for example, in random or mixed effects models, the levels of the main random factors are assumed to be a random sample from a large population of levels.

For the asymptotic results of this paper to hold, we do not require any distributional assumption on the errors. That means the results can be used in real-life applications where normality assumption is not tenable.

As it happens, the asymptotic distributions of the three statistics are normal. The statistics have been found to be asymptotically null robust against the departure from normality in the balanced designs. The expressions for the asymptotic means and variances are fairly simple. That makes the results an

* Corresponding author. Fax: +1 419 372 6092.
E-mail address: gupta@bgnet.bgsu.edu (A.K. Gupta).

doi:10.1016/j.jmva.2004.12.001

attractive alternative to the standard asymptotic results. These statements are favorably supported by the numerical results.

AMS 1991 subject classification: primary 62H10; secondary 62H15

Keywords: Asymptotics; Elliptically contoured distribution; Generalized quadratic forms; Mixed model; Multivariate statistics; Robustness

1. Introduction

Let the $N \times p$ observation matrix Y have the following structure:

$$Y = \mathbf{1}_N \boldsymbol{\mu}' + \sum_{i=1}^{t} X_i B_i + \sum_{i=t+1}^{k-1} A_i \mathcal{T}_i + \mathcal{E}, \tag{1.1}$$

where $\boldsymbol{\mu}$ is a $p \times 1$ vector of the grand mean effect, X_i and A_i are, respectively, $N \times m_i$ and $N \times r_i$ known design matrices of group indicators, B_i is an $m_i \times p$ unknown matrix of fixed effects and $\mathcal{T}_i$ is $r_i \times p$ random effects matrix whose rows are identically and independently distributed random vectors with mean zero and variance Σ_i. Also, $\mathcal{E}$ is an $N \times p$ random error matrix whose rows are identically and independently distributed with mean zero and variance Σ. Further, $\mathcal{T}_i$'s and $\mathcal{E}$ are mutually independent.

We know that when $\mathcal{T}_i$ and $\mathcal{E}$ have normal distributions, there may not always exist a unique partitioning of the total sum of squares and cross products into quadratic forms that have independent Wishart distributions [14]. Even if such partitioning is possible, tests on the various effects in (1.1) can be derived by comparing the expected values of the partitions only if there exists a random effect whose mean square matrix has the same expected value as that of the effect under consideration (see, for example, [11]). In the cases where tests can be derived from partitioned sum of squares, the usual multivariate test criteria, i.e. likelihood ratio (LR), Lawley–Hotelling (LH) and Bartlett–Nanda–Pillai (BNP) can be used as test statistics. It should be remarked that the LR criterion is not the conventional likelihood procedure (See [3,15]). When the error degrees of freedom is large, the null distribution of these statistics can be approximated by chi-square distribution. For an extensive treatment of these statistics under normality one may refer to Anderson [2] and Siotani et al. [16].

Due to their intuitive appeal, these statistics have also been studied under non-normality. Fujikoshi [5] provides a summary of works in this connection.

In this paper, we are concerned with the asymptotic distribution of these statistics under non-normality. The asymptotic frame work is when both hypothesis and error degrees of freedom tend to infinity at the same rate. Fujikoshi [4] is in the same spirit but under normality and fixed effects MANOVA. More recently, Akritas and Arnold [1] derived the asymptotic [large hypothesis degrees of freedom] distribution of F-statistics in the univariate fixed, random and mixed models. Their fixed effects results have been extended to the multivariate situation by Gupta et al. [8]. In this paper, we extend the works of Gupta et al. [8] and Akritas and Arnold [1] to multivariate mixed models. In Section 2, notations and some lemmas needed in the subsequent sections are presented. In Section 3, we consider the balanced case. Detailed results are given for one-way random, two-way random and two-way mixed effects models and the extension to the general case is outlined. The unbalanced mixed model is treated in Section 4. Section 5 contains the simulation study. We summarize our findings and conclusions in Section 6. Proofs and some technical details for results presented in Section 2 are given in Appendix A.

2. Notations and lemmas

In the sequel, the notations E_N, Var_N and Cov_N mean expected value, variance and covariance, respectively, when the random variables involved have normal distribution. Given a partitioned matrix $Y = (Y_1, \ldots, Y_q)$ where $Y_i, i = 1, \ldots, q$, is $n \times q_i$ matrix, we write $Var(Y)$ to refer to the $q \times q$ block partitioned matrix whose (i, j)th block is $Cov(vec(Y_i), vec(Y_j))$. The notations J_n and $\mathbf{1}_n$ stand for $n \times n$ matrix and $n \times 1$ vector of ones, respectively. The notation I_p stands for $p \times p$ identity matrix. In the cases where the dimension is clear we drop the subscript p of I_p. Furthermore, K_p stands for a $p^2 \times p^2$ commutation matrix, which is a block matrix whose block in position (i, j) is $\mathbf{e}_j\mathbf{e}_i'$ where $\mathbf{e}_j$ is a p-dimensional unit vector with unity on the jth entry. For properties of commutation matrix, see [9,12].

We will next present results useful in the subsequent sections. We first generalize a result given in [8] to fit in the settings of this paper. The generalization is developed in four steps. We present the final result below, and defer the details to Appendix A. The result is stated keeping in mind the balanced two-way mixed model. The extension to the general balanced mixed model can easily be figured out by induction.

Lemma 2.1. *Let $\mathcal{E}' = (\boldsymbol{\varepsilon}_1, \ldots, \boldsymbol{\varepsilon}_n)$ be a $p \times n$ random matrix whose columns are identically and independently distributed with $E(\boldsymbol{\varepsilon}_1) = 0$, $Var(\boldsymbol{\varepsilon}_1) = \Sigma_\varepsilon (> 0)$ and finite fourth-order moments (i.e., $E(vec(\boldsymbol{\varepsilon}_1\boldsymbol{\varepsilon}_1')vec(\boldsymbol{\varepsilon}_1\boldsymbol{\varepsilon}_1')') =: \Delta_\varepsilon$ with finite entries). Let $\mathcal{T}' = (\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_s)$ be a $p \times s$ random matrix whose columns are identically and independently distributed with $E(\boldsymbol{\tau}_1) = 0$, $Var(\boldsymbol{\tau}_1) = \Sigma_\tau$ $(\Sigma_\tau \geqslant 0)$ and finite fourth-order moment Δ_τ. Let $\mathcal{Z}' = (\boldsymbol{\zeta}_1, \ldots, \boldsymbol{\zeta}_t)$ be a $p \times t$ random matrix whose columns are identically and independently distributed with $E(\boldsymbol{\zeta}_1) = 0$, $Var(\boldsymbol{\zeta}_1) = \Sigma_\zeta (\geqslant 0)$ and finite fourth-order moment Δ_ζ. Let $B_i, i = 1, \ldots, q$, be $n \times n$ fixed symmetric matrices. Let A_i, L_i and M_i be $n \times p$, $n \times s$ and $n \times t$ fixed matrices. Suppose B_i, $C_i = L_i'B_iL_i$ and $D_i = M_i'B_iM_i$ each has equal diagonal elements. Define $Q_i = Q_i(\mathcal{E}, \mathcal{T}, \mathcal{Z}, A_i) = (\mathcal{E}+A_i+L_i\mathcal{T}+M_i\mathcal{Z})'B_i(\mathcal{E}+A_i+L_i\mathcal{T}+M_i\mathcal{Z})$ and let $Q = (Q_1, \ldots, Q_q)$. If the third-order moments of $\boldsymbol{\varepsilon}_1$, $\boldsymbol{\tau}_1$ and $\boldsymbol{\zeta}_1$ are zero or $A_i = 0, i = 1, \ldots, q$, then*

$$E(Q) = E_N(Q)$$

and

$$\begin{aligned} Var(Q) = {} & Var_N(Q) + n(\mathbf{bb}') \otimes (\Delta - I_{p^2} - K_p - vec(I_p)vec(I_p)') \\ & + s(\mathbf{cc}') \otimes (\Delta_\tau - (I_{p^2} + K_p)(\Sigma_\tau \otimes \Sigma_\tau) - vec(\Sigma_\tau)vec(\Sigma_\tau)') \\ & + t(\mathbf{dd}') \otimes (\Delta_\zeta - (I_{p^2} + K_p)(\Sigma_\zeta \otimes \Sigma_\zeta) - vec(\Sigma_\zeta)vec(\Sigma_\zeta)'), \end{aligned}$$

where $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ are $q \times 1$ vectors with ith elements b_{i11}, c_{i11} and d_{i11} which obviously are the entries in the $(1, 1)$ position of B_i, C_i and D_i, respectively.

As Akritas and Arnold [1] have noted, such a result can be proved in its generality by dropping the restrictions on the diagonal elements of B_i, C_i and D_i. However, the resulting expressions will be far too complicated to be of any practical use.

The next result is concerned with the mean and variance of non-central Wishart random matrix. Its proof is given in [9,12].

Lemma 2.2. *Suppose* $S \sim W_p(n, \Sigma, M)$ *and* $\Sigma > 0$. *Then*,

(i) $E(S) = n\Sigma + M$ *and*
(ii) $Var(S) = (I + K_p)(n\Sigma \otimes \Sigma + \Sigma \otimes M + M \otimes \Sigma)$.

3. The balanced mixed model

For a normal balanced mixed MANOVA model, it is well known that there exists a unique partitioning of the total sum of squares and cross products into quadratic forms that have independent Wishart distributions [14]. This results from the fact that the design matrices X_i and A_i can be expressed as Kronecker products of identity matrices and vectors of ones of appropriate dimensions. In light of this fact and on condition that there exists a suitable random effect, the three multivariate statistics, viz. LR, LH and BNP, can be used to develop tests. We will use these test criteria to develop asymptotic tests under non-normality. Instructively, we discuss the one-way random and two-way random and mixed effects model first.

3.1. One-way random effects model

Consider the balanced one-way random effects model given by

$$\mathbf{y}_{ij} = \mu + L\boldsymbol{a}_i + \Sigma^{1/2}\boldsymbol{\varepsilon}_{ij}, \quad i = 1, \ldots, k, \quad j = 1, \ldots, n, \tag{3.1}$$

where $\boldsymbol{\varepsilon}_{ij}$ are identically independently distributed $p \times 1$ random vectors with $E(\boldsymbol{\varepsilon}_{11}) = 0$, $Var(\boldsymbol{\varepsilon}_{11}) = I_p$ and finite $\Delta := E(vec(\boldsymbol{\varepsilon}_{11}\boldsymbol{\varepsilon}_{11}')vec(\boldsymbol{\varepsilon}_{11}\boldsymbol{\varepsilon}_{11}')')$, $\boldsymbol{a}_i$ are $r \times 1 (r \leqslant p)$ identically and independently distributed random vectors with $E(\boldsymbol{a}_1) = 0$, $Var(\boldsymbol{a}_1) = I_r$ and finite $E(vec(\boldsymbol{a}_1\boldsymbol{a}_1')vec(\boldsymbol{a}_1\boldsymbol{a}_1')')$, $\boldsymbol{a}_i$'s and $\boldsymbol{\varepsilon}_{ij}$'s are assumed mutually independent, L is a $p \times r$ fixed but unknown matrix of parameters with rank $r \leqslant p$ and Σ is a $p \times p$ positive definite matrix of parameters.

We would like to test $\mathrm{H}_0 : LL' = 0$ versus $\mathrm{H}_1 :$ not H_0.

3.1.1. Asymptotic distribution of the sum of squares and products

For the testing problem mentioned above, the MANOVA hypothesis and error sums of squares and products are,

$$S_h = \sum_{i=1}^{k} n(\bar{\mathbf{y}}_{i.} - \bar{\mathbf{y}}_{..})(\bar{\mathbf{y}}_{i.} - \bar{\mathbf{y}}_{..})' \quad \text{and} \quad S_e = \sum_{i=1}^{k}\sum_{j=1}^{n} (\mathbf{y}_{ij} - \bar{\mathbf{y}}_{i.})(\mathbf{y}_{ij} - \bar{\mathbf{y}}_{i.})',$$

where $\bar{\mathbf{y}}_{i.} = \frac{1}{n}\sum_{j=1}^{n} \mathbf{y}_{ij}$ and $\bar{\mathbf{y}}_{..} = \frac{1}{k}\sum_{i=1}^{k} \bar{\mathbf{y}}_{i.}$.

The LR, LH and BNP test statistics given, respectively, by $T_1 = -\log \frac{|S_e|}{|S_e+S_h|}$, $T_2 = tr\ S_h S_e^{-1}$ and $T_3 = tr\ S_h(S_e+S_h)^{-1}$ can be used to construct tests under normality. Besides their intuitive appeal, it will be shown later that they are null-robust for large k against departure from normality. Therefore, they can be considered as viable criteria under non-normality.

Let $U_i^{(h)}$ and $U_i^{(e)}$ be defined as follows:

$$U_i^{(h)} := n(\Sigma^{-1/2}\boldsymbol{L}\boldsymbol{a}_i + \bar{\boldsymbol{\varepsilon}}_{i.})(\Sigma^{-1/2}\boldsymbol{L}\boldsymbol{a}_i + \bar{\boldsymbol{\varepsilon}}_{i.})'$$

and

$$U_i^{(e)} := \sum_{j=1}^{n}(\boldsymbol{\varepsilon}_{ij} - \bar{\boldsymbol{\varepsilon}}_{i.})(\boldsymbol{\varepsilon}_{ij} - \bar{\boldsymbol{\varepsilon}}_{i.})', \quad i = 1, \ldots, k,$$

where $\bar{\boldsymbol{\varepsilon}}_{i.} = \frac{1}{n}\sum_{i=1}^{n} \boldsymbol{\varepsilon}_{ij}$.

Put $\Theta = n\Sigma^{-1/2}LL'\Sigma^{-1/2}$, $U_i = (U_i^{(h)}, U_i^{(e)})$ and $\bar{U}_k = \frac{1}{k}\sum_{i=1}^{k} U_i$.

Lemma 3.1. *The limiting distribution of* $\sqrt{k}(\bar{U}_k - E(\bar{U}_k))$ *is* $N(0, \Omega)$ *with* Ω *given by*

$$\Omega = \Psi + \frac{1}{n}C \otimes \left(\Delta - (I_{p^2} + K_p) - vec(I)vec(I)'\right) \\ +D \otimes \left(\Delta_1 - (I_{p^2} + K_p)(\Theta \otimes \Theta) - vec(\Theta)vec(\Theta)'\right), \tag{3.2}$$

where

$$\Delta_1 = n^2 E(vec(\Sigma^{-1/2}\boldsymbol{L}\boldsymbol{a}_i\boldsymbol{a}_i' L'\Sigma^{-1/2})vec(\Sigma^{-1/2}\boldsymbol{L}\boldsymbol{a}_i\boldsymbol{a}_i' L'\Sigma^{-1/2})'),$$

$\Psi = (\Psi_{ij})$ *is a* 2×2 *block matrix with* $\Psi_{11} = (I + K_p)\left((I_p + \Theta) \otimes (I_p + \Theta)\right)$, $\Psi_{22} = (n-1)(I_{p^2}+K_p)$, $\Psi_{21}' = \Psi_{12} = 0$, $C = (c_{ij})$ *is a* 2×2 *matrix with* $c_{11} = 1$, $c_{22} = (n-1)^2$, $c_{12} = c_{21} = (n-1)$ *and* $D = (d_{ij})$ *is a* 2×2 *matrix with* $d_{11} = 1$, $d_{12} = d_{21} = d_{22} = 0$.

Proof. That the limiting distribution is normal follows from the fact that U_i's are identically and independently distributed. We also note that $\Omega = Var(U_i)$. To derive the expression for $Var(U_i)$, let us express $U_i^{(h)}$ and $U_i^{(e)}$, more conveniently, as

$$U_i^{(h)} = (\mathbf{1}_n\boldsymbol{a}_i' L'\Sigma^{-1/2} + \mathcal{E}_i)'\left(\frac{1}{n}J_n\right)(\mathbf{1}_n\boldsymbol{a}_i' L'\Sigma^{-1/2} + \mathcal{E}_i)$$

and

$$U_i^{(e)} = \mathcal{E}_i'\left(I - \frac{1}{n}J_n\right)\mathcal{E}_i, \tag{3.3}$$

where $\mathcal{E}_i' = (\varepsilon_{i1}, \ldots, \varepsilon_{in})$.

Now, if $\mathcal{E}_i \sim N(0, I_n \otimes I_p)$ then $U_i^{(h)}$ and $U_i^{(e)}$ will be independently distributed as $W_p(1, I + \Theta)$ and $W_p(n-1, I_p)$, respectively. Then by Lemma 2.2 we have,

$$\begin{aligned} Var_N(U_i^{(h)}) &= (I + K_p)\big((I+\Theta) \otimes (I+\Theta)\big), \\ Var_N(U_i^{(e)}) &= (n-1)(I_{p^2} + K_p) \end{aligned}$$

and

$$Cov_N(U_i^{(h)}, U_i^{(e)}) = 0. \tag{3.4}$$

Finally appealing to Corollary A.1 ($q = 1$, $B_1 = \frac{1}{n}J_n$, $B_2 = I - \frac{1}{n}J_n$, $L_1 = \mathbf{1}_n$, $L_2 = 0$ $s = 1$ and $\mathcal{T} = \mathbf{a}_i' L' \Sigma^{-1/2}$), we get the expression for $Var(U_i)$ under non-normality. □

Noting that $E(U_i^{(h)}) = I + \Theta$ and $E(U_i^{(e)}) = (n-1)I$, it can be verified that,

$$\frac{1}{k}\Sigma^{-1/2} S_h \Sigma^{-1/2} = I + \Theta + \frac{1}{\sqrt{k}} Z^{(h)}$$

and

$$\frac{1}{k}\Sigma^{-1/2} S_e \Sigma^{-1/2} = (n-1)I + \frac{1}{\sqrt{k}} Z^{(e)}, \tag{3.5}$$

where $Z^{(h)} = \sqrt{k}\big(\bar{U}_k^{(h)} - E(\bar{U}_k^{(h)})\big) - \sqrt{k}n(\bar{\boldsymbol{\varepsilon}}_{..} + \Sigma^{-1/2}\bar{\boldsymbol{a}}_k)(\bar{\boldsymbol{\varepsilon}}_{..} + \Sigma^{-1/2}\bar{\boldsymbol{a}}_k)'$, $Z^{(e)} = \sqrt{k}\big(\bar{U}_k^{(e)} - E(\bar{U}_k^{(e)})\big)$ and $\bar{\boldsymbol{a}}_k = \frac{1}{k}\sum_{i=1}^{k} \boldsymbol{a}_i$.

Putting $Z = (Z^{(h)}, Z^{(e)})$ we get the following result.

Corollary 3.1. *The random matrix Z is asymptotically normally distributed with mean* 0 *and variance Ω.*

Proof. It can easily be shown that,

$$\sqrt{k}n(L\bar{\boldsymbol{a}}_k + \Sigma^{-1/2}\bar{\boldsymbol{\varepsilon}}_{..})(L\bar{\boldsymbol{a}}_k + \Sigma^{-1/2}\bar{\boldsymbol{\varepsilon}}_{..})' \xrightarrow{p} 0$$

as $k \to \infty$. As a result,

$$Z = \sqrt{k}(\bar{U}_k - E(\bar{U}_k)) + o_p(1). \tag{3.6}$$

Then, the desired result follows from Lemma 3.1. □

More explicit expression can be derived for Ω when the random effect and error terms have spherical distribution.

Example 3.1. Let

$$\boldsymbol{a}_i \sim (-2\phi_a'(0))^{-1/2} ECD_r(0, I, \phi_a)$$

and

$$\boldsymbol{\varepsilon}_{ij} \sim (-2\phi_\varepsilon'(0))^{-1/2} ECD_p(0, I, \phi_\varepsilon)$$

where $ECD_k(0, I, \phi)$ stands for k-variate elliptically contoured distribution with characteristic function generator ϕ. It can be shown [6] that

$$E(\boldsymbol{a}_i\boldsymbol{a}_i') = I$$

and

$$E(vec(\boldsymbol{a}_i\boldsymbol{a}_i')vec(\boldsymbol{a}_i\boldsymbol{a}_i')') = \frac{\phi_a''(0)}{(\phi_a'(0))^2}\Big(I_{r^2} + K_r + vec(I_r)vec(I_r)'\Big).$$

Then one can verify that,

$$\begin{aligned}\Omega = \Psi &+ \frac{1}{n}\left(\frac{\phi_\varepsilon''(0)}{(\phi_\varepsilon'(0))^2} - 1\right)\left[\begin{pmatrix} 1 & n-1 \\ n-1 & (n-1)^2 \end{pmatrix}\right. \\ &\left.\otimes\Big(I_{p^2} + K_p + vec(I)vec(I)'\Big)\right] \\ &+ \left(\frac{\phi_a''(0)}{(\phi_a'(0))^2} - 1\right)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes ((I_{p^2} + K_p)(\Theta \otimes \Theta) + vec(\Theta)vec(\Theta)').\end{aligned}$$

Further, if we use the generator of multivariate t distribution with v degrees of freedom for both a_i and ε_{ij}, we get

$$\frac{\phi_a''(0)}{(\phi_a'(0))^2} - 1 = \frac{\phi_\varepsilon''(0)}{(\phi_\varepsilon'(0))^2} - 1 = \frac{2}{v-4}.$$

3.1.2. Distribution of test statistics

Let us finally derive the distribution of the three test statistics. To that end we will show that T_1, T_2 and T_3 are asymptotically linear functions of elements of Z. Define,

$$\begin{aligned}\tilde{T}_1 &:= \sqrt{k}\left(nT_1 + n\log\frac{|(n-1)I|}{|nI + \Theta|}\right), \\ \tilde{T}_2 &:= \sqrt{k}((n-1)T_2 - tr(I+\Theta))\end{aligned} \tag{3.7}$$

and

$$\tilde{T}_3 := \sqrt{k}\left(\frac{n^2}{n-1}T_3 - \frac{n^2}{n-1}tr(I+\Theta)(nI+\Theta)^{-1}\right).$$

As in [8], $\tilde{T}_1$, $\tilde{T}_2$ and $\tilde{T}_3$ can be expanded as,

$$\tilde{T}_i = tr\, A_i^{(h)}Z^{(h)} + tr\, A_i^{(e)}Z^{(e)} + O_p\left(\frac{1}{\sqrt{k}}\right) \tag{3.8}$$

for $i = 1, 2, 3$, where $A_1^{(h)} = n(nI+\Theta)^{-1}$, $A_1^{(e)} = -(\frac{n}{n-1})I + n(nI+\Theta)^{-1}$, $A_2^{(h)} = I$, $A_2^{(e)} = -\frac{1}{n-1}(I+\Theta)$, $A_3^{(h)} = \frac{1}{n-1}(I+\frac{1}{n}\Theta)^{-1}\Big(nI - (I+\Theta)(I+\frac{1}{n}\Theta)^{-1}\Big)$ and $A_3^{(e)} = \frac{-1}{n-1}(I+\frac{1}{n}\Theta)^{-1}(I+\Theta)(I+\frac{1}{n}\Theta)^{-1}$. Finally, we get the following result.

Theorem 3.1. *$\tilde{T}_i$ is asymptotically normally distributed with mean 0 and variance σ_i^2 given by*

$$\sigma_i^2 = 2tr\left(A_i^{(h)}(I_p + \Theta)\right)^2 + 2(n-1)tr(A_i^{(e)})^2 + R_i + S_i,$$

where

$$R_i = vec(A_i)'[C \otimes \left(\Delta - (I_{p^2} + K_p) - vec(I)vec(I)'\right)]vec(A_i),$$

$$S_i = vec(A_i)'[D \otimes \left(\Delta_1 - (I_{p^2} + K_p)(\Theta \otimes \Theta) - vec(\Theta)vec(\Theta)'\right)]vec(A_i)$$

and

$$A_i = (A_i^{(h)}, A_i^{(e)}).$$

The theorem needs only the assumption of existence of fourth-order moment of the errors and the random effects. Note that $R_i = 0$ and $S_i = 0$ under normality. Hence, the terms R_i and S_i arise due to non-normality in the errors and random effect, respectively. Thus, the effects of non-normality in the errors and non-normality in the random effect are additive. For example, since $C \geqslant 0$ and $D \geqslant 0$, if $\left(\Delta - (I_{p^2} + K_p) - vec(I)vec(I)'\right) \geqslant 0$ and $\left(\Delta_1 - (I_{p^2} + K_p)(\Theta \otimes \Theta) - vec(\Theta)vec(\Theta)'\right) \leqslant 0$, the overall effect of non-normality will be reduced.

Under the null hypothesis ($\Theta = 0$), $A_1^{(h)} = A_2^{(h)} = A_3^{(h)} = I$ and $A_1^{(e)} = A_2^{(e)} = A_3^{(e)} = -\frac{1}{n-1}I$.

Corollary 3.2. *Under the null hypothesis* ($\Theta = 0$),

$$\sigma_i^2 = \frac{2np}{n-1} \quad for\ i = 1, 2, 3.$$

Corollary 3.2 tells us that the three test statistics are asymptotically null invariant to departure from normality. However, it is clear from the expression of Ω in Lemma 3.1 that they are, in general, not non-null robust. What is more, it is apparent from the above example that the lack of non-null robustness remains even when the departure is only towards Elliptically Contoured distributions. As was shown in [17], BNP criterion is locally best invariant (LBI) in the balanced one-way random effect model under normality. It is, therefore, clear that the stability of this optimality is less likely.

3.2. Two-way random effects model

Let $\mathbf{y}_{ijk}$ be a $p \times 1$ vector of observations following the two-way random effects model given by,

$$\mathbf{y}_{ijk} = \boldsymbol{\mu} + L\boldsymbol{a}_i + N\boldsymbol{b}_j + M\boldsymbol{d}_{ij} + \Sigma^{1/2}\boldsymbol{\varepsilon}_{ijk},$$

where $i = 1, \ldots, r$; $j = 1, \ldots, c$; $k = 1, \ldots, n$,; $\boldsymbol{\mu}$ and $\Sigma(> 0)$ are $p \times 1$ and $p \times p$, respectively, fixed unknown parameters, L, N, M are, respectively, $p \times s$, $p \times u$, $p \times t$ fixed unknown parameters of ranks s, u, t $(s, u, t \leqslant p)$, $\boldsymbol{a}_i$'s , $\boldsymbol{b}_j$'s, $\boldsymbol{d}_{ij}$ and $\boldsymbol{\varepsilon}_{ijk}$ are $s \times 1, u \times 1, t \times 1$ and $p \times 1$, respectively, random vectors which are mutually independent. Moreover assume that, for all $i = 1, \ldots, r$; $j = 1, \ldots, c$; and $k = 1, \ldots, n$, $\boldsymbol{a}_i$'s are identically and independently distributed with $E(\boldsymbol{a}_i) = 0$, $Var(\boldsymbol{a}_i) = I_s$ and finite $E(vec(\boldsymbol{a}_i\boldsymbol{a}_i')vec(\boldsymbol{a}_i\boldsymbol{a}_i')')$; $\boldsymbol{b}_j$'s are identically and independently distributed with $E(\boldsymbol{b}_j) = 0$, $Var(\boldsymbol{b}_j) = I_u$ and finite $E(vec(\boldsymbol{b}_i\boldsymbol{b}_i')vec(\boldsymbol{b}_i\boldsymbol{b}_i')')$; $\boldsymbol{d}_{ij}$'s are identically and independently distributed with $E(\boldsymbol{d}_{ij}) = 0$, $Var(\boldsymbol{d}_{ij}) = I_s$ and finite $E(vec(\boldsymbol{d}_{ij}\boldsymbol{d}_{ij}')vec(\boldsymbol{d}_{ij}\boldsymbol{d}_{ij}')')$; and $\boldsymbol{\varepsilon}_{ijk}$'s are identically and independently distributed with $E(\boldsymbol{\varepsilon}_{ijk}) = 0$, $Var(\boldsymbol{\varepsilon}_{ijk}) = I_p$ and finite fourth-order moment $\Delta := E(vec(\boldsymbol{\varepsilon}_{ijk}\boldsymbol{\varepsilon}_{ijk}')vec(\boldsymbol{\varepsilon}_{ijk}\boldsymbol{\varepsilon}_{ijk}')')$.

Often the hypotheses of interest are $\mathrm{H}_{01} : MM' = 0$, $\mathrm{H}_{02} : LL' = 0$ and $\mathrm{H}_{03} : NN' = 0$. We will be restricted to the case in which the levels of only one of the main effects go to infinity. Without loss of generality we consider the case r goes to infinity and, n and c are fixed. Since the hypothesis degrees of freedom for H_{03} stays fixed, we will not be concerned with it.

3.2.1. Asymptotic distribution of the sum of squares and products

The MANOVA sums of squares and products for the above hypotheses are,

$$S_h^{(a)} = nc \sum_{i=1}^{r} (\bar{\mathbf{y}}_{i..} - \bar{\mathbf{y}}_{...})(\bar{\mathbf{y}}_{i..} - \bar{\mathbf{y}}_{...})',$$

$$S_h^{(d)} = n \sum_{i=1}^{r} \sum_{j=1}^{c} (\bar{\mathbf{y}}_{ij.} - \bar{\mathbf{y}}_{i..} - \bar{\mathbf{y}}_{.j.} + \bar{\mathbf{y}}_{...})(\bar{\mathbf{y}}_{ij.} - \bar{\mathbf{y}}_{i..} - \bar{\mathbf{y}}_{.j.} + \bar{\mathbf{y}}_{...})'$$

and

$$S_e = \sum_{i=1}^{r} \sum_{j=1}^{c} \sum_{k=1}^{n} (\mathbf{y}_{ijk} - \bar{\mathbf{y}}_{ij.})(\mathbf{y}_{ijk} - \bar{\mathbf{y}}_{ij.})',$$

where $\bar{\mathbf{y}}_{ij.}, \bar{\mathbf{y}}_{i..}, \bar{\mathbf{y}}_{.j.}, \bar{\mathbf{y}}_{...}$ are defined in the obvious way.

The three multivariate test statistics corresponding to H_{02} are $T_1^{(a)} = -\log \frac{|S_h^{(d)}|}{|S_h^{(d)}+S_h^{(a)}|}$, $T_2^{(a)} = tr\, S_h^{(a)} S_h^{(d)^{-1}}$ and $T_3^{(a)} = tr\, S_h^{(a)}(S_h^{(d)} + S_h^{(a)})^{-1}$. It is easy to see that these three statistics are valid under normality. Hence, we may use them under non-normality. The test statistics corresponding to H_{03} are $T_1^{(d)} = -\log \frac{|S^{(e)}|}{|S_e+S_h^{(d)}|}$, $T_2^{(d)} = tr\, S_h^{(d)} S_e^{-1}$ and $T_3^{(d)} = tr\, S_h^{(d)}(S_e + S_h^{(d)})^{-1}$. They are also valid under normality and, thus, can be considered as candidates under non-normality. Another motivation for the viability of the three test statistics under non-normality is that they all are asymptotically null robust against departure from normality as will be shown later.

Let us denote,

$$U_i^{(a)} = nc(\bar{\boldsymbol{\varepsilon}}_{i..} + \Sigma^{-1/2} L\boldsymbol{a}_i + \Sigma^{-1/2} M\bar{\boldsymbol{d}}_{i.})(\bar{\boldsymbol{\varepsilon}}_{i..} + \Sigma^{-1/2} L\boldsymbol{a}_i + \Sigma^{-1/2} M\bar{\boldsymbol{d}}_{i.})',$$

$$
\begin{aligned}
U_i^{(d)} &= n\sum_{j=1}^{c}(\Sigma^{-1/2}M(\boldsymbol{d}_{ij}-\bar{\boldsymbol{d}}_{i.})+(\bar{\boldsymbol{\varepsilon}}_{ij.}-\bar{\boldsymbol{\varepsilon}}_{i..}))\\
&\quad\times(\Sigma^{-1/2}M(\boldsymbol{d}_{ij}-\bar{\boldsymbol{d}}_{i.})+(\bar{\boldsymbol{\varepsilon}}_{ij.}-\bar{\boldsymbol{\varepsilon}}_{i..}))'
\end{aligned}
$$

and

$$
U_i^{(e)}=\sum_{j}^{c}\sum_{k}^{n}(\boldsymbol{\varepsilon}_{ijk}-\bar{\boldsymbol{\varepsilon}}_{ij.})(\boldsymbol{\varepsilon}_{ijk}-\bar{\boldsymbol{\varepsilon}}_{ij.})',
$$

where $\bar{\boldsymbol{d}}_{i.}$, $\bar{\boldsymbol{d}}_{..}$,$\bar{\boldsymbol{\varepsilon}}_{ij.}$, $\bar{\boldsymbol{\varepsilon}}_{i..}$ are defined in the usual way. Put $U_i=(U_i^{(a)},U_i^{(d)},U_i^{(e)})$, $\Theta^{(a)}=nc\Sigma^{-1/2}LL'\Sigma^{-1/2}$ and $\Theta^{(d)}=n\Sigma^{-1/2}MM'\Sigma^{-1/2}$.

We will need the following lemma in deriving the asymptotic distribution of the test statistics.

Lemma 3.2. *The limiting distribution of* $\sqrt{r}(\bar{U}_r-E(\bar{U}_r))$ *is* $N(0,\Omega)$ *where*

$$
\begin{aligned}
\Omega = \Psi &+ \frac{1}{cn}H_1\otimes(\Delta-(I_{p^2}+K_p)-vec(I)vec(I)')\\
&+H_2\otimes\left(\Delta_1-(I_{p^2}+K_p)(\Theta^{(a)}\otimes\Theta^{(a)})-vec(\Theta^{(a)})vec(\Theta^{(a)})'\right)\\
&+\frac{1}{c}H_3\otimes(\Delta_2-(I_{p^2}+K_p)(\Theta^{(d)}\otimes\Theta^{(d)})-vec(\Theta^{(d)})vec(\Theta^{(d)})'),
\end{aligned}
$$

$\Psi=(\psi_{ij})$ *is a* 3×3 *block matrix with* $\Psi_{11}=(I+K_p)((I_p+\Theta^{(a)}+\Theta^{(d)})\otimes(I_p+\Theta^{(a)}+\Theta^{(d)}))$, $\Psi_{22}=(c-1)(I+K_p)((I_p+\Theta^{(d)})\otimes(I_p+\Theta^{(d)}))$, $\Psi_{33}=c(n-1)(I+K_p)$, $\Psi_{ij}=0$ *for* $i\neq j$,

$$
H_1=\begin{pmatrix}1 & c-1 & c(n-1)\\ c-1 & (c-1)^2 & c(c-1)(n-1)\\ c(n-1) & c(c-1)(n-1) & c^2(n-1)^2\end{pmatrix},
$$

$$
H_2=\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix},\quad H_3=\begin{pmatrix}1 & c-1 & 0\\ c-1 & (c-1)^2 & 0\\ 0&0&0\end{pmatrix},
$$

$$
\Delta_1=n^2c^2E(vec(\Sigma^{-1/2}L\boldsymbol{a}_i\boldsymbol{a}_i'L'\Sigma^{-1/2})vec(\Sigma^{-1/2}L\boldsymbol{a}_i\boldsymbol{a}_i'L'\Sigma^{-1/2})')
$$

and

$$
\Delta_2=n^2E(vec(\Sigma^{-1/2}M\boldsymbol{d}_{ij}\boldsymbol{d}_{ij}'M'\Sigma^{-1/2})vec(\Sigma^{-1/2}M\boldsymbol{d}_{ij}\boldsymbol{d}_{ij}'M'\Sigma^{-1/2})').
$$

Proof. Since U_i's are identically and independently distributed, asymptotic normality follows immediately. It remains to derive the expression for the asymptotic variance. For that purpose, it will be handy, as before, to express $U_i^{(a)}$, $U_i^{(d)}$ and $U_i^{(e)}$ in generalized quadratic forms as

$$
\begin{aligned}
U_i^{(a)} &= (\mathcal{E}_i+(\mathbf{1}_c\otimes\mathbf{1}_n)\boldsymbol{a}_i'L'\Sigma^{-1/2}+(I_c\otimes\mathbf{1}_n)D_iM'\Sigma^{-1/2})'\left[\frac{1}{cn}J_c\otimes J_n\right]\\
&\quad\times(\mathcal{E}_i+(\mathbf{1}_c\otimes\mathbf{1}_n)\boldsymbol{a}_i'L'\Sigma^{-1/2}+(I_c\otimes\mathbf{1}_n)D_iM'\Sigma^{-1/2})),
\end{aligned}
$$

$$U_i^{(d)} = (\mathcal{E}_i + (I_c \otimes \mathbf{1}_n)D_i M'\Sigma^{-1/2})' \left[\frac{1}{n}(I_c - \frac{1}{c}J_c) \otimes J_n\right] \times(\mathcal{E}_i + (I_c \otimes \mathbf{1}_n)D_i M'\Sigma^{-1/2})$$

and

$$U_i^{(e)} = \mathcal{E}_i' \left[I_c \otimes \left(I_n - \frac{1}{n}J_n\right)\right]\mathcal{E}_i,$$

where $\mathcal{E}_i' = (\boldsymbol{\varepsilon}_{i11}, \ldots, \boldsymbol{\varepsilon}_{i1n}, \ldots, \boldsymbol{\varepsilon}_{ic1}, \ldots, \boldsymbol{\varepsilon}_{icn})$ and $D_i' = (\boldsymbol{d}_{i1}, \ldots, \boldsymbol{d}_{ic})$.

Under normality, i.e. $\mathcal{E}_i \sim N(0, I_c \otimes I_n \otimes I_p)$, $\boldsymbol{a}_i \sim N(0, I_s)$ and $D_i \sim N(0, I_c \otimes I_p)$, it is easy to see that $U_i^{(a)} \sim W_p(1, I + \Theta^{(a)} + \Theta^{(d)})$, $U_i^{(d)} \sim W_p(c-1, I + \Theta^{(d)})$ and $U_i^{(e)} \sim W_p(c(n-1), I)$. Moreover, they are mutually independent. Therefore, applying Lemma 2.2,

$$Var_N(U_i^{(a)}) = (I + K_p)((I + \Theta^{(a)} + \Theta^{(d)}) \otimes (I + \Theta^{(a)} + \Theta^{(d)})),$$
$$Var_N(U_i^{(d)}) = (c-1)(I + K_p)((I + \Theta^{(d)}) \otimes (I + \Theta^{(d)}))$$

and

$$Var_N(U_i^{(d)}) = c(n-1)(I + K_p).$$

Now invoking Lemma 2.1 with $q = 3$, $s = 1$, $t = c$, $A_1 = A_2 = A_3 = 0$, $\mathcal{T} = \boldsymbol{a}_i L'\Sigma^{-1/2}$, $\mathcal{Z} = D_i M'\Sigma^{-1/2}$, $L_1 = \mathbf{1}_c \otimes \mathbf{1}_n$, $L_2 = L_3 = 0$, $M_1 = M_2 = I_c \otimes \mathbf{1}_n$, $M_3 = 0$, $B_1 = \frac{1}{cn}J_c \otimes J_n$, $B_2 = \frac{1}{n}(I_c - \frac{1}{c}J_c) \otimes J_n$ and $B_3 = I_c \otimes (I_n - \frac{1}{n}J_n)$, we have the desired result in the non-normal case. □

Note that $E(U_i^{(a)}) = I + \Theta^{(a)} + \Theta^{(d)}$, $E(U_i^{(d)}) = (c-1)(I + \Theta^{(d)})$ and $E(U_i^{(e)}) = c(n-1)I$. One can also see that

$$\frac{1}{r}\Sigma^{-1/2} S_h^{(a)} \Sigma^{-1/2} = I + \Theta^{(a)} + \Theta^{(d)} + \frac{1}{\sqrt{r}}Z^{(a)},$$
$$\frac{1}{r}\Sigma^{-1/2} S_h^{(d)} \Sigma^{-1/2} = (c-1)(I + \Theta^{(d)}) + \frac{1}{\sqrt{r}}Z^{(d)}$$

and

$$\frac{1}{r}\Sigma^{-1/2} S_e \Sigma^{-1/2} = c(n-1)I + \frac{1}{\sqrt{r}}Z^{(e)}, \tag{3.9}$$

where

$$Z^{(a)} = \sqrt{r}(\bar{U}_r^{(a)} - E(\bar{U}_r^{(a)})) - \sqrt{r}nc(\Sigma^{-1/2}L\bar{\boldsymbol{a}} + \Sigma^{-1/2}M\bar{\boldsymbol{d}}_{..} + \bar{\boldsymbol{\varepsilon}}_{...})(\Sigma^{-1/2}L\bar{\boldsymbol{a}} + \Sigma^{-1/2}M\bar{\boldsymbol{d}}_{..} + \bar{\boldsymbol{\varepsilon}}_{...})',$$

$$Z^{(d)} = \sqrt{r}(\bar{U}_r^{(d)} - E(\bar{U}_r^{(d)})) - \sqrt{r}n\sum\nolimits_j(\Sigma^{-1/2}M(\bar{\boldsymbol{d}}_{.j} - \bar{\boldsymbol{d}}_{..}) + (\bar{\boldsymbol{\varepsilon}}_{.j.} - \bar{\boldsymbol{\varepsilon}}_{...})) \times(\Sigma^{-1/2}M(\bar{\boldsymbol{d}}_{.j} - \bar{\boldsymbol{d}}_{..}) + (\bar{\boldsymbol{\varepsilon}}_{.j.} - \bar{\boldsymbol{\varepsilon}}_{...}))',$$

$Z^{(e)} = \sqrt{r}(\bar{U}_r^{(e)} - E(\bar{U}_r^{(e)}))$ and $\bar{\boldsymbol{\varepsilon}}_{.j.}, \bar{\boldsymbol{\varepsilon}}_{...}, \bar{\boldsymbol{d}}_{.j.}$ and $\bar{\boldsymbol{d}}_{..}$ are defined as usual.

Define $Z := (Z^{(a)}, Z^{(d)}, Z^{(e)})$. Then, we get the following Corollary as an immediate consequence of Lemma 3.2.

Corollary 3.3. *Z has asymptotic normal distribution with mean 0 and variance Ω.*

Proof. Clearly $Z = \sqrt{r}(\bar{U}_r - E(\bar{U}_r)) + o_p(1)$. Hence, the desired result follows from the Lemma. $\square$

3.2.2. Distribution of $T_i^{(d)}$ and $T_i^{(a)}$

For the test statistics concerning H_{01}, let us define

$$\tilde{T}_1^{(d)} := \sqrt{r}\Big((cn-1)T_1^{(d)} + (cn-1)\log\frac{|c(n-1)I|)}{|(cn-1)I + (c-1)\Theta^{(d)}|}\Big),$$
$$\tilde{T}_2^{(d)} := \sqrt{r}(c(n-1)T_2^{(d)} - (c-1)tr(I+\Theta^{(d)}))$$

and

$$\tilde{T}_3^{(d)} := \sqrt{r}\Big(\frac{(cn-1)^2}{c(n-1)}T_3^{(d)} - \frac{(cn-1)^2(c-1)}{c(n-1)}tr(I+\Theta^{(d)})((cn-1)I + (c-1)\Theta^{(d)})^{-1}\Big). \quad (3.10)$$

It can be seen that,

$$\tilde{T}_i^{(d)} = tr\, A_i^{(d)} Z^{(d)} + tr\, B_i^{(d)} Z^{(e)} + O_p\left(\frac{1}{\sqrt{r}}\right) \quad \text{for} \quad i = 1, 2, 3, \quad (3.11)$$

where $A_1^{(d)} = (I + \frac{(c-1)}{(cn-1)}\Theta^{(d)})^{-1}$, $B_1^{(d)} = (\frac{-(cn-1)}{c(n-1)}I + (I + \frac{(c-1)}{(cn-1)}\Theta^{(d)})^{-1})$, $A_2^{(d)} = I$, $B_2^{(d)} = \frac{-(c-1)}{c(n-1)}(I + \Theta^{(d)})$, $A_3^{(d)} = \frac{1}{c(n-1)}(I + \frac{(c-1)}{(cn-1)}\Theta^{(d)})^{-1}\Big[(cn-1)I - (c-1)(I+\Theta^{(d)})(I + \frac{(c-1)}{(cn-1)}\Theta^{(d)})^{-1}\Big]$, and $B_3^{(d)} = -\frac{(c-1)}{c(n-1)}(I + \frac{(c-1)}{(cn-1)}\Theta^{(d)})^{-1}(I+\Theta^{(d)})(I + \frac{(c-1)}{(cn-1)}\Theta^{(d)})^{-1}$.

We know that Z has asymptotic normal distribution. Therefore we have proved the following result.

Theorem 3.2. *$\tilde{T}_i^{(d)}$ has asymptotic normal distribution with mean 0 and variance*

$$\sigma_i^{2(d)} = 2(c-1)tr(A_i^{(d)}(I_p + \Theta^{(d)}))^2 + 2c(n-1)tr(B_i^{(d)})^2 + R_i + S_i,$$

where

$$R_i = vec(C_i^{(d)})'\left[\frac{1}{cn}H_1 \otimes (\Delta - (I_{p^2} + K_p) - vec(I)vec(I)')\right]vec(C_i^{(d)}),$$

$$S_i = vec(C_i^{(d)})'\Big[\frac{1}{c}H_3 \otimes (\Delta_2 - (I_{p^2} + K_p)(\Theta^{(d)} \otimes \Theta^{(d)}) - vec(\Theta^{(d)})vec(\Theta^{(d)})')\Big]vec(C_i^{(d)})$$

and

$$C_i^{(d)} = (0, A_i^{(d)}, B_i^{(d)}).$$

It does not appear that the three statistics are, in general, non-null robust. However, as shown in the following corollary, under the null hypothesis ($\Theta^{(d)} = 0$), $\sigma_i^{2(d)}$ does not depend on Δ, Δ_1 and Δ_2. This ensures null-robustness against departure from normality.

Corollary 3.4. *Let* $\Theta^{(d)} = 0$. *Then*

$$\sigma_i^{2(d)} = 2p(c-1)\Big(1 + \frac{(c-1)}{(cn-1)}\Big).$$

Similarly for the test concerning H_{02}, we define

$$\tilde{T}_1^{(a)} := \sqrt{r}\Big(cT_1^{(a)} + c\log\frac{|(c-1)(I+\Theta^{(d)})|}{|c(I+\Theta^{(d)}) + \Theta^{(a)}|}\Big),$$
$$\tilde{T}_2^{(a)} := \sqrt{r}\Big((c-1)T_2^{(a)} - (c-1)tr(I+\Theta^{(a)}+\Theta^{(d)})(I+\Theta^{(d)})^{-1}\Big)$$

and

$$\begin{aligned}\tilde{T}_3^{(a)} := \sqrt{r}\left(\frac{c^2}{c-1}T_3^{(a)} - \frac{c^2}{c-1}tr(I+\Theta^{(a)}+\Theta^{(d)})(\Theta^{(a)}\right.\\ \left. +c(I+\Theta^{(d)}))^{-1}\right).\end{aligned} \tag{3.12}$$

It can also be shown that

$$\tilde{T}_i^{(a)} = tr\,A_i^{(a)}Z^{(a)} + tr\,B_i^{(a)}Z^{(d)} + O_p\left(\frac{1}{\sqrt{r}}\right) \quad \text{for} \quad i = 1, 2, 3, \tag{3.13}$$

where $A_1^{(a)} = (I+\Theta^{(d)} + \frac{1}{c}\Theta^{(a)})^{-1}$, $B_1^{(a)} = (I+\Theta^{(d)} + \frac{1}{c}\Theta^{(a)})^{-1} - \frac{c}{c-1}(I+\Theta^{(d)})^{-1}$, $A_2^{(a)} = (I+\Theta^{(d)})^{-1}$, $B_2^{(a)} = \frac{-1}{c-1}(I+\Theta^{(d)})^{-1}(I+\Theta^{(a)}+\Theta^{(d)})(I+\Theta^{(d)})^{-1}$, $A_3^{(a)} = \frac{1}{c-1}(\frac{1}{c}\Theta^{(a)} + I + \Theta^{(d)})^{-1}[cI - (I+\Theta^{(a)}+\Theta^{(d)})(\frac{1}{c}\Theta^{(a)} + I + \Theta^{(d)})]^{-1}$, and $B_3^{(a)} = \frac{-1}{c-1}(\frac{1}{c}\Theta^{(a)} + I + \Theta^{(d)})^{-1}(I+\Theta^{(a)}+\Theta^{(d)})(\frac{1}{c}\Theta^{(a)} + I + \Theta^{(d)})^{-1})$.

Thus, we have established the following theorem.

Theorem 3.3. $\tilde{T}_i^{(a)}$ *has asymptotic normal distribution with mean* 0 *and variance*

$$\begin{aligned}\sigma_i^{2(a)} &= 2tr(A_i^{(a)}(I_p+\Theta^{(a)}+\Theta^{(d)}))^2 + 2(c-1)tr(B_i^{(a)}(I_p+\Theta^{(d)}))^2\\ &\quad +R_i + S_i + T_i\end{aligned}$$

where

$$R_i = vec(C_i^{(a)})'\left[\frac{1}{cn}H_1 \otimes (\Delta - (I_{p^2}+K_p) - vec(I)vec(I'))\right]vec(C_i^{(a)}),$$

$$S_i = vec(A_i^{(a)})'\big(\Delta_1 - (I_{p^2}+K_p)(\Theta^{(a)}\otimes\Theta^{(a)}) - vec(\Theta^{(a)})vec(\Theta^{(a)})'\big)vec(A_i^{(a)}),$$

$$\begin{aligned}T_i = vec(C_i^{(a)})'\Big[\frac{1}{c}H_3 \otimes (\Delta_2 - (I_{p^2}+K_p)(\Theta^{(d)}\otimes\Theta^{(d)})\\ -vec(\Theta^{(d)})vec(\Theta^{(d)})')\Big]vec(C_i^{(a)})\end{aligned}$$

and

$$C_i^{(a)} = (A_i^{(a)}, B_i^{(a)}, 0).$$

Asymptotic null-robustness follows from the following Corollary.

Corollary 3.5. *When* $\Theta^{(a)} = \Theta^{(d)} = 0$, *we get*

$$\sigma_i^{2(a)} = 2p\left(1 + \frac{1}{c-1}\right)$$

for $i = 1, 2, 3$.

3.3. Two-way mixed effects model

The two-way mixed effects model can be expressed as

$$\mathbf{y}_{ijk} = \boldsymbol{\mu} + \boldsymbol{\alpha}_i + L\boldsymbol{a}_j + M\boldsymbol{d}_{ij} + \Sigma^{-1/2}\boldsymbol{\varepsilon}_{ijk},$$

where $i = 1, \ldots, r$; $j = 1, \ldots, c$; $k = 1, \ldots, n$, $\boldsymbol{\mu}$, M, L, Σ, $\boldsymbol{a}_j$, $\boldsymbol{d}_{ij}$, $\boldsymbol{\varepsilon}_{ijk}$ are as defined in the previous sections and $\boldsymbol{\alpha}_i$ are fixed effects with $\sum_{i=1}^r \boldsymbol{\alpha}_i = 0$.

We consider the unrestricted version of the two-way mixed effects model. That is, we do not assume $\sum_{i=1}^r \boldsymbol{d}_{ij} = 0$. One may be interested in testing $\mathrm{H}_{01} : \boldsymbol{\alpha}_i = 0; i = 1, \ldots, r$, $\mathrm{H}_{02} : LL' = 0$ and $\mathrm{H}_{03} : MM' = 0$. In the asymptotic frame work of this paper two cases can be considered.

The first one is the case when $c \to \infty$ but r and n remain fixed. In this case the testing problem H_{01} will not be of interest to us since its hypothesis degrees of freedom stays fixed. Moreover, we note that the test statistics used for the main random effect and the interaction effects are exactly the same as those for two-way random effects model. Consequently, the results given in Section 3.2 will apply directly to this case.

In the second case, we let $r \to \infty$ [c and n fixed]. In this case, we will not be interested in the hypothesis H_{02}. In the rest of this section we will provide some details for testing H_{01} and H_{03}.

3.3.1. Asymptotic distribution of the sum of squares

Except changing the notation $S_h^{(a)}$ to $S_h^{(\alpha)}$, the sum of squares and products are identical to those given in Section 3.2. Here also let us denote

$$U_i^{(\alpha)} = nc(\bar{\boldsymbol{\varepsilon}}_{i..} + \Sigma^{-1/2}\boldsymbol{\alpha}_i + \Sigma^{-1/2}M\bar{\boldsymbol{d}}_{i.})(\bar{\boldsymbol{\varepsilon}}_{i..} + \Sigma^{-1/2}\boldsymbol{\alpha}_i + \Sigma^{-1/2}M\bar{\boldsymbol{d}}_{i.})',$$

$$\begin{aligned} U_i^{(d)} = n\sum_{j=1}^{c} &(\Sigma^{-1/2}M(\boldsymbol{d}_{ij} - \bar{\boldsymbol{d}}_{i.}) + (\bar{\boldsymbol{\varepsilon}}_{ij.} - \bar{\boldsymbol{\varepsilon}}_{i..})) \\ &\times(\Sigma^{-1/2}M(\boldsymbol{d}_{ij} - \bar{\boldsymbol{d}}_{i.}) + (\bar{\boldsymbol{\varepsilon}}_{ij.} - \bar{\boldsymbol{\varepsilon}}_{i..}))' \end{aligned}$$

and

$$U_i^{(e)} = \sum_{j=1}^{c}\sum_{k=1}^{n}(\boldsymbol{\varepsilon}_{ijk} - \bar{\boldsymbol{\varepsilon}}_{ij.})(\boldsymbol{\varepsilon}_{ijk} - \bar{\boldsymbol{\varepsilon}}_{ij.})',$$

where $\bar{\boldsymbol{d}}_{i.}$, $\bar{\boldsymbol{\varepsilon}}_{i..}$ and $\bar{\boldsymbol{\varepsilon}}_{ij.}$ are defined as before.

Let us put $U_i = (U_i^{(\alpha)}, U_i^{(d)}, U_i^{(e)})$, $\Theta^{(d)} = n\Sigma^{-1/2}MM'\Sigma^{-1/2}$ and $\Theta_i^{(\alpha)} = nc\Sigma^{-1/2}\boldsymbol{\alpha}_i\boldsymbol{\alpha}_i'\Sigma^{-1/2}$. Let $\bar{\Theta}_r^{(\alpha)} = \frac{1}{r}\sum_{i=1}^{r}\Theta_i^{(\alpha)}$. We are, now, ready to prove the following.

Lemma 3.3. *Assume there exists a fixed matrix $\Theta^{(\alpha)}$ such that $\sqrt{r}(\bar{\Theta}_r^{(\alpha)} - \Theta^{(\alpha)}) \to 0$ as $r \to \infty$. Suppose for some $\delta, \beta > 0$, $E|\varepsilon_{111a}\varepsilon_{111b}|^{2+\delta} < \infty$ and $E|d_{11e}d_{11f}|^{2+\beta} < \infty$ for $1 \leqslant a, b \leqslant p$ and $1 \leqslant e, f \leqslant t$ where ε_{111a} is the ath entry of $\boldsymbol{\varepsilon}_{111}$ and d_{11e} is the eth entry of $\boldsymbol{d}_{11}$. Suppose also that the third-order moments of $\boldsymbol{\varepsilon}_{111}$ and $\boldsymbol{d}_{11}$ are zero. Then the limiting distribution of $\sqrt{r}\big(\bar{U}_r - E(\bar{U}_r)\big)$ is $N(0, \Omega)$ and Ω is given by*

$$\begin{aligned}\Omega = \Psi &+ \frac{1}{cn}H_1 \otimes \big(\Delta - (I_{p^2} + K_p + vec(I)vec(I)')\big)\\ &+ \frac{1}{c}H_3 \otimes \big(\Delta_2 - (I_{p^2} + K_p)(\Theta^{(d)} \otimes \Theta^{(d)}) - vec(\Theta^{(d)})vec(\Theta^{(d)})')\big)\end{aligned}$$

where $\Psi = (\psi_{ij})$ is a 3×3 block matrix with $\psi_{11} = (I + K_p)((I + \Theta^{(d)}) \otimes (I + \Theta^{(d)}) + (I + \Theta^{(d)}) \otimes \Theta^{(\alpha)} + \Theta^{(\alpha)} \otimes (I + \Theta^{(d)}))$, $\psi_{22} = (c-1)(I + K_p)((I_p + \Theta^{(d)}) \otimes (I_p + \Theta^{(d)}))$, $\psi_{33} = c(n-1)(I + K_p)$, $\psi_{ij} = 0$ for $i \neq j$, H_1, H_3, Δ and Δ_2 are as defined in Section 3.2.

Proof. It should be noted that unlike the previous cases U_i's are not identically distributed. We will, however, appeal to the Lindeberg Feller Version of the central Limit Theorem, as in [8], to establish asymptotic normality.

To find the expression for the asymptotic variance, we start out by expressing $U_i^{(\alpha)}$, $U_i^{(d)}$ and $U_i^{(e)}$ as quadratic forms as before. That is,

$$\begin{aligned}U_i^{(\alpha)} &= \Big(\mathcal{E}_i + (\mathbf{1}_c \otimes \mathbf{1}_n)\boldsymbol{\alpha}_i'\Sigma^{-1/2} + (I_c \otimes \mathbf{1}_n)D_iM'\Sigma^{-1/2}\Big)'\Big[\frac{1}{nc}J_c \otimes J_n\Big]\\ &\quad\times\Big(\mathcal{E}_i + (\mathbf{1}_c \otimes \mathbf{1}_n)\boldsymbol{\alpha}_i'\Sigma^{-1/2} + (I_c \otimes \mathbf{1}_n)D_iM'\Sigma^{-1/2}\Big),\end{aligned}$$

$$\begin{aligned}U_i^{(d)} &= \Big(\mathcal{E}_i + (I_c \otimes \mathbf{1}_n)D_iM'\Sigma^{-1/2}\Big)'\Big[\frac{1}{n}\left(I_c - \frac{1}{c}J_c\right) \otimes J_n\Big]\\ &\quad\times\Big(\mathcal{E}_i + (I_c \otimes \mathbf{1}_n)D_iM'\Sigma^{-1/2}\Big)\end{aligned}$$

and

$$U_i^{(e)} = \mathcal{E}_i'\left[I_c \otimes \left(I_n - \frac{1}{n}J_n\right)\right]\mathcal{E}_i,$$

where $\mathcal{E}_i$ and D_i are as defined in the previous section.

For the normal model, i.e. $\mathcal{E}_i \sim N(0, I_c \otimes I_n \otimes I_p)$, $\boldsymbol{a}_i \sim N(0, I_s)$ and $D_i \sim N(0, I_c \otimes I_p)$, we see that $U_i^{(\alpha)} \sim W_p(1, I + \Theta^{(d)}, \Theta_i^{(\alpha)})$, $U_i^{(d)} \sim W_p(c-1, I + \Theta^{(d)})$ and $U_i^{(e)} \sim W_p(c(n-1), I)$. Moreover, $U_i^{(\alpha)}$, $U_i^{(d)}$ and $U_i^{(e)}$ are mutually independent.

Now applying Lemma 2.2, we get,

$$\begin{aligned} Var_N(U_i^{(\alpha)}) = &(I + K_p)((I + \Theta^{(d)}) \otimes (I + \Theta^{(d)}) \\ &+ (I + \Theta^{(d)}) \otimes \Theta_i^{(\alpha)} + \Theta_i^{(\alpha)} \otimes (I + \Theta^{(d)})), \end{aligned}$$

$$Var_N(U_i^{(d)}) = (c-1)(I + K_p)((I_p + \Theta^{(d)}) \otimes (I_p + \Theta^{(d)}))$$

and

$$Var_N(U_i^{(e)}) = c(n-1)(I + K_p).$$

Hence, in the non-normal case

$$\lim_{k\to\infty} \frac{1}{k} \sum_{i=1}^{k} Var(U_i) = \Omega. \qquad \square$$

It can easily be seen that $E(U_i^{(\alpha)}) = I + \Theta^{(d)} + \Theta_i^{(\alpha)}$, $E(U_i^{(d)}) = (c-1)(I + \Theta^{(d)})$ and $E(U_i^{(e)}) = c(n-1)I$. Then, as in the previous sections, one can show that

$$\begin{aligned} &\frac{1}{r}\Sigma^{-1/2} S_h^{(\alpha)} \Sigma^{-1/2} = I + \Theta^{(\alpha)} + \frac{1}{\sqrt{r}} Z^{(\alpha)}, \\ &\frac{1}{r}\Sigma^{(-1/2)} S_h^{(d)} \Sigma^{-1/2} = (c-1)(I + \Theta^{(d)}) + \frac{1}{\sqrt{r}} Z^{(d)} \end{aligned}$$

and

$$\frac{1}{r}\Sigma^{-1/2} S_e \Sigma^{-1/2} = c(n-1)I + \frac{1}{\sqrt{r}} Z^{(e)}, \tag{3.14}$$

where

$$\begin{aligned} Z^{(\alpha)} = &\sqrt{r}(\bar{U}_r^{(\alpha)} - E(\bar{U}_r^{(\alpha)})) + \sqrt{r}(\bar{\Theta}_r^{(\alpha)} - \Theta^{(\alpha)}) \\ &- \sqrt{r}nc(\Sigma^{-1/2} L\bar{\boldsymbol{\alpha}}_r + \Sigma^{-1/2} M\bar{\boldsymbol{d}}_{..} + \bar{\boldsymbol{\varepsilon}}_{...})(\Sigma^{-1/2} L\bar{\boldsymbol{\alpha}}_r + \Sigma^{-1/2} M\bar{\boldsymbol{d}}_{..} + \bar{\boldsymbol{\varepsilon}}_{...})', \end{aligned}$$

$$\begin{aligned} Z^{(d)} = &\sqrt{r}(\bar{U}_r^{(d)} - E(\bar{U}_r^{(d)})) - \sqrt{r}n \sum_{j=1}^{c} (\Sigma^{-1/2} M(\bar{\boldsymbol{d}}_{.j} - \bar{\boldsymbol{d}}_{..}) + (\bar{\boldsymbol{\varepsilon}}_{.j.} - \bar{\boldsymbol{\varepsilon}}_{...})) \\ &\times (\Sigma^{-1/2} M(\bar{\boldsymbol{d}}_{.j} - \bar{\boldsymbol{d}}_{..}) + (\bar{\boldsymbol{\varepsilon}}_{.j.} - \bar{\boldsymbol{\varepsilon}}_{...}))' \end{aligned}$$

and

$$Z^{(e)} = \sqrt{r}(\bar{U}_r^{(e)} - E(\bar{U}_r^{(e)}).$$

Denote $Z = (Z^{(\alpha)}, Z^{(d)}, Z^{(e)})$. Since $Z = \sqrt{r}\big(\bar{U}_r - E(\bar{U}_r)\big) + o_p(1)$, we obtain the following corollary.

Corollary 3.6. *Under the assumptions of Lemma 3.3, Z is asymptotically normally distributed with mean* 0 *and variance* Ω.

Let us define $\tilde{T}_i^{(\alpha)}$ and $\tilde{T}_i^{(d)}$, $i = 1, 2, 3$, analogous to (3.10) and (3.12) . By comparing (3.9) and (3.14), it is not hard to see that,

$$\tilde{T}_i^{(\alpha)} = tr\, A_i^{(\alpha)} Z^{(\alpha)} + tr\, B_i^{(\alpha)} Z^{(d)} + O_p\left(\frac{1}{\sqrt{r}}\right)$$

and

$$\tilde{T}_i^{(d)} = tr\, A_i^{(d)} Z^{(d)} + tr\, B_i^{(d)} Z^{(e)} + O_p\left(\frac{1}{\sqrt{r}}\right) \quad \text{for} \quad i = 1, 2, 3,$$

where the coefficient matrices $A_i^{(.)}$ and $B_i^{(.)}$ are as given in (3.11) and (3.13) except that we need to replace every occurrence of a by α.

Hence, we have proved the following Theorem.

Theorem 3.4. *Under the assumptions of Lemma 3.3, $\tilde{T}_i^{(d)}$ and $\tilde{T}_i^{(\alpha)}$ will each have asymptotic normal distribution with means* 0 *and variances, respectively,*

$$\sigma_i^{2(d)} = 2tr(A_i^{(d)}(I_p + \Theta^{(d)}))^2 + 2c(n-1)tr(B_i^{(d)})^2 + R_i^{(d)} + T_i^{(d)}$$

$$\begin{aligned}\sigma_i^{2(\alpha)} &= 2tr(A_i^{(a)}(I + \Theta^{(d)}))^2 + 2c(n-1)tr(B_i^{(a)})^2 \\ &\quad +4tr(A_i^{(\alpha)}(I + \Theta^{(d)})A_i^{(\alpha)}\Theta^{(\alpha)}) + R_i^{(\alpha)} + S_i^{(\alpha)} + T_i^{(\alpha)},\end{aligned}$$

where

$$R_i^{(d)} = vec(C_i^{(d)})' \left[\frac{1}{cn} H_1 \otimes (\Delta - (I_{p^2} + K_p) - vec(I)vec(I')\right] vec(C_i^{(d)}),$$

$$\begin{aligned}T_i^{(d)} &= vec(C_i^{(d)})' \left[\frac{1}{c} H_3 \otimes (\Delta_2 - (I_{p^2} + K_p)(\Theta^{(d)} \otimes \Theta^{(d)}) \right. \\ &\quad \left. -vec(\Theta^{(d)})vec(\Theta^{(d)})')\right] vec(C_i^{(d)}),\end{aligned}$$

$$R_i^{(\alpha)} = vec(C_i^{(\alpha)})' \left[\frac{1}{cn} H_1 \otimes (\Delta - (I_{p^2} + K_p) - vec(I)vec(I')\right] vec(C_i^{(\alpha)}),$$

$$S_i^{(\alpha)} = vec(A_i^{(\alpha)})'\big(\Delta_1 - (I_{p^2} + K_p)(\Theta^{(\alpha)} \otimes \Theta^{(\alpha)}) - vec(\Theta^{(\alpha)})vec(\Theta^{(\alpha)})'\big)vec(A_i^{(\alpha)}),$$

$$T_i^{(\alpha)} = vec(C_i^{(\alpha)})' \left[\frac{1}{c} H_3 \otimes (\Delta_2 - (I_{p^2} + K_p)(\Theta^{(d)} \otimes \Theta^{(d)}), \right.$$
$$\left. - vec(\Theta^{(d)}) vec(\Theta^{(d)})') \right] vec(C_i^{(\alpha)}),$$

where $C_i^{(d)} = (0, A_i^{(d)}, B_i^{(d)})$ *and* $C_i^{(\alpha)} = (A_i^{(\alpha)}, 0, B_i^{(\alpha)})$.

The assumptions of zero third-order moments for $\boldsymbol{\varepsilon}_{111}$ and $\boldsymbol{d}_{11}$ appear strong. However, these assumptions are crucial for the computation of the variances. They are very helpful in simplifying the expression for the variances as given in Lemma 3.3 and Theorem 3.4. Moreover, these assumptions are not needed for the null distributions.

It may be noted that U_i's are identically distributed if $\Theta^{(d)} = 0$ and $\Theta^{(\alpha)} = 0$. Consequently, we can drop the conditions of the theorem and get the following result.

Corollary 3.7. *Suppose* $\Theta^{(d)} = 0$ *and* $\Theta_j^{(\alpha)} = 0$, $j = 1, \ldots, r$. *Then,*

$$\sigma_i^{2(d)} = 2p(c-1)\left(1 + \frac{(c-1)}{(cn-1)}\right)$$

and

$$\sigma_i^{2(\alpha)} = 2p\left(1 + \frac{1}{c-1}\right)$$

for $i = 1, 2, 3$.

Clearly, null robustness is exhibited by the three statistics.

3.4. The general balanced mixed model

Now, we return to the case where there are arbitrarily many effects. We use more involved notations adopted from [10]. In the general balance MANOVA model the $p \times 1$ vector of observations can be expressed as,

$$\boldsymbol{y}_\vartheta = \sum_{j=0}^{v+1} L_j \boldsymbol{g}_{\vartheta_j}^{(j)},$$

where $\vartheta = \{k_1, k_2, \ldots, k_s\}$ is a complete set of subscripts with $k_i = 1, \ldots, a_i$, ϑ_j is the set of subscripts of the jth effect $\boldsymbol{g}_{\vartheta_j}^{(j)}$. For $0 \leqslant j \leqslant t$, $\boldsymbol{g}_{\vartheta_j}^{(j)}$ are fixed effects and $L_j = I_p$. For $t+1 \leqslant j \leqslant v$, $\boldsymbol{g}_{\vartheta_j}^{(j)}$ are random effects and L_j is a $p \times r_j$ matrix of rank $r_j \leqslant p$. And $\boldsymbol{g}_{\vartheta_{v+1}}^{(v+1)}$ is the error term and $L_{v+1} = \Sigma^{-1/2}$ is $p \times p$ positive definite matrix. Note that since ϑ_0 is the set of subscripts for the grand mean effect, it is empty set, and $\vartheta_{v+1} = \vartheta$ since it is the set subscripts for the error term. Due to the balanced property of the model, the total sample size equals $\prod_{i=1}^{s} a_i$. We assume that $\boldsymbol{g}_{\vartheta_j}^{(j)}$ are mutually independent for all possible values of ϑ_j and $t+1 \leqslant j \leqslant v+1$.

The design matrices X_i and A_i of (1.1) can be expressed as Kronecker products of an identity matrices and vectors of ones of appropriate dimensions. More precisely, (1.1) can be written as,

$$Y = \sum_{i=1}^{v+1} H_i \mathcal{B}_i,$$

where $\mathcal{B}_i$ is a matrix whose columns are $\boldsymbol{g}_{\vartheta_i}^{(i)}$, $H_i = \otimes_{j=1}^{s} C_{ij}$ and $C_{ij} = \begin{cases} I_{a_j} & \text{if } k_j \in \vartheta_i, \\ \mathbf{1}_{a_j} & \text{if } k_j \notin \vartheta_i. \end{cases}$

Let $Y'P_jY$ be the sum of squares associated with the *j*th effect. It is known that P_j is symmetric and idempotent. The following lemma is proved in [10].

Lemma 3.4. *Let* $A_j = H_jH_j'$ *and* $b_j = \prod_{k_l \notin \vartheta_j} a_l$. *Then*

$$P_j = \sum_{i=0}^{v+1} \frac{\lambda_{ji}}{b_i} A_i,$$

where $\lambda_{ji} \in \{-1, 0, 1\}$.

We want to study the distribution of the three multivariate statistics when the number of levels of one of the main effects goes to infinity. With out loss of generality, we may consider the case when $a_1 \to \infty$ and $a_2, \ldots, a_s$ are fixed. Let us define $\psi := \{j : 0 \leqslant j \leqslant v+1 \text{ and } k_1 \in \vartheta_j\}$. We will concern ourselves with testing the significance of the *j*th effect for $j \in \psi$ because those are the only effects for which the hypothesis degrees of freedom goes to infinity as $a_1 \to \infty$. We can write,

$$P_j = \sum_{\psi} \frac{\lambda_{ji}}{b_i} A_i + \sum_{\psi^c} \frac{\lambda_{ji}}{b_i} A_i. \tag{3.15}$$

Then by virtue of the above lemma and the structure of A_i for $k_1 \in \vartheta_i$,

$$Y'P_jY = \sum_{k_1=1}^{a_1} Y_{k_1}' \Big[\sum_{\psi} \frac{\lambda_{ji}}{b_i} A_i^* \Big] Y_{k_1} + Y' \left[\sum_{\psi^c} \frac{\lambda_{ji}}{b_i} A_i \right] Y, \tag{3.16}$$

where $Y_{k_1} = (\boldsymbol{y}_{k_1 1 1 \cdots 1}, \ldots, \boldsymbol{y}_{k_1 a_2 a_3 \cdots a_s})'$, $A_j^* = H_j^* H_j^{*\prime}$, $H_j^* = \otimes_{j=2}^{s} C_{ij}$ and C_{ij} is as defined before.

For $j \in \psi$, suppose we are interested in testing the significance of the *j*th effect. Let $S^{(j)}$ be the sum of squares and products associated with the *j*th effect. Suppose there exists a random effect, say *l*th effect, whose associated mean square has the same expectation as that of *j*th effect under the null hypothesis. Needless to say $l \in \psi$. Let $S^{(l)}$ denote the sum of square associated with that random effect. Define

$$U_{k_1}^{(j)} = Y_{k_1}' \Big[\sum_{\psi} \frac{\lambda_{ji}}{b_i} A_i^* \Big] Y_{k_1} \quad \text{and} \quad U_{k_1}^{(l)} = Y_{k_1}' \Big[\sum_{\psi} \frac{\lambda_{li}}{b_i} A_i^* \Big] Y_{k_1}. \tag{3.17}$$

Let us assume that there exist constant matrices $\Theta^{(j)}$ and $\Theta^{(l)}$ such that $\sqrt{a}_1(E(\bar{U}_{a_1}^{(j)}) - (I + \Theta^{(j)})) \to 0$ and $\sqrt{a}_1(E(\bar{U}_{a_1}^{(l)}) - (I + \Theta^{(l)})) \to 0$. It can easily be seen that

$$\frac{1}{a_1}\Sigma^{-1/2}S^{(.)}\Sigma^{-1/2} = I + \Theta^{(.)} + \frac{1}{\sqrt{a_1}}Z^{(.)}, \tag{3.18}$$

where

$$Z^{(.)} = \sqrt{a}_1(\bar{U}_{a_1}^{(.)} - E(\bar{U}_{a_1}^{(.)})) + \sqrt{a}_1(E(\bar{U}_{a_1}^{(.)}) - \Theta^{(.)}) + \frac{1}{\sqrt{a}_1}Y'\Big[\sum_{\psi^c}\frac{\lambda_{.i}}{b_i}A_i\Big]Y. \tag{3.19}$$

Put $U_{k_1} = (U_{k_1}^{(j)}, U_{k_1}^{(j)})$, $Z = (Z^{(j)}, Z^{(l)})$ and $\Theta = (\Theta^{(j)}, \Theta^{(j)})$. Noting the fact that $\frac{1}{\sqrt{a}_1}Y'\Big[\sum_{\psi^c}\frac{\lambda_{.i}}{b_i}A_i\Big]Y \xrightarrow{p} 0$, it is clear that Z and $\sqrt{a}_1(\bar{U}_{a_1} - E(\bar{U}_{a_1}))$ have the same asymptotic distribution. The remaining part of the work is pretty similar to Sections 3.1–3.3.

4. Unbalanced mixed model

We consider the one-way random effect model to show the results for balanced mixed model given in Section 3 can be extended to the unbalanced case. As it turns out we will need some restrictions on the sample sizes to derive our results. It will not be hard to see that the conditions get more stronger as the number of factors in the model increases.

Consider the unbalanced model given by

$$\mathbf{y}_{ij} = \boldsymbol{\mu} + L\boldsymbol{a}_i + \Sigma^{1/2}\boldsymbol{\varepsilon}_{ij}, \quad j = 1, \ldots, n_i, \quad i = 1, \ldots, k, \tag{4.1}$$

where $\boldsymbol{\varepsilon}_{ij}$, $\boldsymbol{a}_i$, L and Σ are as defined in Section 3.1 with all the assumptions there.

Suppose we want to test $H_0 : LL' = 0$ versus H_1: not H_0.

4.1. Asymptotic distribution of the sum of squares and products

The hypothesis and error sum of squares and products are

$$S_h = \sum_{i=1}^{k} n_i(\bar{\mathbf{y}}_{i.} - \bar{\mathbf{y}}_{..})(\bar{\mathbf{y}}_{i.} - \bar{\mathbf{y}}_{..})'$$

and

$$S_e = \sum_{i=1}^{k}\sum_{j=1}^{n_i}(\mathbf{y}_{ij} - \bar{\mathbf{y}}_{i.})(\mathbf{y}_{ij} - \bar{\mathbf{y}}_{i.})',$$

where $\bar{\mathbf{y}}_{i.} = \frac{1}{n_i}\sum_{j=1}^{k}\mathbf{y}_{ij}$ and $\bar{\mathbf{y}}_{..}$ is as defined before.

Let $U_i^{(h)}$, $U_i^{(e)}$ and U_i be defined as follows:

$$U_i^{(h)} := n_i(\Sigma^{-1/2}L\boldsymbol{a}_i + \bar{\boldsymbol{\varepsilon}}_{i.})(\Sigma^{-1/2}L\boldsymbol{a}_i + \bar{\boldsymbol{\varepsilon}}_{i.})',$$
$$U_i^{(e)} := \sum_{j=1}^{n}(\boldsymbol{\varepsilon}_{ij} - \bar{\boldsymbol{\varepsilon}}_{i.})(\boldsymbol{\varepsilon}_{ij} - \bar{\boldsymbol{\varepsilon}}_{i.})'$$

and

$$U_i := (U_i^{(h)}, U_i^{(h)}),$$

where $\bar{\boldsymbol{\varepsilon}}_{i.}$ and $\bar{U}_k$ are defined in the obvious way.

Let $\bar{n}_k = \sum_{i=1}^{k} \frac{n_i}{k}$, $\bar{\bar{n}}_k = \sum_{i=1}^{k} \frac{n_i^2}{k}$ and $\underline{n}_k = \sum_{i=0}^{k} \frac{1}{kn_i}$. Put $\Theta_i = n_i\Sigma^{-1/2}LL'\Sigma^{-1/2}$ and $\bar{\Theta}_k = \bar{n}_k\Sigma^{-1/2}LL'\Sigma^{-1/2}$. We can prove the following result in a similar way as Lemma 3.1.

Lemma 4.1. *Suppose there exist real numbers $\bar{n}$, $\bar{\bar{n}}$ and $\underline{n}$ such that $\sqrt{k}(\bar{n}_k - \bar{n}) \to 0$, $\sqrt{k}(\bar{\bar{n}}_k - \bar{\bar{n}}) \to 0$ and $\sqrt{k}(\underline{n}_k - \underline{n}) \to 0$ as $k \to \infty$. Suppose also that for some $\delta, \beta > 0$, $E|\varepsilon_{11s}\varepsilon_{11t}|^{2+\delta} < \infty$ and $E|a_{1l}a_{1m}|^{2+\beta} < \infty$ for $1 \leqslant s, t \leqslant p$ and $1 \leqslant l, m \leqslant r$ where ε_{11s} is the sth entry of $\boldsymbol{\varepsilon}_{11}$ and a_{1l} is the lth entry of $\boldsymbol{a}_1$. Then, the limiting distribution of $\sqrt{k}(\bar{U}_k - E(\bar{U}_k))$ is $N(0, \Omega)$ and Ω is given by,*

$$\Omega = \Psi + C \otimes (\varDelta - I_{p^2} - (K_p + vec(I)vec(I)'))$$
$$+ \frac{\bar{\bar{n}}}{\bar{n}^2} H \otimes (\varDelta_1 - (I_{p^2} + K_p)(\Theta \otimes \Theta) - vec(\Theta)vec(\Theta)')$$

where

$$\varDelta_1 = \bar{n}^2 E(vec(\Sigma^{-1/2}L\boldsymbol{a}_i\boldsymbol{a}_i'L'\Sigma^{-1/2})vec(\Sigma^{-1/2}L\boldsymbol{a}_i\boldsymbol{a}_i'L'\Sigma^{-1/2})'),$$

$\Theta = \bar{n}\Sigma^{-1/2}LL'\Sigma^{-1/2}$, $\Psi = (\Psi_{ij})$ is a 2×2 block matrix with $\Psi_{11} = (I + K_p)(I_p + \Theta \otimes I + I \otimes \Theta + \frac{\bar{\bar{n}}}{\bar{n}^2}\Theta \otimes \Theta)$, $\Psi_{22} = (\bar{n} - 1)(I_{p^2} + K_p)$, $\Psi_{21}' = \Psi_{12} = 0$, $C = (c_{ij})$ is a 2×2 matrix with $c_{11} = \underline{n}$, $c_{21} = c_{12} = (1 - \underline{n})$ and $c_{22} = (\bar{n} + \underline{n} - 2)$, $H = (h_{ij})$ is a 2×2 matrix with $h_{11} = 1$ and $h_{12} = h_{21} = h_{22} = 0$.

Proof. Normality follows by a similar type of Lindeberg–Feller argument as in [8]. The limiting variance can be derived as in the proof of Lemma 3.1 but in this case,

$$\Omega = \lim_{k \to \infty} \frac{1}{k} \sum_{i=1}^{k} Var(U_i). \qquad \square$$

We note that $E(\bar{U}_k^{(h)}) = I + \bar{\Theta}_k$ and $E(\bar{U}_k^{(e)}) = (\bar{n}_k - 1)I$, it can be verified that

$$\frac{1}{k}\Sigma^{-1/2}S_h\Sigma^{-1/2} = I + \Theta + \frac{1}{\sqrt{k}}Z^{(h)}$$

and

$$\frac{1}{k}\Sigma^{-1/2}S_e\Sigma^{-1/2} = (\bar{n}-1)I + \frac{1}{\sqrt{k}}Z^{(e)},$$

where $Z^{(h)} = \sqrt{k}\big(\bar{U}_k^{(h)} - E(\bar{U}_k^{(h)})\big) + \sqrt{k}(\bar{\Theta}_k - \Theta) - \sqrt{k}\bar{n}_k(\bar{\boldsymbol{\varepsilon}}_{..} + \Sigma^{-1/2}\bar{\boldsymbol{a}}_k)(\bar{\boldsymbol{\varepsilon}}_{..} + \Sigma^{-1/2}\bar{\boldsymbol{a}}_k)'$, $Z^{(e)} = \sqrt{k}\big(\bar{U}^{(e)} - E(\bar{U}^{(e)})\big) + \sqrt{(\bar{n}_k - \bar{n})}$ and $\bar{\boldsymbol{a}}_k = \frac{1}{k}\sum_{i=1}^{k}\boldsymbol{a}_i$.

Putting $Z = (Z^{(h)}, Z^{(e)})$ we can get the following as a consequence of Lemma 4.1.

Corollary 4.1. *Under the assumptions of Lemma* 4.1, *Z is asymptotically normally distributed with mean* 0 *and variance Ω.*

Proof. It can easily be seen that

$$\sqrt{k}n(L\bar{\boldsymbol{a}}_k + \Sigma^{-1/2}\bar{\boldsymbol{\varepsilon}}_{..})(L\bar{\boldsymbol{a}}_k + \Sigma^{-1/2}\bar{\boldsymbol{\varepsilon}}_{..})' \xrightarrow{p} 0 \quad \text{and} \quad \sqrt{k}(\bar{\Theta}_k - \Theta)$$

as $k \to \infty$. As a result

$$Z = \sqrt{k}(\bar{U}_k - E(\bar{U}_k)) + o_p(1). \qquad \square$$

4.2. Distribution of test statistics

Let $\tilde{T}_i$ be as in (3.7) but replacing n with $\bar{n}$. Here also we can write

$$\tilde{T}_i = tr\, A_i^{(h)}Z^{(h)} + tr\, A_i^{(e)}Z^{(e)} + O_p\left(\frac{1}{\sqrt{k}}\right) \tag{4.2}$$

for $i = 1, 2, 3$ where $A_i^{(h)}$ and $A_i^{(e)}$ are defined as in Section 3.1 but replacing n with $\bar{n}$. As in the previous sections we get the following theorem.

Theorem 4.1. *If the assumptions of Lemma* 4.1 *hold, then $\tilde{T}_i$ is asymptotically normally distributed with mean* 0 *and variance σ_i^2 given by*

$$\sigma_i^2 = 2tr(A_i^{(h)})^2 + 4tr(A_i^{(h)}\Theta) + 2\frac{\bar{\bar{n}}}{\bar{n}^2}tr(A_i^{(h)}\Theta)^2 + 2(\bar{n}-1)tr(A_i^{(e)})^2 + R_i + S_i,$$

where

$$R_i = vec(A_i)'\big(C \otimes (\Delta - I_p^2 - K_p - vec(I)vec(I)')\big)vec(A_i),$$

$$S_i = vec(A_i^{(h)})'\big[\Delta_1 - (I_{p^2} + K_p)(\Theta \otimes \Theta) - vec(\Theta)vec(\Theta)'\big]vec(A_i^{(h)})$$

and $A_i = (A_i^{(h)}, A_i^{(e)})$.

Under the null hypothesis ($\Theta = 0$), we get simpler expression for the variances of $\tilde{T}_i$.

Corollary 4.2. *Under the assumptions of Lemma 4.1, the asymptotic variances of* $\tilde{T}_i$, $i = 1, 2, 3$ *under the null hypotheses are*:

$$\sigma_i^2 = \frac{\bar{n}(\bar{n}\underline{n} - 1)}{(\bar{n} - 1)^2}\kappa_4^{(1)} + \frac{2\bar{n}p}{\bar{n} - 1},$$

where $\kappa_4^{(1)} = \sum_{a,b}^{p} \kappa_{aabb}$, $\kappa_{aabb} = E(\varepsilon_{11a}^2 \varepsilon_{11b}^2) - 2\delta_{ab} - 1$ *are fourth-order cumulants and* $\delta_{ab} = 1$ *when* $a = b$ *and* $\delta_{ab} = 0$ *when* $a \neq b$.

It may be noted that $\kappa_4^{(1)}$ is the multivariate measure of kurtosis suggested by Mardia [13]. It is known that $\kappa_4^{(1)} = 0$ when $\boldsymbol{\varepsilon}_{11}$ is normal. Thus, it is clear from Corollary 4.2 that, in general, the null and non-null distributions of the three test statistics are not stable against departure from normality.

The extension to the unbalanced two-way random and mixed effects model can be obtained in a similar way. But it is not hard to imagine that more stronger assumptions will be needed on the sample sizes. The extension to the general unbalanced mixed model (as done in Section 3.4 for the balanced cases) is not easy to come by at this point. In the balanced mixed model the situation is simplified by the fact that the design matrices can be written as the kronecker products of the identity matrices and a vector of ones.

5. Simulation study

In this section, we assess the numerical accuracy of the asymptotic distributions for one-way random effect model. We assume multivariate skew $\boldsymbol{t}$ distribution with 12 degrees of freedom for both the error and random effect. For the definition and properties of multivariate skew t distribution see [7]. Gupta uses the notation $MST_v(\boldsymbol{\alpha})$ for the multivariate skew t distribution with skewness parameter $\boldsymbol{\alpha}$ and degrees of freedom v. It must be noted that values of $\boldsymbol{\alpha}$ away from $\mathbf{0}$ provide higher skewness and larger values of v provide lesser kurtosis.

In our simulation study, we consider the values 2 and 3 for p and the values 15, 20, 30 and 40 for k. The p-dimensional skewness parameter vectors we consider are $\boldsymbol{\alpha} = (0, 0, \ldots, 0)'$ and $(1, 1, \ldots, 1)'$. We will consider three sample size structures. They are $n_i = 5$ for $i = 1, \ldots, k$; $n_1 = 8$ and $n_i = 5$ for $i = 2, \ldots, k$; and $n_1 = \cdots = n_5 = 8$ and $n_i = 5$ for $i = 6, \ldots, k$. We will denote the three structures as $n = 1$, 2 and 3, respectively. For the alternative point we use $\Theta = d\bar{n}\Psi$ with values 0.1 and 0.2 for d, and $diag\{1, 0, \ldots, 0\} + \mathbf{1}_p \otimes \mathbf{1}_p'$ for Ψ.

In Tables 1 and 2 are displayed the achieved test sizes when sampling is done from $MST_{12}(\boldsymbol{\alpha})$ and $MST_{35}(\boldsymbol{\alpha})$, respectively. It is clear from these tables that the asymptotic approximation for the null distribution does a pretty good job for LR statistic. In the cases of LH and BNP, we see that the asymptotic results lead to liberal and conservative rejection regions, respectively. It is also clear that large value of p requires large value of k. We also observe that the effects of skewness and kurtosis are not considerable.

Tables 3–6 display simulated powers and theoretical powers for 5% test size. Note that the theoretical powers do not depend on the skewness parameter $\boldsymbol{\alpha}$. Similar patterns as exhibited

Table 1
Achieved 5% and 1% test sizes when sampling from $MST_{12}(\boldsymbol{\alpha})$

p	k	n	5%						1%					
			$\boldsymbol{\alpha} = (0, 0, 0)'$			$\boldsymbol{\alpha} = (1, 1, 1)'$			$\boldsymbol{\alpha} = (0, 0, 0)'$			$\boldsymbol{\alpha} = (1, 1, 1)'$		
			LR	LH	BNP	LR	LH	BNP	LR	LH	BNP	LR	LH	BNP
2	15	1	4.8	6.3	3.3	5.0	6.5	3.5	1.5	2.4	0.9	1.6	2.6	0.8
2	20	1	4.8	6.3	3.5	4.8	6.3	3.7	1.5	2.1	0.9	1.4	2.2	0.8
2	30	1	4.9	6.1	4.0	5.2	6.4	4.1	1.4	2.1	1.0	1.4	2.0	0.9
2	40	1	4.7	5.5	3.9	4.5	5.5	3.8	1.3	1.7	0.9	1.2	1.8	0.8
2	15	2	4.6	6.1	3.3	4.8	6.5	3.6	1.3	2.3	0.7	1.3	2.4	0.7
2	20	2	4.7	6.0	3.4	4.9	6.3	3.7	1.4	2.0	0.9	1.5	2.2	0.9
2	30	2	4.7	5.7	3.9	4.8	5.8	3.9	1.4	2.0	0.9	1.5	2.0	1.0
2	40	2	4.4	5.3	3.6	4.8	5.8	4.0	1.2	1.5	0.9	1.4	1.7	1.0
2	15	3	4.8	6.1	3.6	4.9	6.4	3.6	1.3	2.1	0.6	1.7	2.4	1.0
2	20	3	4.8	5.8	3.8	4.6	5.7	3.5	1.4	2.1	1.0	1.4	1.9	0.8
2	30	3	4.8	5.6	3.9	5.1	6.0	4.1	1.3	1.8	0.9	1.4	2.0	0.9
2	40	3	4.4	5.4	3.7	4.7	5.4	3.9	1.1	1.5	0.9	1.3	1.6	0.9
3	15	1	5.0	7.5	2.9	4.6	7.2	2.7	1.4	3.0	0.6	1.5	2.7	0.6
3	20	1	4.6	6.8	3.1	5.1	7.2	3.3	1.1	2.2	0.5	1.4	2.6	0.7
3	30	1	4.7	6.5	3.4	5.0	6.8	3.5	1.3	2.3	0.7	1.3	2.1	0.7
3	40	1	5.0	6.5	3.9	4.9	6.3	3.6	1.3	2.0	0.8	1.4	1.9	0.9
3	15	2	4.8	7.1	3.0	4.9	7.6	3.0	1.3	2.7	0.6	1.4	2.7	0.7
3	20	2	4.6	6.4	3.1	4.5	6.5	3.0	1.3	2.1	0.7	1.3	2.3	0.7
3	30	2	4.6	6.2	3.2	4.6	6.1	3.0	1.2	1.9	0.8	1.1	1.8	0.7
3	40	2	4.5	5.9	3.4	4.8	6.1	3.7	1.1	1.7	0.8	1.4	1.9	1.0
3	15	3	4.4	6.4	3.0	4.4	6.5	2.9	1.3	2.3	0.7	1.2	2.2	0.6
3	20	3	4.7	6.4	3.1	4.5	6.4	3.2	1.2	2.2	0.7	1.2	2.1	0.7
3	30	3	4.5	5.9	3.4	4.9	6.6	3.7	1.3	2.0	0.7	1.3	2.0	0.8
3	40	3	4.7	5.9	3.6	4.3	5.3	3.4	1.2	1.7	0.8	1.0	1.6	0.7

by the null case is observed in the non-null case. In a simulation study not reported here, we have been able to observe similar pattern for 1% test size. Also, as expected, numerical and theoretical powers approached 1.000 quickly as d and/or k gets larger.

6. Concluding remarks

We derived the asymptotic distribution of the three commonly used multivariate test statistics, namely LR, LH and BNP statistics, for testing hypotheses on the fixed and random effects of multivariate mixed linear models. The asymptotic framework is as the number of levels of one of the main factors goes to infinity. This essentially means, both the hypothesis and error degrees of freedom go to infinity at a fixed rate.

We found, under no distributional assumptions on the error and random effects, the asymptotic distributions of the three test statistics to be normal. In our approach, it was necessary to make some usual type restrictions on the non-centrality parameter and symmetric error structure to establish normality in the non-null case when both fixed and random effects

Table 2
Achieved 5% and 1% test sizes when sampling from $MST_{35}(\boldsymbol{\alpha})$

p	k	n	5%						1%					
			$\boldsymbol{\alpha} = (0,0,0)'$			$\boldsymbol{\alpha} = (1,1,1)'$			$\boldsymbol{\alpha} = (0,0,0)'$			$\boldsymbol{\alpha} = (1,1,1)'$		
			LR	LH	BNP	LR	LH	BNP	LR	LH	BNP	LR	LH	BNP
2	15	1	4.5	5.9	3.3	5.0	6.7	3.8	1.5	2.5	0.9	1.7	2.7	1.0
2	20	1	4.5	5.9	3.5	4.4	5.8	3.4	1.2	1.9	0.7	1.1	2.0	0.7
2	30	1	4.9	5.9	3.9	5.2	6.4	4.4	1.4	1.9	0.9	1.7	2.2	1.1
2	40	1	4.9	5.9	4.1	4.6	5.4	3.8	1.4	2.1	1.0	1.2	1.7	0.9
2	15	2	4.5	6.1	3.1	4.8	6.5	3.4	1.2	2.2	0.7	1.5	2.5	0.8
2	20	2	5.1	6.4	3.9	4.8	6.2	3.6	1.6	2.3	1.0	1.2	1.8	0.8
2	30	2	4.7	5.7	3.8	5.2	6.3	4.1	1.4	2.0	1.0	1.5	2.2	1.0
2	40	2	5.2	6.1	4.5	4.8	5.9	4.0	1.3	1.9	1.0	1.3	1.8	0.9
2	15	3	4.5	5.5	3.2	4.3	5.7	3.2	1.3	2.0	0.8	1.1	1.8	0.6
2	20	3	4.7	5.9	3.5	4.9	6.0	3.7	1.3	1.9	0.9	1.4	2.1	0.9
2	30	3	5.2	6.0	4.3	4.7	5.7	3.8	1.4	2.0	1.0	1.3	1.8	0.8
2	40	3	4.9	5.6	4.1	5.1	5.8	4.2	1.4	1.9	1.0	1.3	1.8	1.0
3	15	1	4.5	7.1	2.8	4.8	7.3	2.8	1.3	2.7	0.6	1.4	2.6	0.5
3	20	1	4.7	6.6	3.2	4.9	7.0	3.1	1.3	2.4	0.7	1.4	2.5	0.7
3	30	1	4.6	6.3	3.2	4.9	6.8	3.3	1.1	1.8	0.7	1.3	2.1	0.7
3	40	1	4.8	6.1	3.6	5.1	6.4	3.8	1.2	1.9	0.8	1.4	2.1	0.9
3	15	2	4.8	7.0	2.9	4.8	7.1	3.0	1.4	2.6	0.6	1.4	2.7	0.6
3	20	2	4.8	7.2	3.2	4.8	7.0	3.3	1.3	2.3	0.7	1.4	2.4	0.8
3	30	2	4.7	6.4	3.5	4.6	6.2	3.4	1.2	1.9	0.7	1.1	1.9	0.7
3	40	2	5.0	6.3	3.8	4.9	6.4	3.7	1.5	2.1	0.9	1.3	1.9	0.9
3	15	3	4.2	6.3	2.9	4.5	6.7	3.1	1.2	2.1	0.6	1.4	2.4	0.8
3	20	3	4.7	6.5	3.3	4.7	6.7	3.2	1.4	2.2	0.7	1.4	2.3	0.9
3	30	3	4.6	6.2	3.5	5.0	6.5	3.7	1.2	2.0	0.8	1.2	1.9	0.7
3	40	3	5.0	6.2	3.7	5.1	6.7	4.0	1.3	1.8	0.9	1.4	2.0	0.9

exist in the model. Indeed, the conditions were only needed for testing the fixed effects. In the pure random effects model, only the i.i.d. version of Central Limit Theorem is needed. In the unbalanced case, we require the convergence of some partial sums of the sample sizes.

Appendix A.

A.1. Details for Section 2

We first generalize Theorem 2.1 in [8] to the case when the covariance of the random matrix in the quadratic form is non-negative definite.

Lemma A.1. *Let $\mathcal{E} = (\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2, \ldots, \boldsymbol{\varepsilon}_n)'$ be an $n \times p$ matrix whose rows are identically and independently distributed vectors with $E(\boldsymbol{\varepsilon}_1) = 0$, $Var(\boldsymbol{\varepsilon}_1) = \Sigma_\varepsilon(\geqslant 0)$ and finite fourth-order moment $\Delta = E(vec(\boldsymbol{\varepsilon}_1\boldsymbol{\varepsilon}_1')vec(\boldsymbol{\varepsilon}_1\boldsymbol{\varepsilon}_1')')$. Suppose $B_i, i = 1, \ldots, q$, are $n \times n$ symmetric matrices with equal diagonal elements. Define $Q = (Q_1, Q_2, \ldots, Q_q)$ with $Q_i = \mathcal{E}' B_i \mathcal{E}$.*

Table 3
Simulated and theoretical powers when sampling from $MST_{12}(\boldsymbol{\alpha})$ with $p = 2$

d	k	n	Simulated						Theoretical		
			$\boldsymbol{\alpha} = (0, 0, 0)'$			$\boldsymbol{\alpha} = (1, 1, 1)'$					
			LR	LH	BNP	LR	LH	BNP	LR	LH	BNP
0.1	15	1	0.580	0.579	0.569	0.677	0.691	0.655	0.668	0.689	0.637
0.1	20	1	0.699	0.703	0.689	0.771	0.781	0.758	0.745	0.759	0.724
0.1	30	1	0.826	0.832	0.817	0.896	0.903	0.886	0.851	0.854	0.842
0.1	40	1	0.905	0.910	0.899	0.956	0.960	0.949	0.913	0.911	0.911
0.2	15	1	0.879	0.880	0.871	0.921	0.928	0.911	0.899	0.883	0.904
0.2	20	1	0.950	0.950	0.945	0.970	0.972	0.967	0.945	0.928	0.953
0.2	30	1	0.991	0.992	0.990	0.996	0.997	0.995	0.984	0.972	0.989
0.2	40	1	0.997	0.998	0.997	1.000	1.000	1.000	0.995	0.989	0.998
0.1	15	2	0.609	0.617	0.602	0.686	0.698	0.671	0.687	0.706	0.659
0.1	20	2	0.698	0.702	0.689	0.798	0.804	0.783	0.759	0.771	0.740
0.1	30	2	0.837	0.843	0.829	0.911	0.916	0.904	0.858	0.860	0.851
0.1	40	2	0.904	0.910	0.897	0.959	0.964	0.953	0.917	0.915	0.916
0.2	15	2	0.896	0.900	0.888	0.931	0.937	0.924	0.908	0.890	0.915
0.2	20	2	0.949	0.951	0.946	0.975	0.976	0.971	0.949	0.932	0.959
0.2	30	2	0.990	0.991	0.987	0.997	0.997	0.996	0.985	0.974	0.991
0.2	40	2	0.998	0.998	0.998	1.000	1.000	1.000	0.996	0.990	0.998
0.1	15	3	0.687	0.692	0.677	0.772	0.781	0.759	0.753	0.761	0.737
0.1	20	3	0.761	0.762	0.752	0.840	0.847	0.831	0.808	0.812	0.797
0.1	30	3	0.859	0.868	0.853	0.931	0.936	0.924	0.885	0.883	0.882
0.1	40	3	0.936	0.940	0.930	0.969	0.972	0.965	0.933	0.928	0.933
0.2	15	3	0.931	0.934	0.927	0.962	0.964	0.959	0.934	0.913	0.947
0.2	20	3	0.969	0.968	0.967	0.985	0.986	0.983	0.963	0.945	0.974
0.2	30	3	0.992	0.993	0.991	0.998	0.998	0.997	0.989	0.978	0.994
0.2	40	3	0.999	0.999	0.999	1.000	1.000	1.000	0.997	0.991	0.999

Then,

$$E(Q) = E_N(Q) = nB \otimes \Sigma_\varepsilon$$

and

$$Var(Q) = Var_N(Q) + (n\boldsymbol{b}\boldsymbol{b}') \otimes \big(\Delta - (I + K_p)(\Sigma_\varepsilon \otimes \Sigma_\varepsilon) - vec(\Sigma_\varepsilon)vec(\Sigma_\varepsilon)'\big),$$

where $\boldsymbol{b} = (b_{i11})$ *is a* $q \times 1$ *vector and* b_{i11} *is the* $(1, 1)$*th entry of the matrix* B_i.

Proof. That $E(Q) = E_N(Q)$ is obvious. Indeed,

$$E(\mathcal{E}' B_i \mathcal{E}) = (tr\ B_i)Var(\boldsymbol{\varepsilon}_1) = nb_{i11}\Sigma_\varepsilon.$$

For the second part, we note that

$$E(vec(Q_i)vec(Q_j)') = \sum_{l=1}^{n}\sum_{m=1}^{n}\sum_{l'=1}^{n}\sum_{m'=1}^{n} b_{ilm}b_{jl'm'}E(vec(\boldsymbol{\varepsilon}_l\boldsymbol{\varepsilon}_m')vec(\boldsymbol{\varepsilon}_{l'}\boldsymbol{\varepsilon}_{m'}')').$$

Table 4
Simulated and theoretical powers when sampling from $MST_{12}(\boldsymbol{\alpha})$ with $p=3$

d	k	n	Simulated						Theoretical		
			$\boldsymbol{\alpha}=(0,0,0)'$			$\boldsymbol{\alpha}=(1,1,1)'$					
			LR	LH	BNP	LR	LH	BNP	LR	LH	BNP
0.1	15	1	0.619	0.641	0.590	0.764	0.785	0.727	0.704	0.730	0.664
0.1	20	1	0.740	0.757	0.716	0.853	0.870	0.829	0.783	0.799	0.753
0.1	30	1	0.874	0.884	0.854	0.949	0.959	0.933	0.884	0.889	0.870
0.1	40	1	0.936	0.944	0.922	0.984	0.987	0.977	0.939	0.938	0.933
0.2	15	1	0.907	0.918	0.893	0.962	0.968	0.946	0.923	0.906	0.922
0.2	20	1	0.967	0.972	0.957	0.988	0.990	0.983	0.962	0.946	0.965
0.2	30	1	0.996	0.997	0.994	0.999	0.999	0.998	0.991	0.982	0.994
0.2	40	1	0.999	0.999	0.998	1.000	1.000	1.000	0.998	0.994	0.999
0.1	15	2	0.652	0.665	0.633	0.789	0.801	0.760	0.723	0.745	0.687
0.1	20	2	0.773	0.783	0.749	0.874	0.888	0.844	0.797	0.810	0.770
0.1	30	2	0.865	0.878	0.849	0.959	0.966	0.946	0.891	0.894	0.878
0.1	40	2	0.939	0.948	0.924	0.986	0.989	0.980	0.942	0.941	0.937
0.2	15	2	0.924	0.930	0.912	0.966	0.971	0.957	0.930	0.912	0.932
0.2	20	2	0.971	0.974	0.965	0.991	0.992	0.985	0.965	0.949	0.970
0.2	30	2	0.995	0.996	0.993	0.999	1.000	0.999	0.992	0.983	0.994
0.2	40	2	0.999	1.000	0.999	1.000	1.000	1.000	0.998	0.994	0.999
0.1	15	3	0.725	0.733	0.705	0.849	0.862	0.825	0.788	0.798	0.767
0.1	20	3	0.815	0.826	0.796	0.917	0.925	0.900	0.842	0.847	0.827
0.1	30	3	0.907	0.915	0.893	0.972	0.978	0.962	0.914	0.913	0.908
0.1	40	3	0.954	0.960	0.947	0.990	0.992	0.987	0.954	0.951	0.952
0.2	15	3	0.957	0.960	0.949	0.982	0.985	0.976	0.953	0.931	0.962
0.2	20	3	0.983	0.985	0.979	0.995	0.996	0.993	0.976	0.959	0.983
0.2	30	3	0.998	0.998	0.997	1.000	1.000	0.999	0.994	0.986	0.997
0.2	40	3	1.000	1.000	0.999	1.000	1.000	1.000	0.999	0.995	0.999

Further we observe that

$$E(vec(\boldsymbol{\varepsilon}_l\boldsymbol{\varepsilon}_m')vec(\boldsymbol{\varepsilon}_{l'}\boldsymbol{\varepsilon}_{m'}')') = E_N(vec(\boldsymbol{\varepsilon}_l\boldsymbol{\varepsilon}_m')vec(\boldsymbol{\varepsilon}_{l'}\boldsymbol{\varepsilon}_{m'}')')$$

unless $l=m=l'=m'$.

Let the rank of Σ_ε be r. It is obvious that under normality, we can write $\boldsymbol{\varepsilon}_l\boldsymbol{\varepsilon}_l' \overset{d}{=} L\boldsymbol{a}\boldsymbol{a}'L$ where $\boldsymbol{a}\sim N(0, I_r)$ and LL' is rank factorization of Σ_ε. Therefore,

$$E_N(vec(\boldsymbol{\varepsilon}_l\boldsymbol{\varepsilon}_l')vec(\boldsymbol{\varepsilon}_l\boldsymbol{\varepsilon}_l')') = (L\otimes L)\big[E(vec(\boldsymbol{a}\boldsymbol{a}')vec(\boldsymbol{a}\boldsymbol{a}')')\big](L'\otimes L').$$

Noting that $\boldsymbol{a}\boldsymbol{a}'\sim W_r(1, I_r)$, we can appeal to Lemma 2.2 to get,

$$\begin{aligned}&E_N(vec(\boldsymbol{\varepsilon}_l\boldsymbol{\varepsilon}_l')vec(\boldsymbol{\varepsilon}_l\boldsymbol{\varepsilon}_l')')\\&\quad=(L\otimes L)\big[(I+K_r+vec(I_r)vec(I_r)')(I_r\otimes I_r)\\&\qquad+vec(I_r)vec(I_r)'\big](L'\otimes L')\end{aligned} \tag{A.1}$$

Table 5
Simulated and theoretical powers when sampling from $MST_{35}(\boldsymbol{\alpha})$ with $p = 2$

d	k	n	Simulated						Theoretical		
			$\boldsymbol{\alpha} = (0, 0, 0)'$			$\boldsymbol{\alpha} = (1, 1, 1)'$					
			LR	LH	BNP	LR	LH	BNP	LR	LH	BNP
0.1	15	1	0.583	0.589	0.573	0.691	0.699	0.670	0.673	0.696	0.641
0.1	20	1	0.702	0.708	0.687	0.791	0.798	0.773	0.753	0.767	0.730
0.1	30	1	0.822	0.832	0.815	0.914	0.921	0.902	0.859	0.862	0.849
0.1	40	1	0.922	0.927	0.914	0.963	0.967	0.958	0.920	0.919	0.917
0.2	15	1	0.886	0.890	0.879	0.935	0.937	0.927	0.912	0.897	0.916
0.2	20	1	0.954	0.956	0.949	0.977	0.979	0.974	0.955	0.940	0.961
0.2	30	1	0.991	0.991	0.990	0.997	0.998	0.997	0.988	0.979	0.992
0.2	40	1	0.999	0.999	0.998	1.000	1.000	1.000	0.997	0.993	0.999
0.1	15	2	0.602	0.613	0.590	0.710	0.715	0.687	0.694	0.713	0.665
0.1	20	2	0.707	0.717	0.696	0.807	0.821	0.789	0.767	0.779	0.747
0.1	30	2	0.838	0.847	0.827	0.914	0.919	0.905	0.867	0.869	0.859
0.1	40	2	0.919	0.923	0.913	0.963	0.968	0.957	0.925	0.923	0.923
0.2	15	2	0.900	0.905	0.895	0.946	0.949	0.939	0.921	0.905	0.927
0.2	20	2	0.958	0.959	0.954	0.980	0.981	0.976	0.959	0.944	0.966
0.2	30	2	0.992	0.992	0.989	0.997	0.998	0.996	0.989	0.980	0.993
0.2	40	2	0.998	0.998	0.998	1.000	1.000	0.999	0.997	0.993	0.999
0.1	15	3	0.709	0.716	0.701	0.779	0.788	0.768	0.763	0.772	0.746
0.1	20	3	0.764	0.768	0.754	0.856	0.862	0.846	0.818	0.822	0.807
0.1	30	3	0.880	0.884	0.872	0.939	0.943	0.930	0.894	0.893	0.891
0.1	40	3	0.937	0.941	0.932	0.975	0.976	0.969	0.940	0.936	0.940
0.2	15	3	0.942	0.944	0.939	0.967	0.969	0.962	0.947	0.928	0.958
0.2	20	3	0.973	0.974	0.970	0.988	0.988	0.986	0.972	0.957	0.980
0.2	30	3	0.995	0.995	0.995	0.998	0.999	0.998	0.993	0.985	0.996
0.2	40	3	1.000	1.000	1.000	1.000	1.000	1.000	0.998	0.995	0.999

Applying the well-known identity $K_{mn}(A_{n\times p} \otimes B_{m\times q}) = (B \otimes A)K_{qp}$ (see [9]) to (A.1), we get,

$$E_N(vec(\varepsilon_l\varepsilon_l')vec(\varepsilon_l\varepsilon_l')') = (I + K_p + vec(I)vec(I)')(\Sigma_\varepsilon \otimes \Sigma_\varepsilon) + vec(\Sigma_\varepsilon)vec(\Sigma_\varepsilon)'.$$

Consequently,

$$E(vec(Q_i)vec(Q_j)') - E_N(vec(Q_i)vec(Q_j)') = \Delta - (I_{p^2} + K_p + vec(I)vec(I)')(\Sigma_\varepsilon \otimes \Sigma_\varepsilon) - vec(\Sigma_\varepsilon)vec(\Sigma_\varepsilon).$$

Finally, since $E(vec(Q_i)) = E_N(vec(Q_i))$ by the first part, the desired result follows. □

Next we extend Lemma A.1 to the case when the random matrix in the quadratic form is the sum of two-independent random matrices.

Corollary A.1. *In Lemma A.1, let $\Sigma_\varepsilon > 0$ and $Q_i = Q_i(\mathcal{E}, \mathcal{T}) = (\mathcal{E} + L_i\mathcal{T})'B_i(\mathcal{E} + L_i\mathcal{T})$ where $\mathcal{T}' = (\tau_1, \tau_2, \ldots, \tau_s)$ is $p \times s$ matrix whose columns are identically and independently*

Table 6
Simulated and theoretical powers when sampling from $MST_{35}(\boldsymbol{\alpha})$ with $p = 3$

d	k	n	Simulated						Theoretical		
			$\boldsymbol{\alpha} = (0, 0, 0)'$			$\boldsymbol{\alpha} = (1, 1, 1)'$					
			LR	LH	BNP	LR	LH	BNP	LR	LH	BNP
0.1	15	1	0.635	0.650	0.603	0.779	0.800	0.740	0.711	0.738	0.669
0.1	20	1	0.738	0.755	0.710	0.874	0.888	0.842	0.792	0.809	0.760
0.1	30	1	0.883	0.896	0.860	0.960	0.967	0.947	0.893	0.898	0.877
0.1	40	1	0.941	0.948	0.928	0.989	0.992	0.983	0.946	0.946	0.939
0.2	15	1	0.919	0.927	0.902	0.964	0.971	0.948	0.935	0.921	0.932
0.2	20	1	0.967	0.971	0.959	0.991	0.992	0.986	0.970	0.957	0.972
0.2	30	1	0.996	0.997	0.995	0.999	1.000	0.999	0.994	0.987	0.996
0.2	40	1	0.999	1.000	0.999	1.000	1.000	1.000	0.999	0.996	0.999
0.1	15	2	0.641	0.658	0.608	0.794	0.813	0.760	0.732	0.755	0.693
0.1	20	2	0.754	0.772	0.729	0.884	0.899	0.856	0.806	0.821	0.778
0.1	30	2	0.884	0.900	0.864	0.963	0.971	0.949	0.900	0.904	0.886
0.1	40	2	0.944	0.952	0.932	0.987	0.990	0.981	0.949	0.949	0.943
0.2	15	2	0.930	0.938	0.912	0.974	0.979	0.962	0.943	0.927	0.942
0.2	20	2	0.975	0.979	0.965	0.993	0.994	0.989	0.974	0.960	0.976
0.2	30	2	0.997	0.997	0.994	1.000	1.000	0.999	0.995	0.988	0.996
0.2	40	2	1.000	1.000	1.000	1.000	1.000	1.000	0.999	0.996	0.999
0.1	15	3	0.749	0.762	0.732	0.870	0.881	0.850	0.800	0.811	0.777
0.1	20	3	0.815	0.826	0.797	0.918	0.930	0.902	0.854	0.859	0.837
0.1	30	3	0.915	0.924	0.901	0.977	0.981	0.969	0.924	0.924	0.916
0.1	40	3	0.950	0.957	0.942	0.992	0.994	0.989	0.961	0.959	0.958
0.2	15	3	0.962	0.965	0.954	0.988	0.989	0.984	0.964	0.946	0.970
0.2	20	3	0.986	0.988	0.982	0.996	0.997	0.994	0.983	0.970	0.988
0.2	30	3	0.998	0.998	0.997	1.000	1.000	0.999	0.996	0.991	0.998
0.2	40	3	1.000	1.000	0.999	1.000	1.000	1.000	0.999	0.997	1.000

distributed with $E(\boldsymbol{\tau}_1) = 0$, $Var(\boldsymbol{\tau}_1) = \Sigma_\tau$ $(\Sigma_\tau \geqslant 0)$ *and finite fourth-order moment* Δ_τ*. Let* $C_i = L_i' B_i L_i$ *has equal diagonal elements. Then,*

$$E(Q) = E_N(Q)$$

$$\begin{aligned} Var(Q) &= Var_N(Q) + n(\boldsymbol{bb}') \otimes (\Delta - I_{p^2} - 2K_p) \\ &\quad + s(\boldsymbol{cc}') \otimes (\Delta_\tau - (I_{p^2} + K_p)(\Sigma_\tau \otimes \Sigma_\tau) - vec(\Sigma_\tau)vec(\Sigma_\tau)'), \end{aligned}$$

where $\boldsymbol{c} = (c_{i11})$ *is a* $q \times 1$ *vector of the* $(1, 1)th$ *entries* c_{i11} *of* C_i*.*

Proof. The first part is obvious. For the second part we note that

$$Q_i(\mathcal{E}, \mathcal{T}) = Q_i(\mathcal{E}, 0) + Q_i(0, \mathcal{T}) + \mathcal{T}' L_i' B_i \mathcal{E} + \mathcal{E}' B_i L_i \mathcal{T}. \tag{A.2}$$

Now in the computation of $E(vec(Q_i(\mathcal{E}, \mathcal{T}))vec(Q_j(\mathcal{E}, \mathcal{T}))')$ the terms involving the third and fourth terms on the right-hand side of A.2 will have either zero expectation or same expectation in the normal and non-normal cases. As a result,

$$
\begin{aligned}
&E(vec(Q_i(\mathcal{E},\tau))vec(Q_j(\mathcal{E},\mathcal{T}))') - E_N(vec(Q_i(\mathcal{E},\mathcal{T}))vec(Q_j(\mathcal{E},\mathcal{T}))') \\
&\quad = E(vec(Q_i(\mathcal{E},0))vec(Q_j(\mathcal{E},0))') - E_N(vec(Q_i(\mathcal{E},0))vec(Q_j(\mathcal{E},0))') \\
&\qquad + E(vec(Q_i(0,\mathcal{T}))vec(Q_j(0,\mathcal{T}))') - E_N(vec(Q_i(0,\mathcal{T}))vec(Q_j(0,\mathcal{T}))').
\end{aligned}
$$

The desired result follows now from Lemma A.1. $\square$

Finally, we extend Corollary A.1 to cover quadratic forms where the random matrix has location parameter. The proof goes along the same lines and, hence, is omitted.

Corollary A.2. *Let Q_i in Corollary A.1 be redefined as $Q_i = Q_i(\mathcal{E},\mathcal{T},A_i) = (\mathcal{E}+A_i+L_i\mathcal{T})'B_i(\mathcal{E}+A_i+L_i\mathcal{T})$ where A_i is an $n\times p$ fixed matrix. If, in addition to the assumptions in Corollary A.1, the third-order moments of $\boldsymbol{\varepsilon}_1$ are zero, then the results of Corollary A.1 still hold.*

Lemma 2.1 also follows in a similar manner.

Acknowledgments

The authors are thankful to the referees for critically reading the paper.

References

[1] M.A. Akritas, S.F. Arnold, Asymptotics for analysis of variance when the number of levels is large, J. Amer. Statist. Assoc. 95 (2000) 212–226.
[2] T.W. Anderson, An Introduction to Multivariate Analysis, second ed., Wiley, New York, 1984.
[3] B.M. Anderson, T.W. Anderson, I. Olkin, Maximum likelihood estimators and likelihood ratio criteria in multivariate components of variance, Ann. Statist. 14 (1986) 405–417.
[4] Y. Fujikoshi, Asymptotic formulas for the non-null distributions of three statistics for multivariate linear hypothesis, Ann. Inst. Statist. Math. 27 (1975) 99–108.
[5] Y. Fujikoshi, Some recent results on asymptotic expansions of multivariate test statistics for mean vectors under non-normality, Calcutta Statist. Assoc. Bull. (special fourth triennial proceedings volume) 52 (205–208) (2002) 1–46.
[6] L. Gang, Moments of a random vector and its quadratic forms, in: K.T. Fang, T.W. Anderson (Eds.), Statistical Inference in Elliptically Contoured and Related Distributions, Allerton Press Inc., New York, 1990, pp. 433–440.
[7] A.K. Gupta, Multivariate skew *t* distribution, Statistics 37 (2003) 359–363.
[8] A.K. Gupta, S. Harrar, Y. Fujikoshi, Distribution of MANOVA statistics under non-normality and large hypothesis degrees of freedom. Department of Mathematics and Statistics, Bowling Green State University, Technical Report No. 03–06, 2003.
[9] A.K. Gupta, D.K. Nagar, Matrix Variate Distributions, Boca Raton, Chapman & Hall, CRC, Boca Raton, FL, 2000.
[10] A.I. Khuri, Direct products: a powerful tool for the analysis of balanced data, Comm. Statist.—Theory Methods 11 (1982) 2903–2920.
[11] A.I. Khuri, T. Mathew, B.K. Sinha, Statistical Test for Mixed Linear Models, Wiley, New York, 1998.
[12] J.R. Magnus, H. Neudecker, The commutation matrix, some properties and applications, Ann. Statist. 7 (1979) 381–394.

[13] K.V. Mardia, Measures of multivariate skewness and kurtosis with applications, Biometrika 57 (1970) 519–530.
[14] T. Mathew, MANOVA in the multivariate components of variance model, J. Multivariate Anal. 29 (1989) 30–38.
[15] J.R. Schott, J.G. Saw, A multivariate one-way classification model with random effects, J. Multivariate Anal. 15 (1984) 1–12.
[16] M. Siotani, T. Hayakawa, Y. Fujikoshi, Modern Multivariate Statistical Analysis: A Graduate Course and Handbook, American Science Press, Columbus, OH, 1985.
[17] L. Zhou, T. Mathew, Hypothesis tests for variance components in some multivariate mixed models, J. Statist. Planning Inference 37 (1993) 21–227.