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A Duality Theory for Decomposable Systems in a Category

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The Arbib–Manes duality theory for decomposable systems in a category is generalized, making it possible to apply the theory to infinite-dimensional linear systems in reflexive Banach spaces.

1. INTRODUCTION

In the state space approach to linear systems three concepts play a central role: realization—to construct an internal (state space) description or model for a given input–output description; reachability—to determine if it is possible to reach any given final state from any initial state by choosing suitable inputs; and observability—to determine if it is possible to distinguish between any two states by only looking at the outputs if zero inputs are applied.

A duality theory between reachability and observability for finite-dimensional linear systems was developed by Kalman about 20 years ago. Thus for any general theorem about the reachability of systems there is a dual theorem about the observability of systems.

In general an input–output description may have no finite-dimensional realization, and it is therefore of interest to extend the above-mentioned duality to infinite-dimensional systems.

During the last decade there have been several arrow-theoretic approaches to this duality [1, 4, 6, 8] of which those in [1, 4] are briefly discussed below.

We first recapture some basic notions about decomposable systems in a category. Most of the categorical concepts which are used in the present paper are carefully motivated and introduced in [1] while the important concept of "dual equivalent categories," which we also need, is discussed in [4].

A (discrete time) finite-dimensional system is defined by the equations

$$q(t+1) = Fq(t) + Gi(t),$$

$$y(t) = Hq(t),$$

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where F, G and H are matrices. We can associate linear transformations with F, G and H and represent a system by a diagram

$$I \xrightarrow{G} Q \xrightarrow{F} Q \xrightarrow{H} Y, \tag{1}$$

where the (finite-dimensional) vector space I, Q and Y are the input, state and output spaces, respectively.

It is now straightforward to define a (decomposable) system in an arbitrary category $\mathscr{A}$—it is simply a diagram such as (1), where we replace the vector spaces I, Q and Y by $\mathscr{A}$-objects and the linear transformations G, F and H by $\mathscr{A}$-morphisms.

To formulate the concepts of reachability and observability for a system in a category we analyze these concepts in the vector space situation.

We denote the direct sum of a countable number of copies of the vector space I by $I^{\S}$. $I^{\S}$ consists of all left infinite sequences $i = (..., 0, ..., 0, i_k, ..., i_1, i_0)$, where all but a finite number of coordinates are zero. The reachability map r: $I^{\S} \to Q$ is defined by assigning to each sequence i as above the state $\sum_{j=0}^{k} F^j G i_j$ which will result if the system is started in the zero state and the sequence i is fed into the system. The system is called reachable if r is onto; i.e., for every state q there is some input sequence which will transform the system to state q.

The above formulation of reachability admits generalization to other categories. "The direct sum of a countable number of copies of the vector space I" in the category of vector spaces translates to "the coproduct of a countable number of copies of the $\mathscr{A}$-object I" in $\mathscr{A}$. The coproduct $I^{\S}$ in $\mathscr{A}$ is equipped with a family in_n: $I \to I^{\S}$ of morphisms which will be used in the sequel. In the vector space case these are simply the maps $i_n \mapsto (..., 0, ..., 0, i_n, 0, ..., 0)$ which insert i_n into the nth position.

To define the reachability map of a system in $\mathscr{A}$ we need a few more definitions from [1].

A dynamics in $\mathscr{A}$ is a pair (Q, F), where Q is an $\mathscr{A}$-object and F: $Q \to Q$ is an $\mathscr{A}$-morphism. A dynamics morphism g: $(Q, F) \to (Q', F')$ is an $\mathscr{A}$-morphism g: $Q \to Q'$ such that $gF = F'g$.

We supply the $\mathscr{A}$-object $I^{\S}$ with a dynamics z: $I^{\S} \to I^{\S}$ by defining z by $z\, in_k = in_{k+1}$. The reader may easily verify that in the vector space case z is simply the map z: $(..., i_1, i_0) \to (..., i_1, i_0, 0)$, which shifts the sequence one place to the left and inserts a zero.

It can now be verified that the reachability map r: $I^{\S} \to Q$ of a linear system is the unique dynamics morphism r: $(I^{\S}, z) \to (Q, F)$ such that $r\, in_0 = G$, $r\, in_{k+1} = F^{k+1}G$. This characterization of r allows us to define the reachability map of a system in $\mathscr{A}$ in this way. Finally, "r is onto" is generalized by specifying a class $\mathscr{E}$ of epimorphisms in $\mathscr{A}$ and requiring that $r \in \mathscr{E}$.

The observability map of a linear system is defined to be the map σ: $q \to (Hq, HFq, HF^2q, ...)$, i.e., it assigns to a state q the sequence of outputs which will result if the system is in state q and only zero inputs are fed into the system. The set of output sequences is denoted by $Y_{\S}$—the direct product of a countable number of copies of Y.

The system is called observable if σ is one-to-one, i.e., different states give rise to different output sequences.

To define observability for a system in $\mathscr{A}$ we first consider the direct product $(Y_\S, \pi_n\colon Y_\S \to Y)$ of a countable number of copies of Y. In the linear case $\pi_n\colon Y_\S \to Y$ is simply the projection map $\pi_n\colon (y_0, y_1, \ldots) \to y_n$. We also supply $Y_\S$ with a dynamics $z\colon Y_\S \to Y_\S$ by defining z by $\pi_n z = \pi_{n+1}$. In the vector space case z is the map $z\colon (y_0, y_1, y_2, \ldots) \to (y_1, y_2, \ldots)$.

It can be verified that the observability map $\sigma\colon Q \to Y_\S$ of a linear system is the unique dynamics morphism

$$\sigma\colon (Q, F) \to (Y_\S, z) \qquad \text{such that} \quad \pi_0 \sigma = H,\ \pi_{k+1}\sigma = HF^{k+1}$$

and we use this characterization to define the observability map of a system in $\mathscr{A}$. Finally we specify a class $\mathscr{M}$ of monomorphisms and call a system in $\mathscr{A}$ observable if $\sigma \in \mathscr{M}$.

Various kinds of epimorphisms and monomorphisms arise in general categories and it is profitable to axiomatize a class of possibilities. Thus an image-factorization system $(\mathscr{E}, \mathscr{M})$ consists of a class $\mathscr{E}$ of epimorphisms and a class $\mathscr{M}$ of monomorphisms satisfying some axioms [1]. We assume that our category $\mathscr{A}$ is equipped with such an image-factorization system.

In [1] a categorical duality theory is developed for decomposable systems in a category $\mathscr{A}$. A class of "finite-dimensional objects" is specified together with a transposition rule $*\colon \mathscr{A}(A, B) \to \mathscr{A}(B, A)$ subject to certain axioms. A general duality theory for systems in $\mathscr{A}$ results, including finite-dimensional linear systems as a special case. The authors were, however, unable to apply the general result to infinite-dimensional systems (in particular to systems in Hilbert space) because they were unable to find an image-factorization system with the required properties.

In [4] it is mentioned that the approach in [1] "does not appear to be readily extendable to infinite-dimensional systems." A duality theory for infinite-dimensional systems is then developed in a linearly topologized framework. In this approach use is made of either a self-dual category with countable powers and copowers, or dual equivalent categories, each with countable powers and copowers. It is mentioned in [4] that "examples, while simple in principle, are extremely complicated in terms of illustrating the topologies."

In the present paper we follow the approach of [1] in a slightly more general setting. We assume that we have two categories $\mathscr{A}$ and $\mathscr{B}$, each with a subcategory of "finite objects." In this way the dual of a finite system in $\mathscr{A}$ is a finite system in $\mathscr{B}$. By extending some of the results in [1] we obtain a more general duality theory for decomposable systems, and this theory is indeed applicable to infinite-dimensional systems in Banach space. Also, the dualities in [4] appear as special cases. Finally it is possible to extend the known duality results for free, finite-dimensional linear systems over some commutative rings [2] to finitely generated systems over some non-commutative rings. Details of these results will appear in a further publication.

We use the standard notation of [1, 4]. In particular if $\mathscr{F}$ and $\mathscr{G}$ are functors, then

$\mathscr{F} \simeq \mathscr{G}$ means that there is a natural equivalence between $\mathscr{F}$ and $\mathscr{G}$. $\mathscr{A}^{\mathrm{op}}$ is used to denote the dual or opposite category of $\mathscr{A}$. The countable power of A is denoted by $(A_{\S}, \pi_n)$, while $(A^{\S}, in_n)$ denotes the countable copower.

2. Duality for Decomposable Systems

In this section we develop our basic duality theory for decomposable systems. While most of the results are extensions of those in [1], our more general approach enables us to apply our results to infinite-dimensional systems.

Two morphisms f and g in a category $\mathscr{C}$ are called *isomorphic* ($f \sim g$) if there exist isomorphisms i and j such that $f = igj$. We note that in the case of finite-dimensional vector spaces "isomorphic morphisms" are simply "equivalent matrices." Clearly $\sim$ defines an equivalence relation on the class of all morphisms of $\mathscr{C}$. If $\mathscr{C}$ has an image-factorization system $(\mathscr{E}, \mathscr{M})$ then $f \sim g$ and $f \in \mathscr{E}[f \in \mathscr{M}]$ implies $g \in \mathscr{E}[g \in \mathscr{M}]$.

We also note that if $(\mathscr{E}, \mathscr{M})$ is an image-factorization system for $\mathscr{C}$, then $(\mathscr{M}, \mathscr{E})$ is an image-factorization system for the dual category $\mathscr{C}^{\mathrm{op}}$.

We now describe the setting for our duality theory.

Let $\mathscr{A}$ and $\mathscr{B}$ be two categories, each with countable powers and copowers, and suppose $(\mathscr{E}_1, \mathscr{M}_1)$ and $(\mathscr{E}_2, \mathscr{M}_2)$ are image-factorization systems for $\mathscr{A}$ and $\mathscr{B}$, respectively. Furthermore, suppose that $\mathscr{K}$ and $\mathscr{H}$ are full subcategories of $\mathscr{A}$ and $\mathscr{B}$, respectively, such that $\mathscr{K}$ and $\mathscr{H}$ are dual equivalent, i.e., there are functors $\mathscr{F}$: $\mathscr{K}^{\mathrm{op}} \to \mathscr{H}$ and $\mathscr{G}: \mathscr{H}^{\mathrm{op}} \to \mathscr{K}$ such that $\mathscr{G} \circ \mathscr{F}^{\mathrm{op}} \simeq 1_{\mathscr{K}}$ and $\mathscr{F} \circ \mathscr{G}^{\mathrm{op}} \simeq 1_{\mathscr{H}}$. $\mathscr{K}$ and $\mathscr{H}$ are called categories of *finite objects*.

As an example of such a setting we once again consider vector spaces. We take $\mathscr{A} = \mathscr{B} =$ category of all vector spaces. As image-factorization systems we simply take (onto maps, one-to-one maps), while for $\mathscr{K}$ and $\mathscr{H}$ we take the subcategories of finite-dimensional spaces. It is well known that the category of finite-dimensional spaces is equivalent to its own dual via the functors $\mathscr{F}$ and $\mathscr{G}$ which both take a space V and assigns to if the space V' of all linear functionals on V. Also, if we have a map f: $V \to W$ then its dual is a map f': $W' \to V'$.

We remark that e: $V \to W$ is an onto map if and only if its dual e': $W' \to V'$ is a one-to-one map. This situation is now investigated in the general case.

Lemma 1. *Suppose that given e: $K \to L$ with K and L in $\mathscr{K}$ we have $e \in \mathscr{E}_1$ if and only if $\mathscr{F}e \in \mathscr{M}_2$. Then*

(i) *$m \in \mathscr{M}_2$ if and only if $\mathscr{G}m \in \mathscr{E}_1$;*

(ii) *$m \in \mathscr{M}_1$ if and only if $\mathscr{F}m \in \mathscr{E}_2$;*

(iii) *$e \in \mathscr{E}_2$ if and only if $\mathscr{G}e \in \mathscr{M}_1$.*

Proof. (i) Suppose $m \in \mathscr{M}_2$. Then $\mathscr{F} \circ \mathscr{G}^{\mathrm{op}}(m) \sim m$ so that $\mathscr{G}^{\mathrm{op}}(m) \in \mathscr{E}_1$. Thus $\mathscr{G}(m) \in \mathscr{E}_1$. The proof of the converse is similar.

(ii) In any image-factorization system $(\mathscr{E}, \mathscr{M})$, $\mathscr{E}$ and $\mathscr{M}$ uniquely determine each other, i.e.,

(A) $\mathscr{M} = \{f \mid f = h \circ e,\ e \in \mathscr{E}$ implies e is an isomorphism$\}$ and

(B) $\mathscr{E} = \{f \mid f = m \circ h,\ m \in \mathscr{M}$ implies m is an isomorphism$\}$

(see [5, Theorem 33.6, p. 253]).

Consider $m_1 \in \mathscr{M}_1$. Suppose that $\mathscr{F}^{\mathrm{op}} m_1 = e \circ m$, $m \in \mathscr{M}_2$, $e \in \mathscr{E}_2$, in $\mathscr{H}^{\mathrm{op}}$. Then $\mathscr{G} \circ \mathscr{F}^{\mathrm{op}} m_1 = \mathscr{G} e \circ \mathscr{G} m$ with $\mathscr{G} m \in \mathscr{E}_1$ (by (i)). But $\mathscr{G} \circ \mathscr{F}^{\mathrm{op}} m_1 \sim m_1 \in \mathscr{M}_1$ so that from (A) it follows that $\mathscr{G} m$ is an isomorphism. Thus $m \sim \mathscr{F} \circ \mathscr{G}^{\mathrm{op}} m$ is an isomorphism, so that $\mathscr{F}^{\mathrm{op}} m_1 = em \in \mathscr{E}_2$, i.e., $\mathscr{F} m_1 \in \mathscr{E}_2$.

The proof of the converse is similar.

(iii) dual to (i). ∎

It is possible to extend the correspondence between $\mathscr{K}$ and $\mathscr{H}$ to some other morphisms in $\mathscr{A}$ and $\mathscr{B}$.

If A, B are $\mathscr{K}$-objects and $f\colon A^{\S} \to B$, we define $\mathscr{F}'f\colon \mathscr{F}B \to (\mathscr{F}A)_{\S}$ by

$$\begin{tikzcd} \mathscr{F}B \arrow[r, dashed, "\mathscr{F}'f"] \arrow[dr, "\mathscr{F}(f\, in_n)"'] & (\mathscr{F}A)_{\S} \arrow[d, "\pi_n"] \\ & \mathscr{F}A \end{tikzcd}$$

Similarly if X, Y are $\mathscr{H}$-objects and $g\colon X \to Y_{\S}$, we define $\mathscr{G}'g\colon (\mathscr{G}Y)^{\S} \to \mathscr{G}X$ by

$$\begin{tikzcd} (\mathscr{G}Y)^{\S} \arrow[r, dashed, "\mathscr{G}'g"] & \mathscr{G}X \\ \mathscr{G}Y \arrow[u, "in_n"] \arrow[ur, "\mathscr{G}(\pi_n g)"'] & \end{tikzcd}$$

For $h\colon A \to B_{\S}$ and $k\colon X^{\S} \to Y$ we can also define morphisms $\mathscr{F}'h\colon (\mathscr{F}B)^{\S}\mathscr{F}A$ and $\mathscr{G}'k\colon \mathscr{G}Y \to (\mathscr{G}X)_{\S}$ as above.

We prove two fundamental lemmas about these extensions.

LEMMA 2. *Let A, B be in $\mathscr{K}$ and X, Y in $\mathscr{H}$. Then*

(i) *for $f\colon A^{\S} \to B$, $\mathscr{G}'\mathscr{F}'f \sim f$;*

(ii) *for $g\colon X \to Y_{\S}$, $\mathscr{F}'\mathscr{G}'g \sim g$;*

(iii) *for $h\colon A \to B_{\S}$, $\mathscr{G}'\mathscr{F}'h \sim h$;*

(iv) *for $k\colon X^{\S} \to Y$, $\mathscr{F}'\mathscr{G}'k \sim k$.*

Proof. (i) We note that if $in_n\colon A \to A^{\S}$ is a copower and $k\, in_n \sim l\, in_n$ *for all n, then* $k \sim l$.

We have

$$(\mathscr{G}'\mathscr{F}'f)\, in_n = \mathscr{G}(\pi_n \mathscr{F}'f)$$
$$= \mathscr{G} \circ \mathscr{F}^{\mathrm{op}}(f\, in_n)$$
$$\sim f\, in_n \qquad \text{for all } n.$$

Thus $\mathscr{G}'\mathscr{F}'f \sim f$

(ii) This is dual to (i) (repeat (i) in $\mathscr{A}^{\mathrm{op}}$, $\mathscr{K}^{\mathrm{op}}$, etc.).

(iii) and (iv) The proofs are similar to those of (i) and (ii) (repeat (ii) after interchanging $\mathscr{A}$ and $\mathscr{B}$, $\mathscr{K}$ and $\mathscr{H}$, etc.). ∎

LEMMA 3. *Let A, B, C be in $\mathscr{K}$, and X, Y, Z in $\mathscr{H}$. Then*

(i) *for $f\colon A^{\S} \to B$ and $t\colon B \to C$, $\mathscr{F}'(tf) = \mathscr{F}'(f)\mathscr{F}(t)$;*

(ii) *for $g\colon X \to Y_{\S}$ and $u\colon Z \to X$, $\mathscr{G}'(gu) = \mathscr{G}(u)\mathscr{G}'(g)$;*

(iii) *for $h\colon A \to B_{\S}$ and $v\colon C \to A$, $\mathscr{F}'(hv) = \mathscr{F}(v)\mathscr{F}'(h)$;*

(iv) *for $k\colon X^{\S} \to Y$ and $w\colon Y \to Z$, $\mathscr{G}'(wk) = \mathscr{G}'(k)\mathscr{G}(w)$.*

Proof. As in the case of the previous result, we have only to prove (i).
We have

$$\pi_n \mathscr{F}'(tf) = \mathscr{F}(tf\, in_n)$$
$$= \mathscr{F}(f\, in_n)\mathscr{F}(t)$$
$$= \pi_n \mathscr{F}'(f)\mathscr{F}(t) \qquad \text{for all } n.$$

Thus $\mathscr{F}'(tf) = \mathscr{F}'(f)\mathscr{F}(t)$. ∎

We consider the following conditions on the image-factorization systems $(\mathscr{E}_1, \mathscr{M}_1)$ and $(\mathscr{E}_2, \mathscr{M}_2)$.

A1. If $e\colon A \to B \in \mathscr{E}_1$ and A is in $\mathscr{K}$, then B is in $\mathscr{K}$.

A2. If $m\colon A \to B \in \mathscr{M}_1$ and B is in $\mathscr{K}$, then A is in $\mathscr{K}$.

A3. If $e\colon X \to Y \in \mathscr{E}_2$ and X is in $\mathscr{H}$, then Y is in $\mathscr{H}$.

A4. If $m\colon X \to Y \in \mathscr{M}_2$ and Y is in $\mathscr{H}$, then X is in $\mathscr{H}$.

In the vector space case A1 simply states that the image of a finite-dimensional space under a linear map is again finite-dimensional, while A2 essentially states that a subspace of a finite-dimensional space is finite-dimensional.

If we assume that these conditions hold, it is possible to extend the results of Lemma 1.

LEMMA 4. *Let $(\mathscr{E}_1, \mathscr{M}_1)$, $(\mathscr{E}_2, \mathscr{M}_2)$ be such that given $e\colon \mathscr{K}_1 \to \mathscr{K}_2$ in $\mathscr{K}$, $e \in \mathscr{E}_1$ if and only if $\mathscr{F}e \in \mathscr{M}_2$. Suppose A, B are in $\mathscr{K}$; X, Y in $\mathscr{H}$; and $f\colon A^{\S} \to B$, $g\colon X \to Y_{\S}$, $h\colon A \to B_{\S}$ and $k\colon X^{\S} \to Y$.*

If A2 *and* A3 *hold, then*

(i) $f \in \mathscr{E}_1$ *if and only if* $\mathscr{F}'f \in \mathscr{M}_2$;

(ii) $g \in \mathscr{M}_2$ *if and only if* $\mathscr{G}'g \in \mathscr{E}_1$.

If A1 *and* A4 *hold then*

(iii) $h \in \mathscr{M}_1$ *if and only if* $\mathscr{F}'h \in \mathscr{E}_2$;

(iv) $k \in \mathscr{E}_2$ *if and only if* $\mathscr{G}'k \in \mathscr{M}_1$.

Proof. (i) Let $f \in \mathscr{E}_1$. Consider the $(\mathscr{E}_2, \mathscr{M}_2)$ factorization of $\mathscr{F}'f$.

$$\begin{tikzcd} \mathscr{F}B \arrow[rr, "\mathscr{F}'f"] \arrow[dr, "e_2"'] & & (\mathscr{F}A)_{\S} \\ & Z \arrow[ur, "m_2"'] & \end{tikzcd}$$

By A3, Z is in $\mathscr{K}$, so that $\mathscr{G}e_2$ and $\mathscr{G}'m_2$ are defined.
We have

$$\begin{aligned} f &\sim \mathscr{G}'\mathscr{F}'f \\ &= \mathscr{G}'(m_2 e_2) \\ &= \mathscr{G}e_2 \mathscr{G}'m_2 \end{aligned}$$

By Lemma 1(iii), $\mathscr{G}e_2 \in \mathscr{M}_1$.

Since $\mathscr{G}e_2\mathscr{G}'m_2 \in \mathscr{E}_1$, we obtain $\mathscr{G}e_2 \in \mathscr{E}_1$ [1, Lemma 5.6]. Thus $\mathscr{G}e_2$ is an isomorphism, so that $e_2 \sim \mathscr{F} \circ \mathscr{G}^{\mathrm{op}}e_2$ is an isomorphism. Hence $\mathscr{F}'f \in \mathscr{M}_2$.

The proof of the converse is dual.

(ii) We have

$$\begin{aligned} &g \in \mathscr{M}_2 \text{ if anf only if } \mathscr{F}'\mathscr{G}'g \in \mathscr{M}_2 && \text{(Lemma 2(ii))} \\ &\text{if and only if } \mathscr{G}'g \neq \mathscr{E}_1 && \text{(by (i)).} \end{aligned}$$

The proof of the rest of the lemma is similar. ∎

A system $M = (Q, F, I, G, Y, H)$ is called *finite* if Q, I and Y are in $\mathscr{K}$. A similar notation is used for systems in $\mathscr{B}$. The dual of a finite system M in $\mathscr{A}[\mathscr{B}]$ is the finite system $\mathscr{F}M = (\mathscr{F}Q, \mathscr{F}F, \mathscr{F}Y, \mathscr{F}H, \mathscr{F}I, \mathscr{F}G)$ $[\mathscr{G}M = (\mathscr{G}Q, \mathscr{G}F, \mathscr{G}Y, \mathscr{G}H, \mathscr{G}I, \mathscr{G}G)]$ in $\mathscr{B}[\mathscr{A}]$.

Duality Theorem for Finitary Systems. *Let* $(\mathscr{E}_1, \mathscr{M}_1)$, $(\mathscr{E}_2, \mathscr{M}_2)$ *be such that* $e \in \mathscr{E}_1$ *if and only if* $\mathscr{F}e \in \mathscr{M}_2$. *Let* M *be a finitary system in* $\mathscr{A}$ *with reachability morphism* $r\colon I^{\S} \to Q$ *and observability morphism* $\sigma\colon Q \to Y_{\S}$. *Then*

(i) $\mathscr{F}'r\colon \mathscr{F}Q \to (\mathscr{F}I)_{\S}$ *is the observability morphism of* $\mathscr{F}M$:

(ii) $\mathscr{F}'\sigma\colon (\mathscr{F}Y)^{\S} \to \mathscr{F}Q$ *is the reachability morphism of* $\mathscr{F}M$;

(iii) *if* $\mathcal{M}_1, \mathcal{E}_2$ *satisfy* A2, A3, *then* M *is* $\mathcal{E}_1$*-reachable if and only if* $\mathcal{F}M$ *is* $\mathcal{M}_2$*-observable*;

(iv) *if* $\mathcal{M}_2, \mathcal{E}_1$ *satisfy* A1, A4, *then* M *is* $\mathcal{M}_1$*-observable if and only if* $\mathcal{F}M$ *is* $\mathcal{E}_2$*-reachable.*

Similarly if M *is a finitary system in* $\mathcal{B}$ *with reachability and observability morphisms* r *and* σ, *respectively, then*

(v) $\mathcal{G}'r\colon \mathcal{G}Q \to (\mathcal{G}I)_\S$ *is the observability morphism of* $\mathcal{G}M$;

(vi) $\mathcal{G}'\sigma\colon (\mathcal{G}Y)^\S \to \mathcal{G}Q$ *is the reachability morphism of* $\mathcal{G}M$;

(vii) *if* $\mathcal{M}_1, \mathcal{E}_2$ *satisfy* A2, A3, *then* M *is* $\mathcal{M}_2$*-observable if and only if* $\mathcal{G}M$ *is* $\mathcal{E}_1$*-reachable*;

(viii) *if* $\mathcal{M}_2, \mathcal{E}_1$ *satisfy* A1, A4, *then* M *is* $\mathcal{E}_2$*-reachable if and only if* $\mathcal{G}M$ *is* $\mathcal{M}_1$*-observable.*

Proof. (i) We have

$$\pi_0\mathcal{F}'r = \mathcal{F}(r\, in_0) = \mathcal{F}(G)$$

and

$$\begin{aligned}\pi_{n+1}\mathcal{F}'r &= \mathcal{F}(r\, in_{n+1})\\ &= \mathcal{F}(rz\, in_n)\\ &= \mathcal{F}(Fr\, in_n)\\ &= \mathcal{F}(r\, in_n)\mathcal{F}(F)\\ &= \pi_n\mathcal{F}'r\mathcal{F}(F).\end{aligned}$$

Thus $\mathcal{F}'r$ is the observability morphism of $\mathcal{F}M$. The proof of (ii) is similar.

(iii) M is $\mathcal{E}_1$-reachable if and only if $r \in \mathcal{E}_1$ if and only if $\mathcal{G}'r \in \mathcal{M}_2$ (Lemma 4(i)) if and only if $\mathcal{F}M$ is $\mathcal{M}_2$-observable.

The proofs of assertions (iv)–(viii) are similar. ∎

A category is called self-dual if it is dual equivalent to itself. The following special case of the above result is worth noting.

COROLLARY. *Let* $\mathcal{A}$ *be a category with countable powers and copowers. Let* $\mathcal{K}$ *be a self-dual (full) subcategory of* $\mathcal{A}$, *the dual equivalence being given by the functors* $\mathcal{F}\colon \mathcal{K}^{\mathrm{op}} \to \mathcal{K}$, $\mathcal{G}\colon \mathcal{K}^{\mathrm{op}} \to \mathcal{K}$. *Suppose* $(\mathcal{E}_1, \mathcal{M}_1)$ *and* $(\mathcal{E}_2, \mathcal{M}_2)$ *are image-factorization systems for* $\mathcal{A}$ *such that* $e \in \mathcal{E}_1$ *if and only if* $\mathcal{F}e \in \mathcal{M}_2$.

If $\mathcal{E}_2, \mathcal{M}_1$ *satisfy* A2, A3, *then for a* $\mathcal{K}$*-finite system* M *in* $\mathcal{A}$ *we have*:

M *is* $\mathcal{E}_1$*-reachable if and only if* $\mathcal{F}M$ *is* $\mathcal{M}_2$*-observable*;
M *is* $\mathcal{M}_2$*-observable if and only if* $\mathcal{G}M$ *is* $\mathcal{E}_1$*-reachable.*

A similar result holds when $\mathcal{E}_1, \mathcal{M}_2$ *satisfy* A1, A4.

Image factorization systems satisfying the condition "$e \in \mathscr{E}_1$ if and only if $\mathscr{F}e \in \mathscr{M}_2$" for $e: A \to B$ with finite A and B can in general be obtained as follows. Given a "standard" image-factorization system $(\mathscr{E}_1, \mathscr{M}_1)$ for the category (such as (coequalizers, monomorphisms)) let $(\mathscr{E}_2, \mathscr{M}_2)$ be the corresponding dual system (i.e., epimorphisms, equalizers)).

EXAMPLES

(1) *Finite-Dimensional Linear Systems*

The well-known Kalman duality theory for finite-dimensional linear systems is obtained from the above Corollary in the following way. Let $\mathscr{A}$ be the category with all vector spaces over a field K as objects and all linear transformations as morphisms. The full subcategory of all finite-dimensional spaces is self-dual (the dual of a space V is the space of all linear functionals on V). We choose $\mathscr{E}_1 = \mathscr{E}_2 =$ surjective linear transformations, and $\mathscr{M}_1 = \mathscr{M}_2 =$ all injective linear transformations. The condition "$e \in \mathscr{E}_1$ if and only if $\mathscr{F}e \in \mathscr{M}_2$" as well as conditions A1–A4 are satisfied, and we may apply the corollary. "$\mathscr{E}_1$-reachability, $\mathscr{M}_2$-observability," etc., are just the usual concepts of reachability and observability for finite-dimensional systems.

We note that in [1] a vector space V and its dual V^* are identified since they are isomorphic if V is a space over the field of real numbers. Over the field of complex numbers the isomorphism $V \to V^*$ is anti-linear, i.e., it is not an isomorphism in the category of complex vector spaces. The results in [1] are therefore not directly applicable to complex systems. This restriction is eliminated in the above result.

(2) *Infinite-Dimensional Linear Systems in Banach Space*

All the information necessary to apply the corollary to systems in Banach spaces is given in [1]. Let $\mathscr{A}$ be the category with all Banach spaces as objects (the spaces may be either all real or all complex). As morphisms we take all linear maps f: $A \to B$ with the contractive property $\|fa\| \leqslant \|a\|$.

The countable copower $A^{\S}$ is the space of all those sequences (a_i) for which

$$\|(a_i)\| = \sum \|a_i\| < \infty$$

with the usual injections. The countable power $A_{\S}$ is the space of all sequences (a_i) which satisfy

$$\sup\{\|a_i\|: i = 1, 2, \ldots\} < \infty$$

with this supremum as norm. The projections are the restrictions of the usual coordinate projections.

As our subcategory $\mathscr{K}$ of finite objects we take the full subcategory of all reflexive Banach spaces. The dual equivalence $\mathscr{F}$ assigns to each reflexive space A its dual A^* and to each f: $A \to B$ its dual f^*: $B^* \to A^*$.

As image-factorization systems we take (epimorphisms, equalizers) and (coequalizers, monomorphisms), i.e., we take

$\mathscr{E}_1 =$ all $e: A \to B$ such that $e(A)$ is dense in B;

$\mathscr{M}_1 =$ all $m: A \to B$ such that M is an isometry of A onto a closed subspace of B;

$\mathscr{E}_2 =$ all $e: A \to B$ which are surjective and for which the norm on B is the quotient norm $\|b\| = \inf\{\|a\|: e(a) = b\}$;

$\mathscr{M}_2 =$ all $m: A \to B$ which are injective.

$\mathscr{M}_1$ satisfies A2, since a closed subspace of a reflexive space is again reflexive. Similarly $\mathscr{E}_2$ satisfies A3, since the quotient of a reflexive space is again reflexive [3, p. 70]. Also, as we noted before starting with our examples, $e \in \mathscr{E}_1$ if and only if $\mathscr{F}e \in \mathscr{M}_2$. Thus we can apply the corollary.

The various concepts of reachability and observability need some comment. Thus, for instance, $\mathscr{E}_1$-reachability means that we can reach a dense subspace of the state space, while $\mathscr{M}_2$-observability is just the usual concept of observability, i.e., different states produce different output sequences.

(3) *Infinite-Dimensional Systems in Linearly Topologized Spaces*

In [4] it is shown that the category $K - \mathscr{D}\mathscr{P}$ of dual pairs over K, the category $sK - \mathscr{L}\mathscr{C}\mathscr{S}$ of weak linearly topologized spaces, and also the category $kK - \mathscr{L}\mathscr{C}\mathscr{S}$, the category of Mackey linearly topologized spaces, are all self-dual categories, each with countable powers and copowers. Several image-factorization systems for these categories are exhibited. The corresponding duality theory in [4] follows from the above corollary by taking $\mathscr{A} = \mathscr{K} = K - \mathscr{D}\mathscr{P}$, etc. Conditions A1–A4 are then trivially satisfied.

It is also shown that the categories $K - \mathscr{L}\mathscr{S}$ of linear spaces over the field K and $cK - \mathscr{L}\mathscr{C}\mathscr{S}$ of linearly compact, linearly topologized spaces are dual equivalent and both have countable powers and copowers. In this case there is only a single image-factorization system for each category. The resulting systems duality is obtained by taking $\mathscr{A} = \mathscr{K} = K - \mathscr{L}\mathscr{S}$, $\mathscr{B} = \mathscr{H} = cK - \mathscr{L}\mathscr{C}\mathscr{S}$, etc.

(4) *Linear Systems over Rings*

An example where the categories $\mathscr{A}$ and $\mathscr{B}$ in our duality theory are not the same is obtained by considering linear systems over non-commutative rings. Given a non-commutative ring R we let $\mathscr{A}$ be the category of left R-modules, $\mathscr{B}$ the category of right R-modules and $\mathscr{K}$ and $\mathscr{H}$ finitely generated left and right R-modules, respectively. Details of the resulting duality theory will appear in a separate paper [7].

REFERENCES

1. M. A. ARBIB AND E. G. MANES, Foundations of system theory: Decomposable systems, *Automatica* **10** (1974), 285–302.
2. W. S. CHING AND B. F. WYMAN, Duality and the regulator problem for linear systems over commutative rings, *J. Comput. System Sci.* **14** (1977), 360–368.
3. M. M. DAY, "Normed Linear Spaces," Springer-Verlag, Berlin/New York, 1973.
4. S. J. HEGNER, Duality theory for discrete-time linear systems, *J. Comput. System Sci.* **17** (1978), 116–143.
5. H. HERRLICH AND G. E. STRECKER, "Category Theory," Allyn & Bacon, Boston, 1973.
6. R. E. KALMAN, Introduction to the algebraic theory of linear dynamical systems, *in* "Mathematical Systems Theory and Economics" (H. W. Kuhn and G. P. Szegö, Eds.), Vol. 1, Springer-Verlag, New York (1969), 41–65.
7. G. NAUDÉ AND C. NOLTE, A duality theory for linear systems over rings, submitted for publication.
8. J. RISSANEN AND B. WYMAN, Duals of input/output maps, *in* "Category Theory Applied to Computation and Control" (Proceedings of the First International Symposium, San Francisco, February 25–26, 1974) (E.G. Manes, Ed.), Lecture Notes in Computer Science No. **25**, Springer-Verlag, New York, 1975.