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Universal deformation rings for the symmetric group S_5 and one of its double covers

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ABSTRACT

Let S_5 denote the symmetric group on 5 letters, and let $\hat{S}_5$ denote a non-trivial double cover of S_5 whose Sylow 2-subgroups are generalized quaternion groups. We determine the universal deformation rings $R(S_5, V)$ and $R(\hat{S}_5, V)$ of each mod 2 representation V of S_5 that belongs to the principal 2-modular block of S_5 and whose stable endomorphism ring is given by scalars when it is inflated to $\hat{S}_5$. We show that for these V, a question raised by the first author and Chinburg concerning the relation of the universal deformation ring of V to the Sylow 2-subgroups of S_5 and $\hat{S}_5$, respectively, has an affirmative answer.

1. Introduction

Let k be an algebraically closed field of characteristic $p > 0$ and let $W = W(k)$ be the ring of infinite Witt vectors over k. Let G be a finite group, and suppose V is a finitely generated kG-module. It was proved in [5] that if the stable endomorphism ring $\underline{\mathrm{End}}_{kG}(V)$ is one-dimensional over k then V has a universal deformation ring $R(G, V)$. The ring $R(G, V)$ is universal with respect to deformations of V over complete local commutative Noetherian rings with residue field k (for details, see Section 2). In [2–7], the isomorphism types of $R(G, V)$ have been determined for V belonging to cyclic blocks, respectively to various tame blocks with dihedral defect groups. In the present paper, we will consider the principal 2-modular blocks of the symmetric group S_5 and one of its double covers $\hat{S}_5$ whose Sylow 2-subgroups are generalized quaternion groups. One of the main goals is to investigate how the universal deformation rings change when inflating modules from S_5 to $\hat{S}_5$. The key tools used to determine the universal deformation rings in all the above cases have been the results from modular and ordinary representation theory due to Brauer, Erdmann [13], Linckelmann [19], Carlson–Thévenaz [9], and others.

The main motivation for studying universal deformation rings for finite groups is that this case helps one understand ring theoretic properties of universal deformation rings for profinite groups Γ. The latter have become an important tool in number theory, in particular if Γ is a profinite Galois group (see e.g. [10] and its references). In [12], de Smit and Lenstra showed that if Γ is an arbitrary profinite group and V is a finite-dimensional vector space over k with a continuous Γ-action which has a universal deformation ring $R(\Gamma, V)$, then $R(\Gamma, V)$ is the inverse limit of the universal deformation rings $R(G, V)$ when G ranges over all finite discrete quotients of Γ through which the Γ-action on V factors. Thus to answer questions about the ring structure of $R(\Gamma, V)$, it is natural to first consider the case when $\Gamma = G$ is finite.

Suppose now that the characteristic of k is 2 and that S_5 and $\hat{S}_5$ are as above. The Sylow 2-subgroups of S_5 are dihedral groups of order 8, whereas the Sylow 2-subgroups of $\hat{S}_5$ are generalized quaternion groups of order 16. The center Z of $\hat{S}_5$ has 2 elements and $\hat{S}_5/Z \cong S_5$. Since Z acts trivially on the simple $k\hat{S}_5$-modules, they are all inflated from simple kS_5-modules.

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Moreover, the simple modules belonging to the principal block of $k\hat{S}_5$ are inflated from the simple kS_5-modules belonging to the principal block of kS_5. There are precisely two isomorphism classes of simple kS_5-modules belonging to the principal block of kS_5. They are represented by the trivial simple module T_0 and a 4-dimensional simple module T_1.

Our main result is as follows, where $W[\mathbb{Z}/2]$ denotes the group ring over W of the cyclic group $\mathbb{Z}/2$.

Theorem 1.1. *Let B (resp. $\hat{B}$) be the principal block of kS_5 (resp. $k\hat{S}_5$). Let V be an indecomposable kS_5-module belonging to B, and denote its inflation to $k\hat{S}_5$ also by V, so V belongs to both B and $\hat{B}$.*

(a) *Then $\underline{\mathrm{End}}_{k\hat{S}_5}(V) \cong k$ if and only if $\underline{\mathrm{End}}_{kS_5}(V) \cong k$. Moreover, we have $\underline{\mathrm{End}}_{kS_5}(V) \cong k$ if and only if V is either isomorphic to T_0 or a uniserial kS_5-module whose radical series length is at most 3 and which is a submodule or a quotient module of the projective kS_5-cover of T_1.*

(b) *Suppose $\underline{\mathrm{End}}_{kS_5}(V) \cong k$.*
   (i) *If $V \cong T_0$, then $R(S_5, V) \cong W[\mathbb{Z}/2] \cong R(\hat{S}_5, V)$.*
   (ii) *If $V \cong T_1$, then $R(S_5, V) \cong k$ and $R(\hat{S}_5, V) \cong W$.*
   (iii) *If the radical series length of V is 2, then $R(S_5, V) \cong W[\mathbb{Z}/2] \cong R(\hat{S}_5, V)$.*
   (iv) *If the radical series length of V is 3, then $R(S_5, V) \cong W[[t]]/(t^2, 2t)$ and $R(\hat{S}_5, V) \cong W[[t]]/(t^3 - 2t)$.*

In particular, the universal deformation rings $R(\hat{S}_5, V)$ are all complete intersection rings, whereas for V as in part (iv), $R(S_5, V)$ is not a complete intersection. Note that for all cases (i)–(iv), $R(S_5, V)$ (resp. $R(\hat{S}_5, V)$) is isomorphic to a subquotient ring of WD_8 (resp. WQ_{16}) when D_8 is a dihedral group of order 8 (resp. Q_{16} is a generalized quaternion group of order 16). In particular, this gives a positive answer in these cases to a question raised by the first author and Chinburg in [5, Question 1.1] whether the universal deformation ring of a representation of a finite group whose stable endomorphism ring is isomorphic to k is always isomorphic to a subquotient ring of the group ring over W of a defect group of the modular block associated to the representation.

This paper is organized as follows. In Section 2, we give some background on universal deformation rings. In Section 3, we state properties of the principal 2-modular block B (resp. $\hat{B}$) of S_5 (resp. $\hat{S}_5$) and prove part (a) of Theorem 1.1. In Section 4, we determine the universal deformation rings of the B-modules whose endomorphism rings are isomorphic to k and of their inflations to $\hat{B}$ and prove part (b) of Theorem 1.1. In the Appendix, we list the ordinary and the 2-modular character table of $\hat{S}_5$.

This paper is based on the Ph.D. thesis of the second author under the supervision of the first author [15]. We would like to thank the referee for helpful comments.

2. Preliminaries

Let k be an algebraically closed field of characteristic $p > 0$, let W be the ring of infinite Witt vectors over k and let F be the fraction field of W. Let $\mathcal{C}$ be the category of all complete local commutative Noetherian rings with residue field k. The morphisms in $\mathcal{C}$ are continuous W-algebra homomorphisms which induce the identity map on k.

Suppose G is a finite group and V is a finitely generated kG-module. A lift of V over an object R in $\mathcal{C}$ is a finitely generated RG-module M which is free over R together with a kG-module isomorphsim $\phi : k \otimes_R M \to V$. Two lifts (M, ϕ) and (M', ϕ') of V over R are isomorphic if there is an RG-module isomorphism $f : M \to M'$ such that $\phi' \circ (k \otimes_R f) = \phi$. The isomorphism class of a lift of V over R is called a deformation of V over R, and the set of such deformations is denoted by $\mathrm{Def}_G(V, R)$. The deformation functor $F_V : \mathcal{C} \to \mathrm{Sets}$ is defined to be the covariant functor which sends an object R in $\mathcal{C}$ to $\mathrm{Def}_G(V, R)$.

If there exists an object $R(G, V)$ in $\mathcal{C}$ and a lift $(U(G, V), \phi_U)$ of V over $R(G, V)$ such that for each R in $\mathcal{C}$ and for each lift (M, ϕ) of V over R there is a unique morphism $\alpha : R(G, V) \to R$ in $\mathcal{C}$ such that (M, ϕ) is isomorphic to $(R \otimes_{R(G,V),\alpha} U(G, V), \phi_U)$, then $R(G, V)$ is called the universal deformation ring of V and the isomorphism class of the lift $(U(G, V), \phi_U)$ is called the universal deformation of V. In other words, $R(G, V)$ represents the functor F_V in the sense that F_V is naturally isomorphic to $\mathrm{Hom}_{\mathcal{C}}(R(G, V), -)$. For more information on deformation rings see [12,20].

Suppose V has a universal deformation ring $R(G, V)$ and a universal lift $(U(G, V), \phi_U)$ over $R(G, V)$ that represents the universal deformation of V. Then we call $\overline{R} = R(G, V)/p\,R(G, V)$ the universal mod p deformation ring of V and we call the isomorphism class of the lift $(\overline{R} \otimes_{R(G,V)} U(G, V), \phi_U)$ the universal mod p deformation of V. Note that $\overline{R}$ represents the restriction of the deformation functor F_V to the full subcategory of $\mathcal{C}$ of objects that are k-algebras.

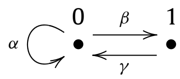
The following two results were proved in [5]. Here Ω denotes the syzygy, or Heller, operator for kG (see for example [1, Section 20]).

Proposition 2.1 ([5, Prop. 2.1])**.** *Suppose V is a finitely generated kG-module whose stable endomorphism ring $\underline{\mathrm{End}}_{kG}(V)$ is isomorphic to k. Then V has a universal deformation ring $R(G, V)$.*

Lemma 2.2 ([5, Cor. 2.5])**.** *Let V be a finitely generated kG-module with $\underline{\mathrm{End}}_{kG}(V) \cong k$. Then $\underline{\mathrm{End}}_{kG}(\Omega(V)) \cong k$, and $R(G, V)$ and $R(G, \Omega(V))$ are isomorphic.*

$$
\begin{array}{c|cc}
 & \varphi_0 & \varphi_1 \\
\chi_1 & 1 & 0 \\
\chi_2 & 1 & 0 \\
\chi_3 & 1 & 1 \\
\chi_4 & 1 & 1 \\
\chi_5 & 2 & 1
\end{array}
\qquad
\left(\text{resp.}\quad
\begin{array}{c|cc}
 & \varphi_0 & \varphi_1 \\
\psi_1 & 1 & 0 \\
\psi_2 & 1 & 0 \\
\psi_3 & 1 & 1 \\
\psi_4 & 1 & 1 \\
\psi_5 & 0 & 1 \\
\psi_6 & 2 & 1 \\
\psi_7 & 2 & 1 \\
\psi_8 & 2 & 1
\end{array}\right).
$$

Fig. 1. The decomposition matrix for B (resp. for $\hat{B}$).

$Q =$ 

$$
\begin{aligned}
I_c &= \langle \beta\gamma, \alpha^2 - c(\gamma\beta\alpha)^2, (\gamma\beta\alpha)^2 - (\alpha\gamma\beta)^2 \rangle, \\
\hat{I}_d &= \langle \gamma\beta\gamma - \alpha\gamma(\beta\alpha\gamma)^3, \beta\gamma\beta - \beta\alpha(\gamma\beta\alpha)^3, \alpha^2 - \gamma\beta(\alpha\gamma\beta)^3 - d(\alpha\gamma\beta)^4, \beta\alpha^2 \rangle.
\end{aligned}
$$

Fig. 2. The algebras $\Lambda_c = kQ/I_c$ ($c \in \{0, 1\}$) and $\hat{\Lambda}_d = kQ/\hat{I}_d$ ($d \in k$).

3. The principal 2-modular blocks of S_5 and $\hat{S}_5$

Let k be an algebraically closed field of characteristic 2, let W be the ring of infinite Witt vectors over k and let F be the fraction field of W.

Let B (resp. $\hat{B}$) be the principal block of kS_5 (resp. of $k\hat{S}_5$). Then the defect groups of B (resp. of $\hat{B}$) are dihedral groups of order 8 (resp. generalized quaternion groups of order 16). Looking at the ordinary and the 2-modular character table of $\hat{S}_5$ (see the Appendix), we see that the decomposition matrix for B (resp. for $\hat{B}$) is as in Fig. 1.

Remark 3.1. The field F is a splitting field for S_5. It follows from the ordinary character table of $\hat{S}_5$ in Fig. 5 and from [14, Thm. A] that the Schur indices of all irreducible characters of $\hat{S}_5$ with respect to F are 1. Hence the characters $\psi_1, \psi_2, \ldots, \psi_6$ (resp. ψ_7, ψ_8) correspond to irreducible representations of $\hat{S}_5$ which are realizable over F (resp. over $F(\sqrt{2})$). Moreover, ψ_7, ψ_8 are conjugate under the action of the Galois group of $F(\sqrt{2})$ over F. Hence the characters of the irreducible representations of $\hat{S}_5$ over F which belong to $\hat{B}$ are

$$\psi_1, \psi_2, \ldots, \psi_5, \psi_6, \psi_7 + \psi_8.$$

If V_6 (resp. V_{78}) is the $F\hat{S}_5$-module whose character is ψ_6 (resp. $\psi_7 + \psi_8$), then $\mathrm{End}_{F\hat{S}_5}(V_6) \cong F$ and $\mathrm{End}_{F\hat{S}_5}(V_{78}) \cong F(\sqrt{2})$.

Using the decomposition matrices in Fig. 1, it follows from [13, p. 294 and p. 303] that there exist $c \in \{0, 1\}$ and $d \in k$ such that B (resp. $\hat{B}$) is Morita equivalent to $\Lambda_c = kQ/I_c$ (resp. $\hat{\Lambda}_d = kQ/\hat{I}_d$) as described in Fig. 2. For the vertices 0, 1 in Q, the radical series of the corresponding projective indecomposable Λ_c-modules P_0, P_1 and the corresponding projective indecomposable $\hat{\Lambda}_d$-modules $\hat{P}_0, \hat{P}_1$ are described in Fig. 3.

Remark 3.2. Let z be the non-trivial central element in $\hat{S}_5$ and let $Z = \langle z \rangle$ be the center of $\hat{S}_5$. In the following, we identify S_5 with $\hat{S}_5/Z$. Let $\pi : k\hat{S}_5 \to kS_5$ be the natural projection given by $\pi(g) = gZ$ for all $g \in \hat{S}_5$. Since Z acts trivially on the simple $k\hat{S}_5$-modules, we can identify the simple kS_5-modules with the simple $k\hat{S}_5$-modules. This implies that the restriction of π to $\hat{B}$ gives a surjective k-algebra homomorphism $\pi_B : \hat{B} \to B$. In particular, if V is a kS_5-module belonging to B, then its inflation to $k\hat{S}_5$ via π belongs to $\hat{B}$. Let $\hat{e}$ be a sum of orthogonal primitive idempotents in $\hat{B}$ such that $\hat{e}\hat{B}\hat{e}$ is basic and Morita equivalent to $\hat{B}$, and let $e = \pi_B(\hat{e})$. Then eBe is basic and Morita equivalent to B, and the restriction of π_B to $\hat{e}\hat{B}\hat{e}$ gives a surjective k-algebra homomorphism $\pi_e : \hat{e}\hat{B}\hat{e} \to eBe$.

If c, d are such that B is Morita equivalent to Λ_c and $\hat{B}$ is Morita equivalent to $\hat{\Lambda}_d$, let $\Lambda = \Lambda_c$ and $\hat{\Lambda} = \hat{\Lambda}_d$. Then $\hat{e}\hat{B}\hat{e} \cong \hat{\Lambda}$ and $eBe \cong \Lambda$, and π_e induces a surjective k-algebra homomorphism $\pi_\Lambda : \hat{\Lambda} \to \Lambda$. It follows from the description of the projective indecomposable Λ-modules P_0, P_1 and the projective indecomposable $\hat{\Lambda}$-modules $\hat{P}_0, \hat{P}_1$ in Fig. 3 that $\Lambda \otimes_{\hat{\Lambda},\pi_\Lambda} \hat{P}_i \cong P_i$ for $i \in \{0, 1\}$. In other words, the simple $\hat{\Lambda}$-module $\hat{P}_i/\mathrm{rad}(\hat{P}_i)$ is isomorphic to the inflation via π_Λ of the simple Λ-module $P_i/\mathrm{rad}(P_i)$ for $i \in \{0, 1\}$.

Let $S_0 = P_0/\mathrm{rad}(P_0)$ and $S_1 = P_1/\mathrm{rad}(P_1)$. Then S_0 corresponds to the trivial simple kS_5-module T_0, and S_1 corresponds to the 4-dimensional simple kS_5-module T_1 which is inflated from either one of the two 2-dimensional simple kA_5-modules.

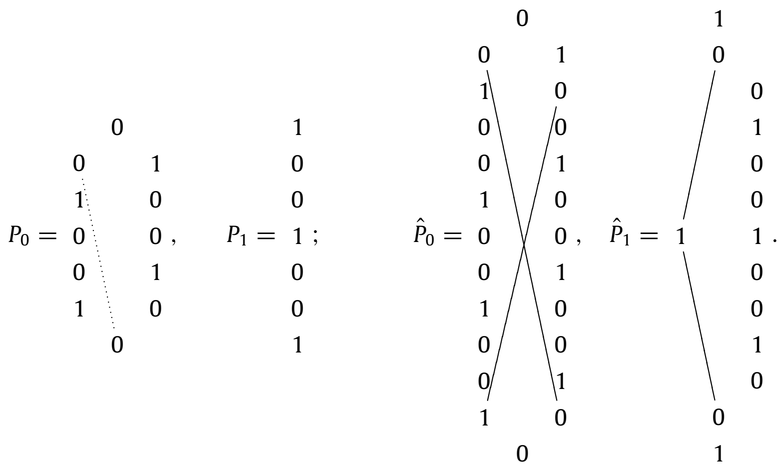


Fig. 3. The projective indecomposable Λ_c-modules P_0, P_1 and the projective indecomposable $\hat{\Lambda}_d$-modules $\hat{P}_0, \hat{P}_1$.

The inflation of T_0 (resp. T_1) to $k\hat{S}_5$ via π corresponds to the inflation of S_0 (resp. S_1) to $\hat{\Lambda}$ via π_Λ. In particular, the former inflations are simple $k\hat{S}_5$-modules, which we again denote by T_0 and T_1, and the latter inflations are simple $\hat{\Lambda}$-modules, which we again denote by S_0 and S_1.

We are now ready to prove part (a) of Theorem 1.1. We assume the above notation.

Proof of Part (a) of Theorem 1.1. Let V be an indecomposable kS_5-module belonging to B, and denote its inflation to $k\hat{S}_5$ also by V. By Higman's criterion (see [16, Thm. 1]), the $k\hat{S}_5$-module endomorphisms of V that factor through projective $k\hat{S}_5$-modules are precisely those in the image of the trace map $\mathrm{Tr}_1^{\hat{S}_5} : \mathrm{End}_k(V) \to \mathrm{End}_{k\hat{S}_5}(V)$, where $\mathrm{Tr}_1^{\hat{S}_5}(\psi)(v) = \sum_{g\in\hat{S}_5} g\,\psi(g^{-1}v)$ for all $\psi \in \mathrm{End}_k(V)$ and all $v \in V$. Because Z acts trivially on V, Tr_1^Z is multiplication by 2. Hence Tr_1^Z is zero, which implies that $\mathrm{Tr}_1^{\hat{S}_5} = \mathrm{Tr}_Z^{\hat{S}_5} \circ \mathrm{Tr}_1^Z$ is also zero. It follows that $\underline{\mathrm{End}}_{k\hat{S}_5}(V) \cong \mathrm{End}_{kS_5}(V)$. In particular, $\underline{\mathrm{End}}_{k\hat{S}_5}(V) \cong k$ if and only if $\mathrm{End}_{kS_5}(V) \cong k$.

Suppose now that $\mathrm{End}_{kS_5}(V) \cong k$. Then V corresponds under the Morita equivalence between B and Λ to an indecomposable Λ-module M whose endomorphism ring is isomorphic to k. It follows from the description of the projective indecomposable Λ-modules P_0 and P_1 in Fig. 3 that M cannot be projective. Therefore, M is inflated from an indecomposable $\Lambda/\mathrm{soc}(\Lambda)$-module whose endomorphism ring is isomorphic to k. Since $\Lambda/\mathrm{soc}(\Lambda)$ is a string algebra, all its indecomposable modules can be described using strings and bands (see for example [8]). It follows that a complete list of isomorphism classes of Λ-modules whose endomorphism rings are isomorphic to k is given by the following 6 uniserial Λ-modules which are uniquely determined, up to isomorphism, by their descending radical series:

$$S_0, S_1, \begin{matrix}0\\1\end{matrix}, \begin{matrix}1\\0\end{matrix}, \begin{matrix}0\\0\\1\end{matrix}, \begin{matrix}1\\0\\0\end{matrix}. \tag{3.1}$$

This completes the proof of part (a) of Theorem 1.1. □

Remark 3.3. Since $\mathrm{Ext}^1_{\hat{\Lambda}}(S_0, S_1) \cong k \cong \mathrm{Ext}^1_{\hat{\Lambda}}(S_1, S_0)$, there is up to isomorphism a unique uniserial $\hat{\Lambda}$-module with descending composition factors (S_0, S_1) (resp. (S_1, S_0)), which we denote by M_{01} (resp. M_{10}). It follows that the inflations via π_Λ of the two-dimensional Λ-modules in the list (3.1) are isomorphic to M_{01} or M_{10}.

Because $\mathrm{Ext}^1_{\hat{\Lambda}}(S_0, M_{01}) \cong k \cong \mathrm{Ext}^1_{\hat{\Lambda}}(M_{10}, S_0)$, there is up to isomorphism a unique uniserial $\hat{\Lambda}$-module with descending composition factors (S_0, S_0, S_1) (resp. (S_1, S_0, S_0)), which we denote by M_{001} (resp. M_{100}). It follows that the inflations via π_Λ of the three-dimensional Λ-modules in the list (3.1) are isomorphic to M_{001} or M_{100}.

4. Universal deformation rings

In this section we prove part (b) of Theorem 1.1. We assume the notation from Section 3. In particular, k is an algebraically closed field of characteristic 2, and B (resp. $\hat{B}$) is the principal block of kS_5 (resp. of $k\hat{S}_5$). We need the following lemma.

Lemma 4.1. *Suppose Λ, $\hat{\Lambda}$ and π_Λ are as in Remark 3.2. Let M be one of the two uniserial $\hat{\Lambda}$-modules M_{001} or M_{100} defined in Remark 3.3. Then* $\mathrm{Ext}^2_{\hat{\Lambda}}(M, M) \cong k$.

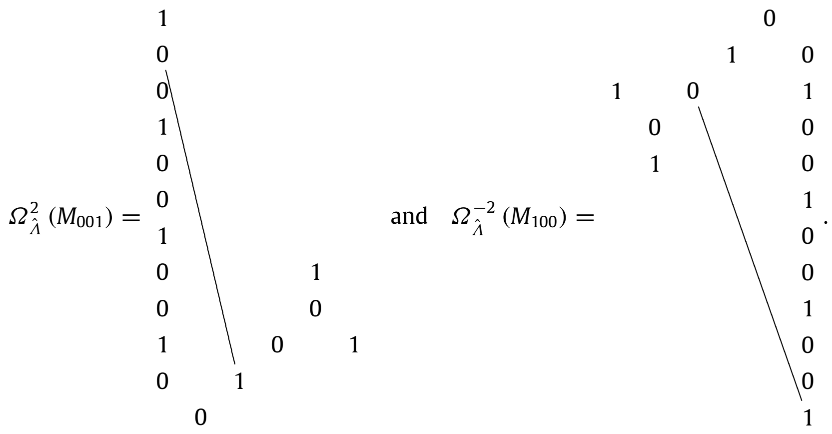


Fig. 4. The syzygies $\Omega^2_{\hat{\Lambda}}(M_{001})$ and $\Omega^{-2}_{\hat{\Lambda}}(M_{100})$.

Proof. It follows from the description of the projective indecomposable $\hat{\Lambda}$-modules in Fig. 3 that $\Omega^2_{\hat{\Lambda}}(M_{001})$ and $\Omega^{-2}_{\hat{\Lambda}}(M_{100})$ can be described as in Fig. 4. This implies that

$$\mathrm{Hom}_{\hat{\Lambda}}(\Omega^2_{\hat{\Lambda}}(M_{001}), M_{001}) \cong k^2 \quad \text{and} \quad \mathrm{Hom}_{\hat{\Lambda}}(M_{100}, \Omega^{-2}_{\hat{\Lambda}}(M_{100})) \cong k^2.$$

Since in both cases there is a one-dimensional subspace of these Hom spaces consisting of homomorphisms factoring through $\hat{P}_0$, we obtain

$$\mathrm{Ext}^2_{\hat{\Lambda}}(M_{001}, M_{001}) \cong \underline{\mathrm{Hom}}_{\hat{\Lambda}}(\Omega^2_{\hat{\Lambda}}(M_{001}), M_{001}) \cong k$$

and

$$\mathrm{Ext}^2_{\hat{\Lambda}}(M_{100}, M_{100}) \cong \underline{\mathrm{Hom}}_{\hat{\Lambda}}(M_{100}, \Omega^{-2}_{\hat{\Lambda}}(M_{100})) \cong k. \quad \square$$

Proof of Part (b) of Theorem 1.1. We go through the four different cases in the statement of the theorem.
Case (i). Since the maximal abelian 2-quotient group of both S_5 and $\hat{S}_5$ is a cyclic group of order 2, it follows e.g. from [20, Section 1.4] that $R(S_5, T_0) \cong W[\mathbb{Z}/2] \cong R(\hat{S}_5, T_0)$.
Case (ii). Let E be one of the two non-isomorphic 2-dimensional simple kA_5-modules, where A_5 denotes the alternating group on 5 letters which is a subgroup of S_5. Then T_1 is isomorphic to the induction $\mathrm{Ind}_{A_5}^{S_5} E$. It follows from the description of the projective indecomposable Λ-modules (resp. $\hat{\Lambda}$-modules) in Fig. 3 that $\mathrm{Ext}^1_{kS_5}(T_1, T_1) = 0 = \mathrm{Ext}^1_{k\hat{S}_5}(T_1, T_1)$. Hence by [4, Prop. 2.1.3] and by [2, Lemma 3.5(c)], we have $R(S_5, T_1) \cong k$. Since it can be seen from the decomposition matrix for $\hat{B}$ in Fig. 1 that T_1 when viewed as a $k\hat{S}_5$-module has a lift over W, we have $R(\hat{S}_5, T_1) \cong W$.

Case (iii). Suppose $V \in \left\{ \begin{smallmatrix} T_0 \\ T_1 \end{smallmatrix}, \begin{smallmatrix} T_1 \\ T_0 \end{smallmatrix} \right\}$. It follows from the description of the projective indecomposable Λ-modules in Fig. 3 that

$$\mathrm{Ext}^1_{kS_5}(V, V) \cong \underline{\mathrm{Hom}}_B(\Omega_B(V), V) \cong k$$

where Ω_B denotes the syzygy in the category of finitely generated B-modules. Moreover, there is a non-split short exact sequence of kS_5-modules $0 \to V \to U \to V \to 0$ where

$$U = \begin{array}{ccc} & T_0 & \\ T_0 & & T_1 \\ T_1 & & \end{array} \text{ if } V = \begin{array}{c} T_0 \\ T_1 \end{array}, \quad \text{and} \quad U = \begin{array}{ccc} T_1 & & \\ T_0 & & T_1 \\ & T_0 & \end{array} \text{ if } V = \begin{array}{c} T_1 \\ T_0 \end{array}. \tag{4.2}$$

Let C be the cyclic subgroup of S_5 of order 2 generated by the transposition $(1, 2)$. Since $T_1 \cong \mathrm{Ind}_{A_5}^{S_5} E$, it follows that $\mathrm{Res}_C^{S_5} T_1$ is a projective kC-module, and hence isomorphic to $kC \oplus kC$. Moreover, if T_{00} is the uniserial kS_5-module $T_{00} = \begin{smallmatrix} T_0 \\ T_0 \end{smallmatrix}$ then $\mathrm{Res}_C^{S_5} T_{00}$ cannot be trivial since $\mathrm{Res}_{A_5}^{S_5} T_{00}$ is trivial. Hence $\mathrm{Res}_C^{S_5} T_{00} \cong kC$. This means that

$$\mathrm{Res}_C^{S_5} V \cong k \oplus (kC)^2, \quad \text{and} \quad \mathrm{Res}_C^{S_5} U \cong (kC)^5.$$

Thus $\mathrm{Res}_C^{S_5} V$ is a kC-module whose stable endomorphism ring is isomorphic to k and whose universal deformation ring is $R(C, \mathrm{Res}_C^{S_5} V) \cong W[\mathbb{Z}/2]$. Let $(U_{V,C}, \phi_{U,C})$ be a universal lift of $\mathrm{Res}_C^{S_5} V$ over $W[\mathbb{Z}/2]$, and let (U_V, ϕ_U) be a universal lift of V over $R(S_5, V)$. Then there exists a unique W-algebra homomorphism $\sigma : W[\mathbb{Z}/2] \to R(S_5, V)$ in $\mathcal{C}$ such that $(\mathrm{Res}_C^{S_5} U_V, \mathrm{Res}_C^{S_5} \phi_U)$ is isomorphic to $(R(S_5, V) \otimes_{W[\mathbb{Z}/2],\sigma} U_{V,C}, \phi_{U,C})$. To prove that σ is surjective, consider all morphisms $\rho : R(S_5, V) \to k[\epsilon]/(\epsilon^2)$. Since $\mathrm{Res}_C^{S_5} U \cong (kC)^5$ for the kS_5-module U from (4.2), $\mathrm{Res}_C^{S_5} U$ defines a non-trivial lift of $\mathrm{Res}_C^{S_5} V$ over $k[\epsilon]/(\epsilon^2)$. Because U defines a non-trivial lift of V over $k[\epsilon]/(\epsilon^2)$ and because $\mathrm{Ext}^1_{kS_5}(V, V) \cong k$, this implies that as ρ ranges over the morphisms $R(S_5, V) \to k[\epsilon]/(\epsilon^2)$, $\rho \circ \sigma$ ranges over the morphisms $W[\mathbb{Z}/2] \to k[\epsilon]/(\epsilon^2)$. Hence σ is surjective. It follows from the decomposition matrix for B in Fig. 1 and [11, Prop. (23.7)] that V has two non-isomorphic lifts over W whose F-characters are χ_3 and χ_4, respectively. Thus there are two distinct morphisms $R(S_5, V) \to W$ in $\mathcal{C}$, which implies that $\mathrm{Spec}(R(S_5, V))$ contains both points of the generic fiber of $\mathrm{Spec}(W[\mathbb{Z}/2])$. Since the Zariski closure of these points is all of $\mathrm{Spec}(W[\mathbb{Z}/2])$, it follows that $R(S_5, V)$ is isomorphic to $W[\mathbb{Z}/2]$.

Viewing V as a $k\hat{S}_5$-module by inflation, it follows from the description of the projective indecomposable $\hat{\Lambda}$-modules in Fig. 3 that

$$\mathrm{Ext}^1_{k\hat{S}_5}(V, V) \cong \underline{\mathrm{Hom}}_{\hat{B}}(\Omega_{\hat{B}}(V), V) \cong k$$

where $\Omega_{\hat{B}}$ denotes the syzygy in the category of finitely generated $\hat{B}$-modules. Moreover, the module U from (4.2) when viewed as a $k\hat{S}_5$-module by inflation defines a non-trivial lift (U, ν) of V over $k[\epsilon]/(\epsilon^2)$ when viewed as a $k\hat{S}_5$-module. Hence there exists a surjective k-algebra homomorphism

$$\tau : R(\hat{S}_5, V)/2R(\hat{S}_5, V) \to k[t]/(t^2)$$

corresponding to (U, ν). We now show that τ is a k-algebra isomorphism. Suppose this is false. Then there exists a surjective k-algebra homomorphism $\tau_1 : R(\hat{S}_5, V)/2R(\hat{S}_5, V) \to k[t]/(t^3)$ such that $\delta \circ \tau_1 = \tau$ where $\delta : k[t]/(t^3) \to k[t]/(t^2)$ is the natural projection. Let (U_1, ν_1) be a lift of V over $k[t]/(t^3)$ relative to τ_1. Then $k[t]/(t^2) \otimes_{k[t]/(t^3),\delta} U_1 \cong U$ and $t^2 U_1 \cong V$. Thus we have a short exact sequence of $k[t]/(t^3)\,\hat{S}_5$-modules

$$0 \to t^2 U_1 \to U_1 \to U \to 0. \tag{4.3}$$

Since $\mathrm{Ext}^1_{k\hat{S}_5}(U, V) = 0$, the sequence (4.3) splits as a sequence of $k\hat{S}_5$-modules. Thus $U_1 \cong V \oplus U$ as $k\hat{S}_5$-modules. Since V and U are kS_5-modules, U_1 is inflated from a kS_5-module. Because there is no lift of V over $k[t]/(t^3)$ when V is viewed as a kS_5-module, this implies that U_1 does not exist. Hence τ is a k-algebra isomorphism and $R(\hat{S}_5, V)/2R(\hat{S}_5, V) \cong k[t]/(t^2) \cong R(S_5, V)/2R(S_5, V)$. Since $R(S_5, V)$ is a W-algebra quotient of $R(\hat{S}_5, V)$ which is free as a W-module, this implies that $R(\hat{S}_5, V) \cong R(S_5, V) \cong W[\mathbb{Z}/2]$.

Case (iv). Suppose $V \in \left\{\begin{matrix} T_0 & T_1 \\ T_0, & T_0 \\ T_1 & T_0 \end{matrix}\right\}$. It follows from the description of the projective indecomposable modules in Fig. 3 that

$$\mathrm{Ext}^1_{kS_5}(V, V) \cong k \cong \mathrm{Ext}^1_{k\hat{S}_5}(V, V).$$

Moreover, we see from Fig. 3 that there is a uniserial kS_5-module X with descending composition factors

$$(T_0, T_0, T_1, T_0, T_0, T_1) \quad (\text{resp. } (T_1, T_0, T_0, T_1, T_0, T_0))$$

such that X defines a lift (X, ξ) of V over $k[t]/(t^2)$ when the descending composition factors of V are (T_0, T_0, T_1) (resp. (T_1, T_0, T_0)). Additionally, there is a uniserial $k\hat{S}_5$-module Y with descending composition factors

$$(T_0, T_0, T_1, T_0, T_0, T_1, T_0, T_0, T_1) \quad (\text{resp. } (T_1, T_0, T_0, T_1, T_0, T_0, T_1, T_0, T_0))$$

such that Y defines a lift (Y, ζ) of V over $k[t]/(t^3)$ when V is viewed as a $k\hat{S}_5$-module by inflation. Since

$$\mathrm{Ext}^1_{kS_5}(X, V) = 0 = \mathrm{Ext}^1_{k\hat{S}_5}(Y, V),$$

we see that

$$R(S_5, V)/2R(S_5, V) \cong k[t]/(t^2) \quad \text{and} \quad R(\hat{S}_5, V)/2R(\hat{S}_5, V) \cong k[t]/(t^3).$$

Moreover, the isomorphism class of the lift (X, ξ) is the universal mod 2 deformation of V when V is viewed as a kS_5-module, and the isomorphism class of the lift (Y, ζ) is the universal mod 2 deformation of V when V is viewed as a $k\hat{S}_5$-module.

It follows from the decomposition matrix for B in Fig. 1 that V has a lift over W. Hence by [6, Lemma 2.1], there exist $\mu \in \{0, 1\}$, $m \in \mathbb{Z}^+$ and $\lambda \in W$ such that

$$R(S_5, V) \cong W[[t]]/(t^2 - 2\lambda t, \mu 2^m t).$$

Since $X \cong \Omega_B^i(T_1)$ for either $i = 1$ or $i = -1$, it follows that X has a universal deformation ring when viewed as a kS_5-module and $R(S_5, X) \cong k$ by the proof of Case (ii) and by Lemma 2.2. If $\mu = 0$ (resp. $\mu = 1$), then $R(S_5, V)$

class :	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}	C_{11}	C_{12}
order :	1	2	4	3	6	5	10	4	8	8	12	12
length :	1	1	30	20	20	24	24	20	30	30	20	20
ψ_1	1	1	1	1	1	1	1	1	1	1	1	1
ψ_2	1	1	1	1	1	1	1	-1	-1	-1	-1	-1
ψ_3	5	5	1	-1	-1	0	0	1	-1	-1	1	1
ψ_4	5	5	1	-1	-1	0	0	-1	1	1	-1	-1
ψ_5	4	-4	0	-2	1	-1	1	0	0	0	0	0
ψ_6	6	6	-2	0	0	1	1	0	0	0	0	0
ψ_7	6	-6	0	0	0	1	-1	0	$\sqrt{2}$	$-\sqrt{2}$	0	0
ψ_8	6	-6	0	0	0	1	-1	0	$-\sqrt{2}$	$\sqrt{2}$	0	0
ψ_9	4	4	0	1	1	-1	-1	2	0	0	-1	-1
ψ_{10}	4	4	0	1	1	-1	-1	-2	0	0	1	1
ψ_{11}	4	-4	0	1	-1	-1	1	0	0	0	$\sqrt{3}$	$-\sqrt{3}$
ψ_{12}	4	-4	0	1	-1	-1	1	0	0	0	$-\sqrt{3}$	$\sqrt{3}$

Fig. 5. The ordinary character table of $\hat{S}_5$.

(resp. $(W/2^mW) \otimes_W R(S_5, V)$) is free over W (resp. $W/2^mW$). This implies that X, when regarded as a kS_5-module, has a lift over W (resp. $W/2^mW$). Hence $\mu = 1$ and $m = 1$, and so $R(S_5, V) \cong W[[t]]/(t^2 - 2\lambda t, 2t) \cong W[[t]]/(t^2, 2t)$.

Since $\mathrm{Ext}^2_{k\hat{S}_5}(V, V) \cong k$ by Lemma 4.1, it follows from [20, Section 1.6] that there exists an element $f(t) \in W[[t]]$ such that $R(\hat{S}_5, V) \cong W[[t]]/(f(t))$. Since $R(\hat{S}_5, V)/2R(\hat{S}_5, V) \cong k[t]/(t^3)$, it follows by the Weierstrass Preparation Theorem (see e.g. [18, Thm. IV.9.2]) that $f(t)$ can be taken to be of the form $f(t) = t^3 + at^2 + bt + c$ for certain $a, b, c \in 2W$. In particular, $R(\hat{S}_5, V)$ is free as a W-module. Let (Y^W, ζ_W) be a universal lift of V over $R(\hat{S}_5, V)$ when V is viewed as a $k\hat{S}_5$-module. Since the isomorphism class of (Y, ζ) is the universal mod 2 deformation of V as a $k\hat{S}_5$-module, it follows that Y^W defines a lift (Y^W, ω) of Y over W when Y is viewed as a $k\hat{S}_5$-module. If $Y/\mathrm{rad}(Y) \cong T_1$ then Y is a quotient module of the projective indecomposable $k\hat{S}_5$-module $\hat{P}_{T_1}$ with $\hat{P}_{T_1}/\mathrm{rad}(\hat{P}_{T_1}) \cong T_1$. Hence Y^W must be a quotient module of the projective indecomposable $W\hat{S}_5$-module $\hat{P}^W_{T_1}$ which is a lift of $\hat{P}_{T_1}$ over W, and we define $Z^W = Y^W$. If $\mathrm{soc}(Y) \cong T_1$ then $\Omega^{-1}(Y)$ is a quotient module of $\hat{P}_{T_1}$, and by Lemma 2.2, $\Omega^{-1}(Y)$ has a lift (Y'^W, ω') over W. Hence Y'^W must be a quotient module of $\hat{P}^W_{T_1}$. But then the kernel of the surjection $\hat{P}^W_{T_1} \to Y'^W$ is a W-pure submodule of $\hat{P}^W_{T_1}$, and we define Z^W to be this kernel. Therefore we have for both cases of Y that Z^W defines a lift of Y over W and that Z^W is either a quotient module or a submodule of $\hat{P}^W_{T_1}$. Thus it follows from the decomposition matrix for $\hat{B}$ in Fig. 1 that the F-character of Z^W is equal to

$$\chi_Z = \psi_6 + (\psi_7 + \psi_8).$$

This implies by Remark 3.1 that the endomorphism ring of $F \otimes_W Z^W \cong V_6 \oplus V_{78}$ is isomorphic to $F \times F(\sqrt{2})$. Let u be an element in $\hat{S}_5$ of order 8 belonging to the conjugacy class C_9 in Fig. 5 and let K_u be its class sum in $W\hat{S}_5$. Because K_u lies in the center of $W\hat{S}_5$, multiplication by K_u defines a $W\hat{S}_5$-module endomorphism κ_u of Z^W. Since Z^W is free as a W-module, the endomorphism ring $\mathrm{End}_{W\hat{S}_5}(Z^W)$ embeds naturally into

$$\begin{aligned} F \otimes_W \mathrm{End}_{W\hat{S}_5}(Z^W) &\cong \mathrm{End}_{F\hat{S}_5}(F \otimes_W Z^W) \\ &\cong \mathrm{End}_{F\hat{S}_5}(V_6) \times \mathrm{End}_{F\hat{S}_5}(V_{78}) \cong F \times F(\sqrt{2}). \end{aligned}$$

Hence κ_u corresponds to an element in $F \times F(\sqrt{2})$ which we can read off from the ordinary character table of $\hat{S}_5$. Namely, the endomorphism κ_u in $\mathrm{End}_{W\hat{S}_5}(U^W)$ corresponds to the element

$$(0, 5\sqrt{2}) \in F \times F(\sqrt{2}).$$

Because $(0, 5\sqrt{2})$ generates a W-subalgebra of $F \times F(\sqrt{2})$ which is isomorphic to $W[[t]]/(t^3 - 2t)$, it follows that Z^W is a $W[[t]]/(t^3 - 2t)\hat{S}_5$-module. Taking a $k[t]/(t^3)$-basis $\{b_1, \ldots, b_6\}$ of Y, we can lift this basis to a subset $\{c_1, \ldots, c_6\}$ of Z^W which generates Z^W as a $W[[t]]/(t^3 - 2t)$-module. Since $F \otimes_W Z^W$ is a free $(F \times F(\sqrt{2}))$-module of rank 6, it follows that $c_1, \ldots, c_6$ must be linearly independent over $W[[t]]/(t^3 - 2t)$. Thus Z^W defines a lift of V over $W[[t]]/(t^3 - 2t)$. Since $Z^W/2Z^W \cong Y$ is an indecomposable $k\hat{S}_5$-module, $W[[t]]/(t^3 - 2t)$ is a quotient algebra of $R(\hat{S}_5, V)$. This implies that we can take $f(t) = t^3 - 2t$, and hence $R(\hat{S}_5, V) \cong W[[t]]/(t^3 - 2t)$. □

Remark 4.2. Suppose G and $\hat{G}$ are two finite groups such that the Sylow 2-subgroups of G are dihedral groups of order 8 and the Sylow 2-subgroups of $\hat{G}$ are generalized quaternion groups of order 16 and such that $\hat{G}$ is an extension of G by a central subgroup of order 2. Moreover, assume that there exist $c \in \{0, 1\}$ and $d \in k$ such that the principal block B (resp. $\hat{B}$) of kG

class :	C_1	C_4	C_6
order :	1	3	5
φ_0	1	1	1
φ_1	4	−2	−1
φ_2	4	1	−1

Fig. 6. The 2-modular character table of S_5 and $\hat{S}_5$.

(resp. $k\hat{G}$) is Morita equivalent to Λ_c (resp. $\hat{\Lambda}_d$) as in Fig. 2. Many of the arguments in this paper work for this more general case. However, when computing the universal deformation rings for cases (iii) and (iv) of part (b) of Theorem 1.1, one runs into the following issues. First, one needs to prove in general that there is an element of order 2 in G that can take the place of the transposition $(1, 2) \in S_5$ when computing the universal deformation ring $R(G, V)$ in case (iii). Second, one needs to establish similar facts to the ones in Remark 3.1 for the irreducible representations of $\hat{G}$ over F which belong to $\hat{B}$, including the values of the ordinary characters on certain conjugacy classes, when computing the universal deformation ring $R(\hat{G}, V)$ in case (iv).

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Appendix. The ordinary and the 2-modular character table of $\hat{S}_5$

The ordinary character table of $\hat{S}_5$ can be found, for example, in [17, p. 289]. It is then straightforward to determine the ordinary character table of S_5 (see Fig. 5) and also the 2-modular character table of S_5 and $\hat{S}_5$ (see Fig. 6). The ordinary characters $\chi_1, \dots, \chi_4, \chi_5$ of S_5 in Fig. 1 correspond to the ordinary characters $\psi_1, \dots, \psi_4, \psi_6$ of $\hat{S}_5$ in Figs. 1 and 5.

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