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High Subgroups of Primary Abelian Groups of Length $\omega + n$

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Let α be a limit ordinal, n a positive integer and A an abelian p-group of length $\alpha + n$. We give a characterization of those subgroups of $A/p^{\alpha}A$ that are images of $p^{\alpha+i}$-high subgroups of A. Using this we show that the study of abelian p-groups of length $\omega + n$ having all high subgroups isomorphic is equivalent to the study of groups G of length ω having a specified set of pure dense subgroups isomorphic. This set of pure dense subgroups of G is determined by a dense subgroup P of $G[p^n]$ modulo a maximal p^{n-1}-bounded summand of G. For each positive integer n we give an example of a p-group A such that all the high subgroups of A are isomorphic but not all the high subgroups of $A/p^{\omega+n-1}A$ are isomorphic.

For a primary abelian group A, a high subgroup H of A is a subgroup maximal with respect to $H \cap p^{\omega}A = \{0\}$. Let $\eta: A \to A/p^{\omega}A$ be the natural homomorphism. Then η maps every high subgroup of A isomorphically onto a pure dense subgroup of $A/p^{\omega}A$. Hence the study of the set of high subgroups of A reduces to the study of a specified set of pure dense subgroups of a group $(A/p^{\omega}A)$ with no elements of infinite height. To make this reduction of the problem satisfactory one needs reasonable necessary and sufficient conditions for a set of pure dense subgroups of a p-group G with no elements of infinite height to be the set of images of the set of high subgroups of a group A under a homomorphism φ from A onto G with $\ker \varphi = p^{\omega}A$. One would also like to reclaim A from G and information about G.

Call an abelian p-group A an *IH-group* (isomorphic highs) if all of its high subgroups are isomorphic. Until recently (see [2] and [7]), the only IH-groups known were $\sum$-groups, i.e., groups for which the high subgroups are direct sums of cyclic groups (see [6]). Accomplishing the above program would reduce the problem of determining the class of IH-groups to the problem of determining those p-groups with no elements of infinite height having a *specified* set of pure dense subgroups isomorphic. The techniques in [2], [7] and [8], relevant to the problem in question, essentially use this reduction.

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In this paper we shall give a satisfactory reduction for p-groups A of length $\omega+n$. Hence we will reduce the study of IH-groups of length $\omega+n$ to the study of groups with no elements of infinite height having a *specified* set of pure dense subgroups isomorphic. Using this reduction we will show, for each integer $n \geqslant 2$, the extence (and actually the means of constructing) a p-group A with the property that all the high subgroups of A are isomorphic but not all the high subgroups of $A/p^{\omega+n}A$ are isomorphic.

We will actually consider abelian p-groups A of length $\alpha+n$, α a limit ordinal, and characterize those pure dense (in the α-topology) subgroups of $A/p^{\alpha}A$ that are images of $p^{\alpha+i}$-high subgroups of A under the natural homomorphism. We will show that the map $A \to (A/p^{\alpha}A, A[p^n]/p^{\alpha}A)$ is a one-to-one correspondence between the class of isomorphism classes of p-groups of length $\alpha+n$ and the class of equivalence classes of pairs (G, P), where G is a p-group of length α and P is a subgroup of $G[p^n]$ that contains a maximal p^{n-1}-bounded summand of G and is dense in the α-topology with $P[p] \neq G[p]$.

In this paper all groups will be additively written abelian p-groups, p a fixed but arbitrarily prime. For a group A, $A[p^n]=\{x \in A: p^n x=0\}$ is the p^n-socle of A. By the socle of A we mean the p-socle of A. If α is an ordinal $p^{\alpha}G=p(p^{\beta}G)$ if $\alpha=\beta+1$, and $p^{\alpha}G=\bigcap_{\beta\in\alpha} p^{\beta}G$ if α is a limit ordinal. The length of a reduced p-group A is the least ordinal λ such that $p^{\lambda}A=\{0\}$. The height of an element $x \in A$ is the ordinal α such that $x \in p^{\alpha}A - p^{\alpha+1}A$. A subgroup H of A is said to be p^{α}-high in A if H is maximal with respect to $H \cap p^{\alpha}A=\{0\}$. Thus high means p^{ω}-high. The α-topology on a group A is the linear topology on A having $\{p^{\beta}A\}_{\beta\in\alpha}$ as a base for the neighborhoods of $\{0\}$. If no mention is made of the topology we shall mean the ω-topology. My notation and terminology will in general be the same as that used in [3].

1. The Proposition and Other Preliminaries

Let A be a p-group of length $\alpha+n$, α a limit ordinal. Let $\eta: A \to A/p^{\alpha}A$. For $i=0, 1, \ldots, n-1$ we will determine which subgroups of $A/p^{\alpha}A$ are images of $p^{\alpha+i}$-high subgroups of A. In particular, if $p^{\alpha}A$ is homogeneous (i.e., $p^{\alpha}A=\bigoplus Z(p^n)$) and H is a p^{α}-high subgroup of A, then the set of pure subgroups of $A/p^{\alpha}A$ supported by $\eta(H[p^n])$ is the image of the set of p^{α}-high subgroups of A. We will need the following lemma which is Lemma 1.1 in [1].

Lemma 1. *Let A be a reduced p-group, α a limit ordinal and H a $p^{\alpha+n}$-high subgroup of A, n a nonnegative integer. Then $A[p^s]=H[p^s]\oplus K[p^s]$ for $s \leqslant n+1$ and K any complementary summand of a maximal p^n-bounded summand of $p^{\alpha}A$.*

In the following proposition we shall assume that the maximal p^{n-1}-bounded summand of our group A is zero in order to make the statement simpler. The simplest way to remove this restriction is to demand that K_0 contain the image of a maximal p^{n-1}-bounded summand of A. But this does not suit our purpose here.

PROPOSITION 2. *Let A be a p-group of length $\alpha+n$, α a limit ordinal and n a positive integer, such that the maximal p^{n-1}-bounded summand of A is zero. Let $\eta: A \to A/p^\alpha A$ be the natural homomorphism. Let $K_0 \subseteq K_1 \subseteq \cdots \subseteq K_{n-1}$ be a chain of pure subgroups of $A/p^\alpha A$ with the property that $\eta(A[p^n]) = \sum_{i=0}^{n-1} K_i[p^{n-i}]$. Then there exists a chain of subgroups $H_0 \subseteq H_1 \subseteq \cdots \subseteq H_{n-1}$ in A such that for $n = 0, 1,..., n-1$,*

(1) *H_i is $p^{\alpha+i}$-high in A and*

(2) *$\eta(H_i) = K_i$.*

Proof. The proof will be by induction on n where $\alpha+n$ is the length of A. If $n=1$ then $\eta(A[p]) = K_0[p]$. We will show that $p^\alpha A$ is a summand of $L = \eta^{-1}(K_0)$. Note that $p^\alpha A \subseteq A[p]$. Since $p^\alpha A$ is bounded we only need to show that $p^\alpha A$ is pure in L. Let $x \in L$ such that $px \in p^\alpha A$. Then $\eta(x) \in K_0[p]$. Thus there exists $y \in A[p]$ such that $\eta(y) = \eta(x)$. Hence $y - x = z \in p^\alpha A$ and $px = p(y-z) = 0$. Therefore $p^\alpha A$ is pure in L and hence a summand of L. Write $L = H_0 \oplus p^\alpha A$. Then $\eta | H_0$ is an isomorphism from H_0 onto K_0. Moreover, H_0 is pure in A since η preserves heights less than α, $\eta(H_0) = K_0$ and K_0 is pure in $A/p^\alpha A$. Also $A[p] = H_0[p] \oplus p^\alpha A$ and the purity of H_0 in A implies that H_0 is p^α-high in A. (The proof for the case $n=1$ is essentially the same as that in Corollary 5 of [9].)

Assume that the result is true for groups of length $\alpha+n-1$. Let A be a group of length $\alpha+n$, and let $K_0 \subseteq K_1 \subseteq \cdots \subseteq K_n$ be a chain of pure subgroups of $A/p^\alpha A$ satisfying the hypothesis of the proposition. Decompose $p^\alpha A = \bigoplus_{i=1}^n A_i$, where $A_i = \bigoplus Z(p^i)$. We will first show that A_n is a pure subgroup (and hence a summand) of $\eta^{-1}(K_{n-1})$. To this end first note that, by Lemma 1, any $p^{\alpha+n-1}$-high subgroup M of A satisfies $A[p^n] = M[p^n] \oplus A_n$ and hence $\eta(A[p^n]) = \eta(M[p^n])$. Also note that to show the purity of A_n we need only show that if $x \in \eta^{-1}(K_{n-1})$ with $p^r x \in A_n$ then $p^r x \in p^r A_n$. We will induct on r. If $x \in \eta^{-1}(K_{n-1})$ such that $px \in A_n$ then $\eta(x) \in K_{n-1}[p] = \eta(M[p])$. Hence $x \in A[p^n]$. Thus $p^n x = 0$ which implies that $px \in pA_n$. Let r be a positive integer such that $1 < r \leqslant n$. Assume that if $x \in \eta^{-1}(K_{n-1})$ such that $p^n x \in A_n$, $1 \leqslant m < r$, then $p^m x \in p^m A_n$. Let $x \in \eta^{-1}(K_{n-1})$ such that $p^r x \in A_n$. Since $p^{r-1}(px) \in A_n$ we may choose $y \in A_n$ such that $p^{r-1} y = p^r x$. Thus $px - y \in A[p^{r-1}] = M[p^{r-1}] \oplus A_n[p^{r-1}]$. Thus $px - y = s + t$ with $s \in M[p^{r-1}]$ and $t \in A_n[p^{r-1}]$. Let

$u \in M[r]$ such that $pu = s$. Then $px - pu = t + y \in A_n$. Since $u \in M[p^r]$ with $r \leqslant n$ we have $u \in \eta^{-1}(K_{n-1})$. Thus there exists $w \in A_n$ such that $pw = p(x-u)$. Hence $p^r w = p^r x$ since $p^r u = 0$. Note that if $r = n$ then $p^r x \in A_n$ implies that $p^r x = 0$. If $r > n$ we come to the same conclusion. Thus A_n is a summand of $\eta^{-1}(K_{n-1})$.

Decompose $\eta^{-1}(K_{n-1}) = H_{n-1} \oplus A_n$. We will show that H_{n-1} is $p^{\alpha+n-1}$-high in A. Since $p^{\alpha+n-1}A = p^{n-1}A_n$ and $A[p^n] = H_{n-1}[p^n] \oplus A_n$ this follows easily if we can show that H_{n-1} is pure in A. In order to show that H_{n-1} is pure in A, let $L = H_{n-1} \cap p^\alpha A$. Note that $p^\alpha A = L \oplus A_n$ and $\ker(\eta \mid H_{n-1}) = L$. We will first show that $L \subseteq p^\omega(H_{n-1})$. *Let* $x \in L$ such that the height of x in A is $\alpha + k$ $(0 \leqslant k < n-1)$. If M is a $p^{\alpha+n-1}$-high subgroup of A containing L then for every positive integer $m > 0$ there exists $y \in M[p^n]$ such that $h_A(y) \geqslant m$ and $p^{k+1}y = x$. Since $M[p^n] \subseteq \eta^{-1}(K_{n-1})$, we may write $y = z + g$ with $z \in H_{n-1}[p^n]$ and $g \in A_n$. Thus $p^{k+1}(z+g) = x$. Hence $p^{k+1}z - x = -p^{k+1}g \in H_{n-1} \cap A_n = \{0\}$. Thus $p^{k+1}z = x$ and $h_A(z) \geqslant m$. Assume $k = n-2$. Since $\eta(H_{n-1}) = K_{n-1}$ and K_{n-1} is pure in $A/p^\alpha A$ there exists $u \in H_{n-1}$ such that $p^m u = z + t$ with $t \in L$. Thus $p^{m+n-1}u = p^{n+1}z + p^{n-1}t = x$. Hence x has infinite height in H_{n-1}. That is to say the elements in L of height $\alpha + n - 2$ in A have infinite height in H_{n-1}. Assume that we have shown that the elements in L of height greater than $\alpha + k$ in A have infinite height in H_{n-1} where $0 \leqslant k < n-2$. Let $x \in L$ with $h_A(x) = \alpha + k$. Let m be a positive integer and $z \in H_{n-1}$ with $h_A(z) \geqslant m$ and $p^{k+1}z = x$ (as above). Let $u \in H_{n-1}$ such that $p^m u - z = t \in L$ (this follows from the purity of K_{n-1} and the fact that $\eta(H_{n-1}) = K_{n-1}$ with kernel L). Then $p^{m+k+1}u = p^{k+1}z + p^{k+1}t = x + p^{k+1}t$. But $h_A(p^{k+1}t) \geqslant \alpha + k + 1$ and hence $p^{k+1}t$ has infinite height in H_{n-1}. Thus x has infinite height in H_{n-1}.

Next suppose that $x \in H_{n-1}$ such that the height of x in A is $<\alpha$. Suppose that $p^m u = x$ for some $u \in A$ and positive integer m. Then the height of $\eta(x)$ is $\geqslant m$ and hence by the purity of K_{n-1} there exists $y \in H_{n-1}$ such that $p^m \eta(y) = \eta(x)$. Thus $p^m y - x = t \in L$. Thus x has height $\geqslant m$ in H_{n-1} since t has infinite height in H_{n-1}. Therefore H_{n-1} is pure in A. Since $A[p] = H_{n-1}[p] \oplus A_n[p]$ and H_{n-1} is pure in A we can conclude that H_{n-1} is $p^{\alpha+n-1}$-high in A.

We will now obtain the rest of the chain. Note that $\eta(H_{n-1}[p^{n-1}]) = \sum_{i=0}^{n-2} K_i[p^{n-i-1}]$. To see this, observe that $\eta(p(H_{n-1}[p^n])) = p\sum_{i=0}^{n-1} K_i[p^{n-i}] = \sum_{i=0}^{n-2} pK_i[p^{n-i}] = \sum_{i=1}^{n-2} K_i[p^{n-i-1}]$. Hence by the induction hypothesis there exists a chain $H_0 \subseteq H_1 \subseteq \cdots \subseteq H_{n-2}$ of subgroups of H_{n-1} such that for $i = 0, 1, \ldots, n-2$, H_i is $p^{\alpha+i}$-high in H_{n-1} and $\eta(H_i) = K_i$.

Corollary 3. *Let A be a p-group of length $\alpha + n$, α a limit ordinal and n a positive integer, such that the maximal p^{n-1}-bounded summand of A is*

$\{0\}$. *Let* $\eta: A \to A/p^\alpha A$ *be the natural homomorphism. Then a pure subgroup* K *of* $A/p^\alpha A$ *is the image of a* p^α*-high subgroup of* A *if and only if* $\eta(A[p^n]) = K[p^n] \oplus P$ *with* $p^{n-1}P = \{0\}$.

Proof. It is easy to show that K is dense in $A/p^\alpha A$. Decompose $P = \bigoplus_{i=0}^{n-1} P_i$ with $P_i = \bigoplus Z(p^{n-i})$. Choose a chain of pure subgroups $K = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_{n-1}$ of $A/p^\alpha A$ such that $K_i[p^{n-i}] = K_i[p^{n-i}] \oplus P_i$. Then $\eta(A[p^n]) = \sum_{n=0}^{n-1} K_i[p^{n-i}]$. Hence the result follows from the proposition.

Let A be a p-group. Decompose $A = P \oplus A'$, where P is a p^n-bounded summand of A. Then A is an IH-group if and only if A' is an IH-group. Thus in studying IH-groups we may restrict our attention (without loss of generality) to those p-groups whose maximal p^n-bounded summand is zero (for some fixed n).

COROLLARY 4. *Let* A *be a* p*-group of length* $\omega + n$, n *a positive integer, such that the maximal* p^{n-1}*-bounded summand of* A *is* $\{0\}$. *Let* $\eta: A \to A/p^\omega A$ *be the natural homomorphism. Then* A *is an* IH*-group if and only if all the pure subgroups* K *of* $A/p^\omega A$ *such that* $\eta(A[p^n]) = K[p^n] \oplus P$ *with* $p^{n-1}P = 0$ *are isomorphic.*

2. A CONSTRUCTION OF ELONGATIONS AND THE MAIN THEOREM

We will need the following construction of a group A which, in the terminology of [10], is an α-elongation of a p-group G of length α by a bound group B. That is to say $A/p^\alpha A \cong G$ and $p^\alpha A \cong B$. This construction is a special case of Remark 1.15 in [1]. Let α be a limit ordinal and G a p-group of length α. Let $K_0 \subseteq K_1 \subseteq \cdots \subseteq K_i \subseteq \cdots \subseteq K_n = G$ be a chain of pure subgroups of G that are dense in G with respect to the α-topology. Let $B = \bigoplus_{i=0}^n B_i$, where $B_i = \bigoplus_{A_i} Z(p^i)$ with $|A_i| = \operatorname{rank} K_i/K_{i-1}$. Let D_i be the divisible hull of B_i and $D = \bigoplus_{i=1}^n D_i$. Let $\sigma: G \to G/K_0$ be the natural homomorphism and $\tau: D \to \bigoplus_{i=1}^n D_i/B_i$ be the homomorphism such that $\tau \mid D_i = p^i$. Let φ be an isomorphism from G/K_0 onto $\bigoplus_{i=1}^n D_i/B_i$ such that $\varphi(K_i/K_0) = \bigoplus_{j=1}^i D_j/B_j$, $i = 1, 2,..., n$. Let $A = \{(x, y) \in G \oplus D$: $\varphi\sigma(x) = \tau(y)\}$. Then A is an α-elongation of G by B. It is straightforward to show that $H_i = \{(x, y) \in A \mid x \in K_i$ and $y \in \bigoplus_{j=1}^i D_j\}$ is a $p^{\alpha+i}$-high subgroup of A, $i = 0, 1,..., n-1$. Also, if π is the projection of A onto G then $\ker \pi = \bigoplus_{i=1}^n B_i$ and $\pi H_i = K_i$ for $i = 0, 1,..., n-1$ ($\ker \pi = p^\alpha A$).

Next let G' be a p-group of length α, α a limit ordinal. Let n be a positive integer and P a maximal p^{n-1}-bounded summand of G'. Decompose $G' = P \oplus G$. Let S be a dense subgroup of $G[p^n]$ with $S[p] \neq G[p]$. Decompose $S = \bigoplus_{n=0}^{n-1} S_i$, where $S_i = \bigoplus Z[p^{n-i}]$. Choose a chain of pure subgroups $K_0 \subseteq K_1 \subseteq \cdots \subseteq K_{n-1}$ of G such that $K_i[p^{n-i}] = \bigoplus_{j=0}^{i} S_j[p^{n-i}]$. Using the above construction we can construct a group A of length $\alpha + n$ with a chain of $p^{\alpha+i}$-high subgroups $H_0 \subseteq H_1 \subseteq \cdots \subseteq H_{n-1}$ such that $A/p^\alpha A \cong G$ under an isomorphism φ such that $\varphi(H_i + p^\alpha A)/p^\alpha A = K_i$, $i = 0, 1, \ldots, n-1$. By Theorem 1.7 in [1], A is determined up to isomorphism by $K_0 \subseteq K_1 \subset \cdots \subset K_{n-1} \subseteq G$. Hence for $\alpha = \omega$ the study of IH-groups of length $\omega + n$ is reduced to the study of groups of length ω having a prescribed set of pure subgroups isomorphic. This of course has a natural generalization to groups of length α.

We will now formalize what we have said in the following theorems. We will need several definitions before stating the theorems. Let α be a limit ordinal and n a positive integer. For $i = 1, 2$, let G_i be a p-group of length α and P_i a subgroup of $G_i[p^n]$ containing a maximal p^{n-1}-bounded summand of G and dense in the α-topology. We will say that (G_1, P_1) and (G_2, P_2) are *equivalent* if there is an isomorphism ψ from G_1 onto G_2 such that $\psi(P_1) = P_2$. In the case $\alpha = \omega$, such a pair (G, P) is said to be of *IH-type* if given any two pure dense subgroups H_1 and H_2 of G such that $P = H_1[p^n] \oplus Q_1 = H_2[p^n] \oplus Q_2$ with $p^{n-1}Q_1 = \{0\}$ and $p^{n-1}Q_2 = \{0\}$, we have H_1 and H_2 isomorphic.

THEOREM 5. *Let α be a limit ordinal and n a positive integer. Let $\mathscr{G}_{\alpha,n}$ be the class of equivalence classes of pairs (G, P), where G is a p-group of length α and P is a subgroup of $G[p^n]$ containing a maximal p^{n-1}-bounded summand of G and dense in the α-topology with $P[p] \neq G[p]$. Let $\mathscr{A}_{\alpha+n}$ be the class of isomorphism classes of p-groups of length $\alpha + n$. Then the map $\varphi: \mathscr{A}_{\alpha+n} \to \mathscr{G}_{\alpha,n} : A \to (A/p^\alpha A, A[p^n]/p^\alpha A)$ is one-to-one and onto.*

Proof. The construction of elongations given above shows that the map is onto. Theorem 1.7 of [1], together with Proposition 2, shows that φ is one-to-one (this also follows from [6]).

THEOREM 6. *Let $\hat{\mathscr{A}}_{\omega+n}$ be the subclass of $\mathscr{A}_{\omega+n}$ of IH-groups. Let $\hat{\mathscr{G}}_{\omega,n}$ be the subclass of $\mathscr{G}_{\omega,n}$ of IH-types. Then $\varphi \mid \hat{\mathscr{A}}_{\omega+n}$ is onto $\hat{\mathscr{G}}_{\omega,n}$.*

Proof. This follows from Theorem 5, Proposition 2, and the construction of elongations given above.

3. EXAMPLES

For each integer $n \geqslant 2$ we will give an example of a p-group G of length ω with a dense subgroup P of $G[p^n]$ such that all the pure subgroups K of G such that $K[p^n] = P$ are isomorphic but not all the pure subgroups L of G such that $L[p^{n-1}] = P[p^{n-1}]$ are isomorphic. Using the construction of elongations given above we can construct an example of a group A of length

$\omega+n$ having all its high subgroups isomorphic, but $A/p^{\omega+n-1}A$ does not have all of its high subgroups isomorphic.

Let $B=\oplus_{n=1}^{\infty} Z(p^n)$ and $\bar{B}$ be its torsion completion. Let H_1 be a pure dense subgroup of $\bar{B}$ containing B such that $\bar{B}/H_1$ has rank 1. Let H_2 be a pure subgroup of $\bar{B}$ containing B such that $H_1[p^{n-1}]=H_2[p^{n-1}]$ but no pure subgroup of $\bar{B}$ supported by $H_1[p^n]$ is isomorphic to any pure subgroup of $\bar{B}$ supported by $H_2[p^n]$. (One can construct such an H_2 using only a slight generalization of part of the proof of Theorem 66.4 in [3].) Note also that there can be no isomorphism from $H_1[p^n]$ onto $H_2[p^n]$ that preserves heights in $\bar{B}$. (To prove this one can use the same argument as that at the bottom of page 469 in [2].) Let $0 \to K \to F \to \bar{B} \to 0$ be a pure exact sequence with F a direct sum of cyclic p-groups. Let $G=F/K[p^n]$. Then G is a $p^{\omega+n}$-projective p-group such that $G[p^n]=S[p^n]\check{\oplus}\bar{B}[p^n]$ with $S=K/K[p^n]$ and $B[p^n]$ identified with $F[p^n]/K[p^n]$. (By $\check{\oplus}$ we mean direct as a valuated group.) Note that $G/\bar{B}[p^n]$ is a direct sum of cyclic groups. Let $\varphi\colon G \to G/S \equiv \bar{B}$, where φ is natural and G/S is identified with $\bar{B}$. Let $A_1=\varphi^{-1}(H_1)$ and $A_2=\varphi^{-1}(H_2)$. Note that $A_1[p^{n-1}]=A_2[p^{n-1}]$ since $H_1[p^{n-1}]=H_2[p^{n-1}]$.

We will show that A_1 is not isomorphic to A_2. To see this, suppose not and let $\rho\colon A_1 \to A_2$ be an isomorphism. Now for $i=1,2$, $A_i[p^n]=S[p^n]\check{\oplus} P_i$, where $P_i=H_i[p^n]$ and A_i/P_i is a direct sum of cyclic groups. Thus it follows from [4] that $A_2/(\rho P_1 \cap P_2)$ is a direct sum of cyclics. Let $\eta\colon A_2 \to A_2/(\rho P_1 \cap P_2)$ be natural. Since $\eta \mid P_2[p]$ is height nondecreasing and the image of η is a direct sum of cyclic groups we have $P_2[p]=T\check{\oplus}\ker(\eta \mid P_2[p])$, where T supports a pure direct sum of cyclic groups in A_2 (see Lemma 1 in [5]). Since $\bar{B}/H_1$ has finite rank, the elements of T must have bounded height. Hence there is an integer n_1 such that $(p^{n_1}H_2)[p] \subseteq \rho P_1 \cap P_2$. Decompose $H_2=S_1 \oplus H_2'$, where S_1 is a maximal p^{n_1}-bounded summand of H_2. Let $P_2'=H_2'[p^n]$. Note that $P_2' \subseteq P_2$. Define $\eta'\colon A_1/P_2'[p] \to A_1/(\rho P_1 \cap P_2)$: $a+P_2'[p] \to \eta(a)$. Now since $\eta' \mid (P_2'/P_2'[p])[p]$ is height nondecreasing and the image of η' is a direct sum of cyclic groups we have $(P_2'/P_2'[p])[p]=T' \oplus \ker \eta' \mid (P_2'/P_2'[p])[p]$, where as above T supports a pure direct sum of cyclic groups in $A_2/P_2'[p]$. Again we can conclude that the elements of T' have bounded height. Thus there is an integer n_2 such that $(p^{n_2}(H_2'/H_2'[p] \subseteq (\rho P_1 \cap P_2)/H_2'[p]$. Hence there is an integer n_2' such that $(p^{n_2}H_2')[p^2] \subseteq \rho P_1 \cap P_2$. Continuing in this manner we can find an integer m such that $p^m H_2[p^n] \subseteq \rho P_1 \cap P_2$. By the same arguments we may assume that m has been chosen such that $p^m H_1[p^n] \subseteq P_1 \cap \rho^{-1}P_2$. Since η and η^{-1} preserve heights we must have $\eta(p^m H_1[p^n]) \subseteq p^m H_2[p^n]$ and $\eta^{-1}(p^m H_2[p^n]) \subseteq p^m H_1[p^n]$. Thus η is a height-preserving isomorphism from $p^m H_1[p^n]$ onto $p^m H_2[p^n]$. Thus η can be extended to a height-preserving isomorphism from $H_1[p^n]$ onto $H_2[p^n]$. This contradicts our choice of $H_2[p^n]$. Thus A_1 and A_2 are not isomorphic.

Next recall that every subgroup of a $p^{\omega+n}$-projective group is $p^{\omega+n}$-projective. Thus by [4] every pure subgroup of G supported by $A_1[p^n]$ is isomorphic to A_1.

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