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A Remark Concerning Embeddability of Rings in Fields

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In previous papers [1] and [2], we have dealt with the following necessary conditions for embedding rings into fields:

N_m: *If a matrix C of order m is nilpotent then* $C^m = 0$.

The result obtained in [1] led us to the question whether the set of conditions N_m, $m = 1, 2, \ldots$ is sufficient for an integral domain to be embeddable in a field. This question is reasonable also since these conditions are quasi-identities, and it is known [3] that the class of rings embeddable in fields can be characterized as the class of integral domains satisfying certain quasi-identities; i.e., universal formulae of the form $A_1 \wedge \cdots \wedge A_n \Rightarrow B$ (also called Horn sentences), where A_i, B are equations.

Our object is to show that the conditions N_m, $m = 1, 2, \ldots$, are stronger than one might have thought, and the results we shall derive, lead us to the conjecture* that they are sufficient for embeddability.

We shall consider two other sets of necessary conditions for embedding rings into fields which seem to be stronger, and we shall prove that they are equivalent to the set of conditions N_m, $m = 1, 2, \ldots$. One of these sets of conditions was suggested to the author by Professor Amitsur. He asked whether the following property on the product of square matrices of order m over an integral domain is stronger than N_m:

P_m: *If the product of k permutable matrices,* $k > m$, *is zero, then there exist m of these matrices having zero product.*

It is easy to show that P_m is necessary for embedding in a field. Also it is clear that P_m implies N_m in any ring. At a first glance, P_m seems to be stronger than N_m and it is really stronger for arbitrary rings. For instance, any direct product of at least two fields clearly satisfies all the conditions N_m. However,

* *Added in proof*: A counterexample to this conjecture has been found by G. M. Bergman.

it does not satisfy any of the conditions P_m since it can be proved that a ring with 1 which satisfies P_m also satisfies P_n for $n = 1,..., m - 1$. In particular, it satisfies P_1 and a ring which satisfies P_1 is an integral domain. One can also show directly, by an example, that P_m does not hold in a direct product of at least two fields. We shall see that for integral domains N_m implies P_m.

The other set of conditions was suggested by Professor Bergman. To define these conditions, we first need another definition. Let R be a ring with 1 and R^m the right R-module of rows of length m over R.

DEFINITION. A submodule U of R^m is called *closed* if it is the intersection of the kernels of a family of linear functionals on R^m.

The condition defined by Bergman is:

C_m: *The length k of any chain of nonzero closed submodules of R^m: $U_1 \subsetneqq U_2 \subsetneqq \cdots \subsetneqq U_k$ is $\leqslant m$.*

It can be proved directly that C_m is necessary for embedding in a field. Also, N_m is the condition C_m restricted to chains of a special form and C_m seems to be much stronger than N_m. We shall soon see that C_m seems to be even stronger than P_m and this will, in particular, show that a ring satisfying C_m is necessarily an integral domain.

We shall identify the set of all linear functionals $\hom_R(R^m, R)$ with the left R-module of rows of length m over R. The identification is made such that if $x = (x_1, ..., x_m) \in R^m$ and g is a linear functional identified with $(a_1, ..., a_m)$, then $g(x) = \sum_{i=1}^{m} a_i x_i$. We shall also identify $\hom_R(R^m, R^m)$ with the ring of $m \times m$ matrices R_m acting on the left, and this clearly shows that if f is an R-endomorphism of R^m, then $\ker f$ is the intersection of the kernels of m linear functionals, and hence it is a closed submodule of R^m. Using this identification, the definition of P_m for rings with 1 can be given as above with "endomorphisms" replacing "matrices".

Now, we shall show that C_m implies P_m. By induction on k, $k > m$, it follows that it is enough to prove P_m for $k = m + 1$. Let $f_1, ..., f_{m+1}$ be permutable endomorphisms having zero product. If P_m is not satisfied, then $\prod_{j \neq i} f_j \neq 0$ and the following is a chain of $m + 1$ nonzero closed submodules of R^m:

$$\ker f_1 \subsetneqq \ker(f_1 f_2) \subsetneqq \cdots \subsetneqq \ker(f_1 \cdots f_{m+1}) = R^m.$$

This shows that if C_m holds, then P_m also holds, and we have one part of our result which is

THEOREM. *For integral domains with* 1 *and for each $m > 1$, the conditions N_m, P_m and C_m are equivalent.*

Proof. We have proved that C_m implies P_m and it is trivial that P_m implies N_m. It remains to prove that N_m implies C_m. For $m = 1$, the result is clear, since C_1 holds in any integral domain with 1. Let $m \geq 2$ and assume that R does not satisfy C_m. Thus, there exists a chain of more than m nonzero closed submodules of R^m. Let $U_1, \ldots, U_m$ be the first m elements of this chain; so we have

$$U_1 \subsetneqq U_2 \subsetneqq \cdots \subsetneqq U_m \subsetneqq R^m.$$

Considering the definition of a closed submodile, it is clear that if U is closed and $Z(U)$ is the set of all linear functionals annihilating U, then $U = \bigcap_{g \in Z(U)} \ker g$. Hence, for the elements of our chain we have $U_i = \bigcap_{g \in Z(U_i)} \ker g$, $i = 1, \ldots, m$. Since $U_i \neq U_{i+1}$, it follows that $Z(U_i) \neq Z(U_{i+1})$. Also $Z(U_{i+1}) \supseteq Z(U_i)$; hence we have

$$Z(U_1) \supsetneqq Z(U_2) \supsetneqq \cdots \supsetneqq Z(U_m) \supsetneqq \{0\}.$$

Now, choose $0 \neq x_1 \in U_1$ and $x_i \in U_i - U_{i-1}$, $i = 2, \ldots, m$. Choose also $0 \neq g_m \in Z(U_m)$ and $g_{i-1} \in Z(U_{i-1})$ such that $g_{i-1}(x_i) \neq 0$, $i = 2, \ldots, m$. Let $x_k = (x_{1k}, \ldots, x_{mk})$, $k = 1, \ldots, m$, and let g_i be identified with $(a_{i1}, \ldots, a_{im})$, $i = 1, \ldots, m$. Define the following two $m \times m$ matrices: $A = (a_{ij})$ and $X = (x_{jk})$. By the choice of $g_1, \ldots, g_m$ and $x_1, \ldots, x_m$ it follows that

$$AX = (g_i(x_k)) = \begin{pmatrix} 0 & g_1(x_2) & & & * \\ & 0 & g_2(x_3) & & \\ & & \ddots & \ddots & \\ & 0 & & 0 & g_{m-1}(x_m) \\ & & & & 0 \end{pmatrix}$$

From this we obtain that $(AX)^m = 0$, hence $(XA)^{m+1} = 0$. Also $g_{i-1}(x_i) \neq 0$ for $i = 2, \ldots, m$ and since $x_1 \neq 0$ and $g_m \neq 0$, there exist p and q such that $x_{1p} \neq 0$ and $a_{mq} \neq 0$. The matrix $(AX)^{m-1}$ has $g_1(x_2) \cdots g_{m-1}(x_m)$ in its $(1, m)$ place and zeros elsewhere. This implies that the (p, q) entry of $(XA)^m = X(AX)^{m-1}A$ is $x_{1p}g_1(x_2) \cdots g_{m-1}(x_m)\, a_{mj}$ and it is $\neq 0$, since all the terms are $\neq 0$ and R is an integral domain. Thus, we have that the $m \times m$ matrix XA is nilpotent and $(XA)^m \neq 0$. This shows that C_m holds whenever N_m holds and the result of the theorem follows.

An immediate corollary of the theorem is

COROLLARY 1. *For integral domains with* 1 *the conditions* $\bigcap_{m=1}^{\infty} N_m$, $\bigcap_{m=1}^{\infty} P_m$ *and* $\bigcap_{n=1}^{\infty} C_m$ *are equivalent.*

Another interesting corollary is:

COROLLARY 2. *If R is an integral domain with 1 satisfying N_m, then each nonzero closed submodule of R^m is the intersection of the kernels of less than m linear functionals.*

We have seen that the kernel of an R-endomorphism of R^m is closed. Corollary 2 shows that each closed submodule of R^m is the kernel of one endomorphism. Thus, for integral domains with 1, we have that in the presence of N_m the set of closed submodules of R^m coincides with the set of the kernels of the R-endomorphisms of R^m.

We conclude with the following remark. The condition C_m is easily seen to be equivalent to the following condition on matrices: *Any chain of nonzero right (or left) annihilators in the ring R_m has length $\leqslant m$.* This condition applies also to rings without 1. Let us use the same notation C_m for this condition defined for arbitrary rings. Following the proof of our theorem one can prove that the conditions N_m, P_m and C_m are equivalent for arbitrary integral domains.

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