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“Teleparallel” dark energy

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ABSTRACT

Using the “teleparallel” equivalent of General Relativity as the gravitational sector, which is based on torsion instead of curvature, we add a canonical scalar field, allowing for a nonminimal coupling with gravity. Although the minimal case is completely equivalent to standard quintessence, the nonminimal scenario has a richer structure, exhibiting quintessence-like or phantom-like behavior, or experiencing the phantom-divide crossing. The richer structure is manifested in the absence of a conformal transformation to an equivalent minimally-coupled model.

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1. Introduction

The “teleparallel” equivalent of General Relativity (TEGR) [1,2] is an equivalent formulation of classical gravity, in which instead of using the torsionless Levi–Civita connection one uses the curvatureless Weitzenböck one. The dynamical objects are the four linearly independent vierbeins (these are *parallel* vector fields represented by the appellation “teleparallel”). The advantage of this framework is that the torsion tensor is formed solely from products of first derivatives of the tetrad. As described in [2], the Lagrangian density T can be constructed from this torsion tensor under the assumptions of invariance under general coordinate transformations, global Lorentz transformations, and the parity operation, along with requiring the Lagrangian density to be second order in the torsion tensor. Thus, apart from possible conceptual differences, TEGR is completely equivalent and indistinguishable from General Relativity (GR) at the level of equations, both background and perturbation ones.

On the other hand, in General Relativity one can add the quintessence scalar field in order to acquire a dynamical dark energy sector, a scenario that exhibits a very interesting cosmological behavior and has gained a huge amount of research [3]. Amongst others, one can generalize it by including a nonminimal coupling between the quintessence field and gravity [4], or more generally extend it to the scalar-tensor paradigm [5]. One can also use a

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phantom instead of a canonical field [6], or the combination of both these fields in a unified scenario called quintom [7].

In this letter we are interested in formulating “teleparallel dark energy”, adding a canonical scalar field in TEGR. In the minimal coupling case the resulting theory is identical to the ordinary quintessence, both at the background and perturbation levels. However, when the nonminimal coupling is switched on, teleparallel dark energy is different from its GR counterpart, and the cosmological behavior of such a new scenario proves to be very interesting.

2. Quintessence in general relativity

Let us review very briefly the quintessence paradigm in general relativity. In such a scenario the dark energy sector is attributed to a homogeneous scalar field ϕ , and the action is given by [4]

$$S = \int d^4x \sqrt{-g} \left[\frac{R}{2\kappa^2} + \frac{1}{2} (\partial_\mu \phi \partial^\mu \phi + \xi R \phi^2) - V(\phi) + \mathcal{L}_m \right], \quad (1)$$

with $\kappa^2 = 8\pi G$, $c = 1$, $V(\phi)$ the scalar-field potential, ξ the non-minimal coupling parameter, R the Ricci scalar, and \mathcal{L}_m the matter Lagrangian. Note the difference in the metric signature that exists amongst the various works in the literature and the corresponding sign changes in the action, since a change in the metric signature leads $g_{\mu\nu}$, \square and $R_{\mu\nu}$ to change sign, while R and the energy-momentum tensor remain unaffected [4]. In this letter we use the signature $(+, -, -, -)$ in all sections, just to be closer to the literature of teleparallel gravity. Thus, under this convention, the conformal value of ξ is $-1/6$.

In the case of a flat Friedmann–Robertson–Walker (FRW) background metric

$$ds^2 = dt^2 - a^2(t)\delta_{ij} dx^i dx^j, \quad (2)$$

where t is the cosmic time, x^i are the comoving spatial coordinates and $a(t)$ is the scale factor, the Friedmann equations write as

$$\begin{aligned} H^2 &= \frac{\kappa^2}{3}(\rho_\phi + \rho_m), \\ \dot{H} &= -\frac{\kappa^2}{2}(\rho_\phi + p_\phi + \rho_m + p_m), \end{aligned} \quad (3)$$

where $H = \dot{a}/a$ is the Hubble parameter, a dot denotes differentiation with respect to t , and ρ_m and p_m are the matter energy density and pressure, respectively, following the standard evolution equation $\dot{\rho}_m + 3H(1 + w_m)\rho_m = 0$, with $w_m = p_m/\rho_m$ the matter equation-of-state parameter. Additionally, we have introduced the energy density and pressure of the nonminimally coupled scalar field, given by [4]:

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) - 6\xi H\phi\dot{\phi} - 3\xi H^2\phi^2, \quad (4)$$

$$\begin{aligned} p_\phi &= \frac{1}{2}(1 + 4\xi)\dot{\phi}^2 - V(\phi) + 2\xi(1 + 6\xi)\dot{H}\phi^2 \\ &\quad - 2\xi H\phi\dot{\phi} + 3\xi(1 + 8\xi)H^2\phi^2 - 2\xi\phi V'(\phi), \end{aligned} \quad (5)$$

where a prime denotes derivative with respect to ϕ . We note that the above relations have been simplified by using the useful expression $R = 6(\dot{H} + 2H^2)$ in the FRW geometry.

As mentioned above, in such a scenario, dark energy is attributed to the scalar field, and thus its equation-of-state parameter reads:

$$w_{DE} \equiv w_\phi = \frac{p_\phi}{\rho_\phi}. \quad (6)$$

Finally, the equations close by considering the evolution equation for the scalar field [4]:

$$\ddot{\phi} + 3H\dot{\phi} - 6\xi(\dot{H} + 2H^2)\phi + V'(\phi) = 0, \quad (7)$$

which can alternatively be written in the standard form $\dot{\rho}_\phi + 3H(1 + w_\phi)\rho_\phi = 0$.

3. Teleparallel equivalent to general relativity (TEGR)

We now briefly review TEGR. The notation is as follows: Greek indices μ, ν, \dots and capital Latin indices A, B, \dots run over all coordinate and tangent space-time $0, 1, 2, 3$, while lower case Latin indices (from the middle of the alphabet) i, j, \dots and lower case Latin indices (from the beginning of the alphabet) a, b, \dots run over spatial and tangent space coordinates $1, 2, 3$, respectively.

As stated in *Introduction*, the dynamical variable of “teleparallel” gravity is the vierbein field $\mathbf{e}_A(x^\mu)$. This forms an orthonormal basis for the tangent space at each point x^μ of the manifold, that is $\mathbf{e}_A \cdot \mathbf{e}_B = \eta_{AB}$, where $\eta_{AB} = \text{diag}(1, -1, -1, -1)$. Furthermore, the vector \mathbf{e}_A can be analyzed with the use of its components e_A^μ in a coordinate basis, that is $\mathbf{e}_A = e_A^\mu \partial_\mu$.

In such a construction, the metric tensor is obtained from the dual vierbein as

$$g_{\mu\nu}(x) = \eta_{AB} e_A^\mu(x) e_B^\nu(x). \quad (8)$$

Contrary to GR, which uses the torsionless Levi–Civita connection, in TEGR one takes the curvatureless Weitzenböck connection [8], whose torsion tensor reads

$$T_{\mu\nu}^\lambda \equiv \Gamma_{\nu\mu}^\lambda - \Gamma_{\mu\nu}^\lambda = e_A^\lambda (\partial_\mu e_\nu^A - \partial_\nu e_\mu^A), \quad (9)$$

where $\Gamma_{\nu\mu}^\lambda \equiv e_A^\lambda \partial_\mu e_\nu^A$. Moreover, the contorsion tensor, which equals to the difference between Weitzenböck and Levi–Civita connections, is defined as $K^{\mu\nu\rho} \equiv -\frac{1}{2}(T^{\mu\nu\rho} - T^{\nu\mu\rho} - T^{\rho\mu\nu})$ and we also define $S_{\rho}^{\mu\nu} \equiv \frac{1}{2}(K^{\mu\nu\rho} + \delta_\rho^\mu T^{\alpha\nu\alpha} - \delta_\rho^\nu T^{\alpha\mu\alpha})$.

In the present formalism all the information concerning the gravitational field is included in the torsion tensor $T_{\mu\nu}^\lambda$. Using the above quantities one can extract the form of the “teleparallel Lagrangian”, which is nothing else than the torsion scalar, namely [1, 2,9]:

$$\begin{aligned} \mathcal{L} = T &\equiv S_{\rho}^{\mu\nu} T^{\rho}_{\mu\nu} = \frac{1}{4} T^{\rho\mu\nu} T_{\rho\mu\nu} \\ &\quad + \frac{1}{2} T^{\rho\mu\nu} T_{\nu\mu\rho} - T_{\rho\mu}{}^\rho T^{\nu\mu}{}_\nu. \end{aligned} \quad (10)$$

In summary, the simplest action in a universe governed by teleparallel gravity is

$$I = \int d^4x e \left[\frac{T}{2\kappa^2} + \mathcal{L}_m \right], \quad (11)$$

where $e = \det(e_\mu^A) = \sqrt{-g}$ (one could also include a cosmological constant). Variation with respect to the vierbein fields gives equation of motion

$$e^{-1} \partial_\mu (e e_A^\rho S_{\rho}^{\mu\nu}) - e_A^\lambda T^{\rho}_{\mu\lambda} S_{\rho}{}^{\nu\mu} - \frac{1}{4} e_A^\nu T = \frac{\kappa^2}{2} e_A^\rho \mathbf{em}^\nu_{T\rho}, \quad (12)$$

where $\mathbf{em}^\nu_{T\rho}$ stands for the usual energy-momentum tensor. These equations are exactly the same as those of GR for every geometry choice. In particular, for the FRW background metric (2), the vierbein choice of the form

$$e_\mu^A = \text{diag}(1, a, a, a) \quad (13)$$

is an exact solution [2] of the field equation in Eq. (12), which does not generate a divergent energy for the whole space-time. Furthermore, it is easily seen that the corresponding Friedmann equations are identical to the GR ones, both at the background and perturbation levels [1,2,9].

4. Teleparallel dark energy

Let us now construct teleparallel dark energy. This will be done by adding a scalar field in the equivalent, teleparallel, formulation of GR. Thus, the action will simply read:

$$S = \int d^4x e \left[\frac{T}{2\kappa^2} + \frac{1}{2} (\partial_\mu \phi \partial^\mu \phi + \xi T \phi^2) - V(\phi) + \mathcal{L}_m \right]. \quad (14)$$

We emphasize that in the above action a nonminimal coupling between the scalar field and gravity is allowed. Although in the nonminimal case one could use alternative torsion scalars, we prefer to keep the standard one for simplicity. We also note that the action in (14) with the torsion formulation of GR is similar to the standard nonminimal quintessence where the scalar field couples to the Ricci scalar.

Variation of action (14) with respect to the vierbein fields yields equation of motion

$$\begin{aligned} \left(\frac{2}{\kappa^2} + 2\xi\phi^2 \right) \left[e^{-1} \partial_\mu (e e_A^\rho S_{\rho}^{\mu\nu}) - e_A^\lambda T^{\rho}_{\mu\lambda} S_{\rho}{}^{\nu\mu} - \frac{1}{4} e_A^\nu T \right] \\ - e_A^\nu \left[\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right] + e_A^\mu \partial^\nu \phi \partial_\mu \phi \\ + 4\xi e_A^\rho S_{\rho}^{\mu\nu} \phi (\partial_\mu \phi) = e_A^\rho \mathbf{em}^\nu_{T\rho}. \end{aligned} \quad (15)$$

Therefore, imposing the FRW geometry of the form (13) (that is (2)) we obtain the same Friedmann equations as in the conventional quintessence, namely (3), however in this case the scalar field energy density and pressure become:

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) - 3\xi H^2\phi^2, \quad (16)$$

$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi) + 4\xi H\phi\dot{\phi} + \xi(3H^2 + 2\dot{H})\phi^2. \quad (17)$$

Additionally, variation of the action with respect to the scalar field provides its evolution equation, namely:

$$\ddot{\phi} + 3H\dot{\phi} + 6\xi H^2\phi + V'(\phi) = 0. \quad (18)$$

Note that in the above expressions we have used the useful relation $T = -6H^2$, which straightforwardly arises from the calculation of (10) for the FRW geometry.

In this scenario, similar to the standard quintessence, dark energy is attributed to the scalar field, and thus its equation-of-state parameter (w_{DE}) is defined to be the same as that in (6), but ρ_ϕ and p_ϕ are now given by (16) and (17), respectively. Finally, one can see that the scalar field evolution (18) leads to the standard relation $\dot{\rho}_\phi + 3H(1 + w_\phi)\rho_\phi = 0$.

5. Cosmological implications

We now explore the cosmological implications of the scenario at hand. Firstly, we immediately observe that in the case of the minimal coupling, teleparallel dark energy coincides with quintessence (see (4)–(5) and (16)–(17)), and one can verify that at the level of perturbations too. This is expected since, concerning the gravitational sector, TEGR is identical with GR, and in the minimal case one just adds a distinct scalar sector, thus making no difference whether it is added in either of the two theories. However, things are different if we switch on the nonminimal coupling. In this case the additional scalar sector is coupled to gravity, with the curvature scalar in GR and with the torsion scalar in TEGR, and thus the resulting coupled equations do not coincide. Clearly, teleparallel dark energy, under the nonminimal coupling, is a different theory.

Let us proceed in presenting some basic and general features of the nonminimal coupling of the scalar-torsion theory. Apart from the straightforward results that dark energy possesses a dynamical nature as well as it can drive the universe acceleration, the most interesting and direct consequence of the dark energy density and pressure relations (16)–(17) is that the dark energy equation-of-state parameter can lie in the quintessence regime ($w_{DE} > -1$), in the phantom regime ($w_{DE} < -1$), or exhibit the phantom-divide crossing during cosmological evolution. This is a radical difference with the quintessence scenario and reveals the capabilities of the construction.

In order to present the above features in a more transparent way, we evolve numerically the cosmological system for dust matter ($w_m \approx 0$), using the redshift $z = a_0/a - 1$ as the independent variable, imposing the present scale factor a_0 to be equal to 1, the dark energy density $\Omega_{DE} \equiv \kappa^2\rho_\phi/(3H^2)$ at present to be ≈ 0.72 and its initial value to be ≈ 0 . Finally, concerning the scalar field potential we use the exponential ansatz of the form $V = V_0 e^{\lambda\phi}$.

In Fig. 1 we depict the w_{DE} -evolution for three realizations of the scenario at hand. In the case of the black-solid curve the teleparallel dark energy behaves like quintessence, in the red-dashed curve it behaves like a phantom, while in blue-dotted curve the dark energy exhibits the phantom-divide crossing during the evolution. Note that the crossing behavior in Fig. 1 is the one

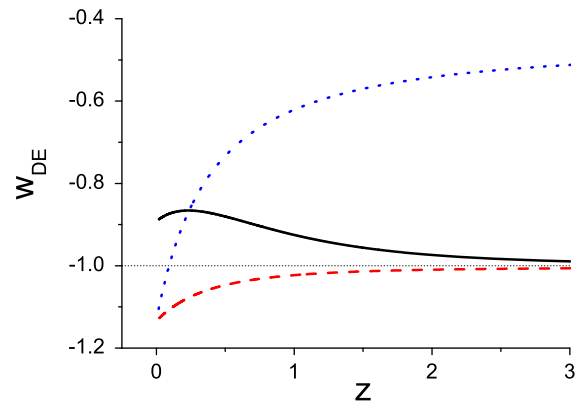


Fig. 1. Evolution of the dark energy equation-of-state parameter w_{DE} as a function of the redshift z , for three cases of the teleparallel dark energy scenario, in the exponential scalar-field potential ansatz of the form $V = V_0 e^{\lambda\phi}$. The black-solid curve presents quintessence-like behavior and corresponds to $\xi = -0.4$, $\lambda = 1.5$ and $V_0 \approx 2 \times 10^{-13}$, the red-dashed curve presents phantom-like behavior and corresponds to $\xi = -0.8$, $\lambda = 0.05$ and $V_0 \approx 10^{-13}$, and the blue-dotted curve presents the phantom-divide crossing and corresponds to $\xi = -0.25$, $\lambda = 40$ and $V_0 \approx 10^{-12}$. λ and V_0 are measured in κ^2 -units and the -1 -line is depicted for convenience.

avored by the observational data, in contrast with viable $f(R)$ -gravity models where it is the opposite one [10]. We remark that in the above graphs we focus on their qualitative features, and in particular we maintain the same potential just to stress that in principle one can obtain the various behaviors with the same potential. Clearly, one could be quantitatively more accurate and impose the observational $w_{DE}(z)$ as an input, reconstructing the corresponding potential. However in the present work we desire to remain as general as possible.

6. Discussion–conclusions

In the present scenario of “teleparallel” dark energy we have added a scalar field to the teleparallel equivalent to general relativity (TEGR), allowing for a nonminimal coupling between the field and gravity. In the minimally-coupled case the cosmological equations coincide with those of the standard quintessence. However when the nonminimal coupling is switched on the resulting theory exhibits different behavior. In particular, although the scalar field is canonical, one can obtain a dark energy sector being quintessence-like, phantom-like, or experiencing the phantom-divide crossing during evolution, a behavior that is much richer comparing to General Relativity (GR) with a scalar field. Moreover, the fact that the phantom regime can be described without the need of phantom fields, which have ambiguous quantum behavior [11], is a significant advantage.

The physical reason for the aforementioned difference, despite the equivalence of pure GR and pure TEGR, is that while in GR one couples the scalar field with the only suitable gravitational scalar, namely the Ricci scalar R , in the later one couples the scalar field with the only suitable gravitational scalar, namely the torsion scalar T . The richness of the resulting theory comparing to GR quintessence is additionally manifested in the fact that, although in the later one can perform a conformal transformation and transit to an “equivalent”, minimally-coupled, theory with transformed field and potential [4], in the former such a transformation does not exist since one obtains extra terms depending on the torsion tensor itself, as can be easily verified transforming the vierbeins as $e_A^\mu \rightarrow \Omega \tilde{e}_A^\mu$ (one applies in our case the similar analysis of [12] of the case of $f(T)$ scenarios). Thus, teleparallel dark energy cannot be transformed to an “equivalent” minimally coupled form, which

is known to be able to describe only the quintessence regime, and this indicates its richer structure. Such an absence of conformal transformation exists in other cosmological scenarios too, for example in scalar-field models with non-minimal derivative couplings, where it is also known that the resulting theories possess a richer structure [13].

The addition of a scalar field to TEGR was inspired by the corresponding procedure in GR. However, although in GR one can alternatively and equivalently generalize the action to $f(R)$, freeing himself of the need to add the scalar field, in the teleparallel formulation of GR the generalization to $f(T)$ [14] seems to spoil the local Lorentz invariance for all functions apart from the linear one [15]. However, at the background level no new degrees of freedom are present, while at linear perturbation the new vector degree of freedom only satisfies constraint equations [16]. Similarly, in our generalization of TEGR, in the case of non-minimal coupling, a Lorentz-violating term appears (the last term in the left hand side of (15)), despite the fact that the theory is linear in T . However, no new degree of freedom will appear at the background level on which we focus on this work. Clearly, going beyond background evolution and examine whether the Lorenz violations do indeed appear under cosmological geometries and scales (we have checked that at the low-energy limit, the theory's basic Parametrized Post Newtonian parameters are consistent with Solar System observations), and if they can be detected, is an interesting and open subject, as it is in $f(T)$ gravity too, and will be incorporated in more details elsewhere.

In summary, the rich behavior of teleparallel dark energy makes it a promising cosmological scenario. In this work we have desired to remain as general as possible, and present its basic and novel features. Clearly, before it can be considered as a good candidate for the description of nature, one needs to investigate various subjects, such as to perform a detailed perturbation analysis, to use observational data in order to constrain the parameters of the model, to examine the phase-space behavior in order to reveal the late-time cosmological features, etc. Such aspects, although necessary, lie outside the goal of the present work and are left for future investigations.

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