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# The $\beta$-Meixner model

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ABSTRACT

We propose to approximate the Meixner model by a member of the $\beta$-family introduced by Kuznetsov (2010) in [2]. The advantage of the approximation is the *semi-explicit* formulae for the running extrema under the $\beta$-family processes which enables us to produce more efficient algorithms for pricing path dependent options through the Wiener–Hopf factors. We will explore the performance of the approximation both in an equity framework and in the credit risk setting, where we use the approximation to calibrate a surface of credit default swaps. The paper follows the approach of the study made by Schoutens and Damme (2010) in [1], where the aim was to approximate the variance gamma. We will contextualize the results by Schoutens and Damme (2010) in [1] and the ones here with respect to the approach taken by Jeannin and Pistorius (2010) in [15]. An asymptotic expression for the rate of convergence of the approximation is derived.



## 1. Introduction

Schoutens and Damme [1] explore the numerical performance of the $\beta$-family introduced by Kuznetsov in [2], both in the equity and in the credit risk field, as an approximation to the variance gamma (VG) process. The VG process is a very popular model in financial mathematics that has now been around for more than 20 years. Their conclusion is that, thanks to the *semi-explicit* formulae for the running extrema under the $\beta$-family, they are able to produce faster and more accurate results for pricing credit default swaps (CDSs). In fact, the formulae for the running extrema are derived from explicit expressions of the Wiener–Hopf factorization. Under the VG process, the CDSs are priced using a partial differential integral equation (PDIE) approach described by Cariboni and Schoutens in [3]. The prices under both processes are equivalent and hence the methodology serves as an alternative approximate algorithm.

The aim of the present paper is to reproduce the same sort of results with respect to the Meixner process. This is also a widespread model in the financial literature. In this case, the CDS spreads under Meixner model will be computed by an inverse Fourier method. More precisely, the one described by Fang et al. in [4] and based on the cosine series expansion of the density of a Lévy process, which is called COS method (see [5,6]). Recall that apart from Monte Carlo simulation, the most general methodologies for pricing path dependent options under Lévy models are PDIEs and Fourier methods. Together with the paper of Schoutens and Damme [1], the present work shows that there is a potential use of Wiener–Hopf theory to price path dependent options as an alternative for classical approaches.

The Wiener–Hopf factorization for Lévy processes has lately been receiving an increasing attention for numerical purposes since the papers of Kuznetsov [2,7] and Kuznetsov et al. [8], which describe a wide range of Lévy processes for which the Wiener–Hopf factorization is known. Some other studies have been devoted to study the numerical tractability of the Wiener–Hopf factorization to price path dependent options, see for instance the work of Kudryavtsev and

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Levendorskiĭ [9,10]. Recall that the Wiener–Hopf factors give a description of the distributions of the extrema under an independent exponential time change. It is worth remarking that explicit expressions of the Wiener–Hopf factorization were a rare result except for particular cases such as one sided Lévy processes in Rogers [11], double sided exponential processes in Kou and Wang [12] or the cases treated in Boyarchenko and Levendorskiĭ [13] or Lewis and Mordecki [14]. In the present work we will show that the asymptotic approximation in Schoutens and Damme [1] and the one described here are particular cases of the more general technique of approximating generalized hyper-exponential Lévy processes by hyper-exponential jump-diffusion models, which was used for pricing digital options with barriers in Jeannin and Pistorius [15]. We will give an asymptotic rate of convergence for the simulation of the infinite divisible distributions derived from the Wiener–Hopf factors.

The purpose of this paper is therefore twofold. From one side the results here and the ones reported in Schoutens and Damme [1] compare the Wiener–Hopf methodology with respect to the PDIE and the Fourier methods to price options depending on the extrema of the process. On the other hand, although the Wiener–Hopf approach is just valid for a particular family of processes, we will contextualize the methodology with respect to the papers of Jeannin and Pistorius [15], Kuznetsov [2,7] and Kuznetsov et al. [8], which describe a rich family of Lévy processes.

The paper is organized as follows. In Section 2 we present the Meixner model and the $\boldsymbol{\beta}$-family, we also construct the $\boldsymbol{\beta}$-$M$ process. Section 3 will relate the present work to the general setting of Jeannin and Pistorius [15] and Kuznetsov et al. [8]. We also give the rate of convergence of the approximation. Section 4 will derive the expressions to price vanilla options and CDS showing the numeric results. We will calibrate the Meixner and the $\boldsymbol{\beta}$-$M$ process to a surface of vanilla options using the Carr and Madan formula (see [16]). After that, we will calibrate both models to a surface of CDS spreads. The spreads are computed under the Meixner model with the COS method, and under the $\boldsymbol{\beta}$-$M$ process with the Wiener–Hopf factorization. Finally, we conclude the paper with some remarks.

## 2. The $\boldsymbol{\beta}$-family and the Meixner process

Let $X = \{X_t\}_{t\geq 0}$ be a Lévy process and recall that the law of every Lévy process is characterized by the triplet $(\mu, \sigma, \nu)$, where $\mu \in \mathbb{R}$, $\sigma \geq 0$ is the Brownian component and $\nu$ is a measure, concentrated in $\mathbb{R} \setminus \{0\}$ and such that $\int_{\mathbb{R}}(1 \wedge x^2)\nu(dx) < \infty$. More precisely, the process is described by its Lévy exponent, $\Psi_{X_1}(z)$, as

$$\varphi_{X_t}(z) = \mathbb{E}[e^{izX_t}] = e^{-t\Psi_{X_1}(z)} \quad \forall z \in \mathbb{C}.$$

The Lévy–Khintchine representation gives the relation between the Lévy exponent and the triplet $(\mu, \sigma, \nu)$:

$$\Psi_{X_1}(z) = -i\mu z + \frac{\sigma^2}{2}z^2 - \int_{-\infty}^{\infty}(e^{izx} - 1 - izh(x))\nu(dx), \tag{1}$$

where $h$ is the cut-off function. In the following we can consider $h(x) \equiv x$ for the Lévy measures we are interested in.

The Meixner process is a pure jump process often used in the financial literature, we refer to Schoutens [17] and the references therein for a variety of examples where this model has been used. The construction of the Meixner process starts from an infinite divisible distribution with characteristic function

$$\varphi(u) = \left(\frac{\cos(b/2)}{\cosh((au - ib)/2)}\right)^{2d},$$

where $a > 0$, $-\pi < b < \pi$ and $d > 0$. This distribution characterizes the law of the process at one unit time and hence the Lévy exponent. The Meixner process does not have a Brownian component and the Lévy measure is absolutely continuous, hence its triplet is given by $(\mu, 0, \nu)$ where

$$\mu = ad\tanh(b/2) - 2d\int_1^{\infty}\frac{\sinh(bx/a)}{\sinh(\pi x/a)}dx$$

$$\nu(x) = d\frac{\exp(bx/a)}{x\sinh(\pi x/a)}. \tag{2}$$

We will make an abuse of notation by using the same name for the Lévy measure and its density if there is no confusion.

The $\beta$-family is a parametric family of Lévy processes introduced by Kuznetsov [2] which belongs to the more general family of processes called meromorphic Lévy processes ($M$-processes) introduced by Kuznetsov et al. [8]. A member of the $\boldsymbol{\beta}$-family is a 10-parameter process with triplet given by $(\mu, \sigma, \nu)$ where the Lévy measure is absolutely continuous with density

$$\nu(x) = c_1\frac{e^{-\alpha_1\beta_1 x}}{(1 - e^{-\beta_1 x})^{\lambda_1}}1_{x>0} + c_2\frac{e^{\alpha_2\beta_2 x}}{(1 - e^{\beta_2 x})^{\lambda_2}}1_{x<0}, \tag{3}$$

where $\alpha_i > 0$, $\beta_i > 0$, $c_i \geq 0$ and $\lambda_i \in (0, 3)$. For the sake of completeness we reproduce here the expression of the characteristic exponent which is derived by Kuznetsov [2, Proposition 9] and satisfies

$$\Psi_{X_1}(z) = -i\mu z + \frac{\sigma^2}{2}z^2 - [c_1 I(z; \alpha_1, \beta_1, \lambda_1) + c_2 I(-z; \alpha_2, \beta_2, \lambda_2)], \tag{4}$$

where

$$I(z;\alpha,\beta,\lambda)=\begin{cases} I_1(z;\alpha,\beta,\lambda);\ \lambda\in(0,3)\setminus\{1,2\};\\ I_2(z;\alpha,\beta,\lambda);\ \lambda=1;\\ I_3(z;\alpha,\beta,\lambda);\ \lambda=2,\end{cases}$$

and

$$I_1(z;\alpha,\beta,\lambda)=\frac{1}{\beta}\mathbb{B}\left[\alpha-\frac{iz}{\beta},1-\lambda\right]-\frac{1}{\beta}\mathbb{B}[\alpha,1-\lambda]\left(1+\frac{iz}{\beta}[\psi(1+\alpha-\lambda)-\psi(\alpha)]\right)$$
$$I_2(z;\alpha,\beta,\lambda)=-\frac{1}{\beta}\left[\psi\left(\alpha-\frac{iz}{\beta}\right)-\psi(\alpha)\right]-\frac{iz}{\beta^2}\psi'(\alpha)$$
$$I_3(z;\alpha,\beta,\lambda)=-\frac{1}{\beta}\left(1-\alpha+\frac{iz}{\beta}\right)\left[\psi\left(\alpha-\frac{iz}{\beta}\right)-\psi(\alpha)\right]-\frac{iz(1-\alpha)}{\beta^2}\psi'(\alpha),$$

and $\mathbb{B}(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ is the Beta function and $\psi(x)=\frac{d}{du}\log(\Gamma(u))\Big|_x$ the Digamma function.

### *2.1. The $\beta$-M process*

Now we make a particular choice of the parameters in the Lévy measure of a general $\boldsymbol{\beta}$-process. To be precise we consider that $\lambda_1=\lambda_2=2$, $\beta_1=\beta_2=1$ and $c_1=c_2=c$, therefore the 8-parameter Lévy density in (3) has now become a 3-parameter density following the expression

$$\nu(x)=c\frac{e^{-\alpha_1 x}}{(1-e^{-x})^2}1_{x>0}+c\frac{e^{\alpha_2 x}}{(1-e^{x})^2}1_{x<0}. \tag{5}$$

We claim that the above measure matches the features of the Lévy measure in (2) which belongs to the Meixner model. From one side the right choice of $c$ will make both densities asymptotically equivalent at the origin. Note that

$$\lim_{x\to 0^+}\frac{(1-e^{-x})^2}{x\sinh(x)}=1 \tag{6}$$

and hence, for a given set of parameters $(a,b,d)$ in (2), the choice of $c=ad/\pi$ in the triplet $(c,\alpha_1,\alpha_2)$ of (5) will give an equivalent density in a neighbourhood of zero. Indeed, for $x>0$

$$\begin{aligned} d\frac{\exp(bx/a)}{x\sinh(\pi x/a)} &= 2d\frac{e^{bx/a}}{x(e^{\pi x/a}-e^{-\pi x/a})}\\ &= 2d\frac{e^{(b-\pi)x/a}}{x(1-e^{-2\pi x/a})}\\ &\approx \frac{ad}{\pi}\frac{e^{(b-\pi)x/a}}{x(1-e^{-x})}\\ &\approx \frac{ad}{\pi}\frac{e^{(b-\pi)x/a}}{(1-e^{-x})^2}, \end{aligned} \tag{7}$$

where the approximate equalities stand for the asymptotic limits as $x\to 0^+$. Same sort of derivations hold for $x<0$.

Observe that both densities decay exponentially outside zero, therefore we claim that both densities behave similar and expect equivalent prices. We denote by $\boldsymbol{\beta}$-$M$ process a Lévy process with triplet $(\mu,0,\nu)$ where $\nu$ is absolutely continuous with density given by (5). For the $\boldsymbol{\beta}$-$M$ process we set the volatility equal zero since we want to mimic the pure jump behaviour of the Meixner model. Although $\alpha_1$ and $\alpha_2$ can be chosen freely and still have an asymptotic equivalence of densities around zero, the limit (7) suggests the rule of thumb

$$c=ad/\pi,\qquad \alpha_1=(\pi-b)/a\quad\text{and}\quad \alpha_2=(\pi+b)/a$$

to convert parameters from one model to the other.

**Remark 2.1.** The particular choice $\beta_1=\beta_2$ and $\lambda_1=\lambda_2$ in the Lévy measure (3) as well as $\sigma=0$ in (1) makes the output family of Lévy processes belong to the Lamperti stable family. Many fluctuation identities related to the Wiener–Hopf factorization are available in close form solution for such processes. We refer to Caballero and Chaumont [18] and Caballero et al. [19] for the definition and properties of Lamperti stable processes. In particular we have the following succession of inclusions:

$\boldsymbol{\beta}$-$M$ processes $\subset$ Lamperti stable $\subset$ $\boldsymbol{\beta}$-processes $\subset$ $M$-processes

**Remark 2.2.** A pure jump Lévy process has infinite variation if and only if $\int_{-1}^{1}|x|\nu(dx)=\infty$. It is clear from the asymptotic equality (6) that the Meixner process is of infinite variation. Therefore the $\boldsymbol{\beta}$-*M* process is also of infinite variation. On the other hand the $\boldsymbol{\beta}$-*VG* process defined in Schoutens and Damme [1] is of finite variation which proves the rich variety of behaviours that the $\boldsymbol{\beta}$-processes can exhibit.

### *2.2. The running extrema under the $\boldsymbol{\beta}$-M process*

The Wiener–Hopf factorization is an analytical decomposition of the Lévy exponent, $\Psi_{X_1}(z)$, associated to the process. For every $q>0$ there exist a pair of characteristic functions $\varphi_q^+(z)$ and $\varphi_q^-(z)$ of infinitely divisible laws such that

$$\frac{q}{q+\Psi_{X_1}(z)}=\varphi_q^+(z)\varphi_q^-(z),\quad z\in\mathbb{R}.$$

More precisely,

$$\varphi_q^+(z)=\mathbb{E}[e^{iz\overline{X}_{\tau(q)}}]\quad\text{and}\quad\varphi_q^-(z)=\mathbb{E}[e^{iz\underline{X}_{\tau(q)}}]\tag{8}$$

where $z\in\mathbb{R}$, $\overline{X}_t=\sup\{X_s\mid 0\le s\le t\}$, $\underline{X}_t=\inf\{X_s\mid 0\le s\le t\}$ and $\tau(q)$ is an exponential distributed random variable with parameter $q$ independent of $X$. The Wiener–Hopf factors are essentially the characteristic function for the running supremum and infimum of a process at independent and exponentially distributed random times and for this reason are of great importance in pricing CDS. For meromorphic Lévy processes there is an explicit expression of the factors in (8) with respect to the poles and the zeros of the meromorphic function $\Psi_{X_1}(z)$. We refer to Kuznetsov et al. [8] for the exact properties and definition of meromorphic Lévy processes but for the purpose of this paper it is enough to say that the Wiener–Hopf factors are of the form

$$\varphi_q^+(z)=\frac{1}{1+\frac{iz}{\zeta_0^-}}\prod_{n\le -1}\frac{1+\frac{iz}{\rho_n}}{1+\frac{iz}{\zeta_n}}\quad\text{and}\quad\varphi_q^-(z)=\frac{1}{1+\frac{iz}{\zeta_0^+}}\prod_{n\ge 1}\frac{1+\frac{iz}{\rho_n}}{1+\frac{iz}{\zeta_n}},\tag{9}$$

where $\{\rho_n\}_n$ are the poles and $\{\zeta_n\}_n$, $\zeta_0^-$ and $\zeta_0^+$ the zeros of the equation

$$\Psi_{X_1}(i\zeta)+q=0\tag{10}$$

which are all real and respect

$$\ldots\zeta_{-2}<\rho_{-2}<\zeta_{-1}<\rho_{-1}<\zeta_0^-<0<\zeta_0^+<\rho_1<\zeta_1<\rho_2<\zeta_2\cdots\tag{11}$$

A related work of one of the authors treat the topic of inverting analytic characteristic functions in [20] – the family of measures described there can be used to construct nontrivial examples of meromorphic Lévy processes. Moreover, that point of view is going to be used to derive the results of Section 3.1.

Let us now adapt here the exact expressions of the Wiener–Hopf factors (9) for a $\boldsymbol{\beta}$-*M* process (cf. Kuznetsov [2, Theorem 10] for the exact expressions of (9) in the case of a process from the $\boldsymbol{\beta}$-family):

$$\varphi_q^+(z)=\frac{1}{1+\frac{iz}{\zeta_0^-(q)}}\prod_{n\le -1}\frac{1+\frac{iz}{(n+1-\alpha_1)}}{1+\frac{iz}{\zeta_n(q)}}\quad\text{and}\quad\varphi_q^-(z)=\frac{1}{1+\frac{iz}{\zeta_0^+(q)}}\prod_{n\ge 1}\frac{1+\frac{iz}{(n-1+\alpha_2)}}{1+\frac{iz}{\zeta_n(q)}},$$

where $\{\zeta_n(q)\}_n$, $\zeta_0^+(q)$ and $\zeta_0^-(q)$ are the zeros of the Eq. (10) where $\Psi_{X_1}$ is given in (4), with the appropriate choice of parameters, and where we have made explicit the dependence with respect to $q$. The interlacing property (11) is now reduced to

$$\begin{aligned}&\zeta_0^-(q)\in(-\alpha_1,0)\\&\zeta_0^+(q)\in(0,\alpha_2)\\&\zeta_n(q)\in(\alpha_2+n-1,\alpha_2+n),\quad n\ge 1\\&\zeta_n(q)\in(-\alpha_1+n,-\alpha_1+n+1),\quad n\le -1.\end{aligned}\tag{12}$$

It turns out that the expressions $\varphi_q^-(z)$ and $\varphi_q^+(z)$ are invertible, and the distribution for the running infimum can be written as (cf. Kuznetsov [2, Theorem 11])

$$\mathbb{P}\left[\underline{X}_{\tau(q)}>x\right]=1-c_0^+(q)e^{\zeta_0^+(q)x}-\sum_{n\ge 1}c_n(q)e^{\zeta_n(q)x},\tag{13}$$

where

$$c_0^+(q)=\prod_{n\ge 1}\frac{1-\frac{\zeta_0^+(q)}{(n-1+\alpha_2)}}{1-\frac{\zeta_0^+(q)}{\zeta_n(q)}},\qquad c_k(q)=\frac{1-\frac{\zeta_k(q)}{(k-1+\alpha_2)}}{1-\frac{\zeta_k(q)}{\zeta_0^+(q)}}\prod_{\substack{n\ge 1\\ n\ne k}}\frac{1-\frac{\zeta_k(q)}{(n-1+\alpha_2)}}{1-\frac{\zeta_k(q)}{\zeta_n(q)}}\quad\text{for } k\ge 1.\tag{14}$$

The derivations by Kuznetsov [2] show that $\mathbb{P}[\underline{X}_t > x]$ is the inverse Laplace transform of $\mathbb{P}[\underline{X}_{\tau(q)} > x]$, therefore one can recover the distribution of the running infimum up to a deterministic time $t$, i.e. we have the equality

$$\frac{d}{dx}\mathbb{P}\left[\underline{X}_{\tau(q)} \leq x\right] = \frac{d}{dx}\int_0^\infty qe^{-qt}\mathbb{P}\left[\underline{X}_t \leq x\right] dt. \tag{15}$$

Therefore in order to compute the running infimum at a deterministic point using the above equality we need to invert the Laplace transform. The general methods to invert a Laplace transform require to evaluate the transformation at complex points, this means evaluating the right hand side of expression (13) at complex points $q$. This expression essentially depend on $\zeta_0^+(q)$ and $\{\zeta_n(q)\}_{n\geq 1}$. Unfortunately, the intervals of localization of such zeros given above are only valid for $q > 0$. One way to overcome this problem is to use the Gaver–Stehfest algorithm, which was also used by Schoutens and Damme [1]. This method only requires to compute the zeros for $q > 0$. Under the Meixner model the computation of $\mathbb{P}[\underline{V}_t > B]$ will be given by the COS method. This method is described in Fang et al. [4] and based in the studies of Fang and Oosterlee [5,6]. This algorithm is based on the fact that the Fourier cosine expansion of the conditional density for a Lévy process is close related to its characteristic function.

**Remark 2.3.** A remark on whether the law of the extrema have an atom is made in the derivations of Kuznetsov [2, Remark 6]. Recalling Remark 2.2 we can conclude that the law of the infimum of the $\boldsymbol{\beta}$-$M$ process is atomless.

**Remark 2.4.** The expression (13) is a generalized Dirichlet series. As a consequence, if it is convergent for some $x_0 < 0$, then it is so for all $x < x_0$ and the convergence is uniform on every compact subset of the half-line (see [21, p. 9]). It is clear from (12) that (13) is convergent for $x < 0$, therefore it differentiable and we recover the expression in Kuznetsov [2, Theorem 11].

**Remark 2.5.** The notation of the first positive and negative zero, $\zeta_0^-$ and $\zeta_0^+$, of the Eq. (10) might not become clear here, but we decided to use it as it the one found in Kuznetsov [2].

## 3. Generalized hyper-exponential and meromorphic Lévy processes

A Lévy process is said to a be generalized hyper-exponential process if its Lévy measure has a density which can be written as

$$k(x) = k_+(x)1_{\{x>0\}} + k_-(-x)1_{\{x<0\}},$$

where $k_+$ and $k_-$ are completely monotone functions on $(0, \infty)$. Bernstein's theorem on completely monotone functions give the following representation

$$k(x) = 1_{x>0}\int_0^\infty e^{-ux}\mu_+(du) + 1_{x<0}\int_{-\infty}^0 e^{-ux}\mu_-(du), \tag{16}$$

for measures $\mu_+$ and $\mu_-$ supported on $(0, \infty)$ and $(-\infty, 0)$ respectively. Jeannin and Pistorius [15] give several examples which belong to this family, some of them are the double exponential model described in Kou and Wang [12], the variance gamma, the Meixner process or the normal-inverse Gaussian process. For the double exponential model the Lévy measure can be written as a sum of exponentials because $\mu_+$ and $\mu_-$ are point mass measures. In this particular case the Wiener–Hopf factors are known in explicit form and this result can be generalized to the case where $\mu_+$ and $\mu_-$ are point mass measures concentrated in a finite number of points. That is the model used in Jeannin and Pistorius [15] to approximate generalized hyper-exponential processes. The idea is to use a Riemann sum to approximate the density in (16) as

$$k(x) \approx 1_{x>0}\sum_{i\in I} w_i e^{-\xi_i x} + 1_{x<0}\sum_{j\in J} w_j e^{-\xi_j x}, \tag{17}$$

where $I, J$ are finite partitions of $(0, \infty)$ and $(-\infty, 0)$ respectively, and $w_i$, $w_j$ are weights. For instance, one could choose $\xi_i \in [t_i, t_{i+1}]$, $\xi_j \in [t_{j+1}, t_j]$, $w_i = \mu_+([t_i, t_{i+1}])$ and $w_j = \mu_-([t_{j+1}, t_j])$ for $t_i \in I$ and $t_j \in J$. A process with a Lévy density as the right hand side of (17) is called hyper-exponential jump-diffusion Lévy processes. A similar study from another point of view that precedes the work of Jeannin and Pistorius [15] can be found in Asmussen et al. [22].

The only drawback in Jeannin and Pistorius [15] methodology is that the intensities of the approximation are fixed in advance and the computation of the weights are done by minimizing the square error with respect to the original measure. The need for imposed intensities makes the algorithm weak. Jeannin and Pistorius [15] comment on the possibility of using Feldmann and Whitt [23] algorithm. A more systematic approach can be found in Crosby et al. [24]. There, the approximation is done at the level of the Lévy exponents but at the end the algorithm also approximates an infinite integral using the Gaussian quadrature. This methodology leads to a ill-posed linear problem, solved using Tikhonov regularization.

Our idea is to use the family of meromorphic Lévy processes described by Kuznetsov et al. [8]. Note that the characterization of meromorphic Lévy process is that the Lévy measure has a density of the form (16) and the measures $\mu_+$ and $\mu_-$ are point mass measures with infinite support but without finite accumulation points. This implies that the density is an infinite mixture of exponentials. Kuznetsov et al. [8] show that in this case an explicit representation for the

Wiener–Hopf factorization is also possible. To fix ideas let us restrict our discussion to the $\boldsymbol{\beta}$-$M$ process. The numerical implementation of the formulae (13) and (14) must be done by a truncation of the infinite sum and the infinite product. This means that essentially we are approximating the Wiener–Hopf factors of the process by a finite product. It turns out that these expressions for the Wiener–Hopf factors generate hyper-exponential jump-diffusion processes. Here though the particular choices of the intensities and the weights for the approximation are given by the way we approximated the Lévy measure. To show that, consider Newton's generalized binomial theorem which states the equality

$$(1-e^{-x})^{-n} = \sum_{k\geq 0} \binom{n+k-1}{k} e^{-kx} \quad x \geq 0, n \in \mathbb{N}.$$

Therefore, in our case of study, the Lévy measure of the Meixner model is being approximated by the measure of the $\boldsymbol{\beta}$-$M$ that can be written as

$$\begin{aligned}\nu(x) &= c\frac{e^{-\alpha_1 x}}{(1-e^{-x})^2} 1_{x>0} + c\frac{e^{\alpha_2 x}}{(1-e^{x})^2} 1_{x<0} \\ &= 1_{x>0} \sum_{k\geq 1} cke^{-(k+\alpha_1-1)x} + 1_{x<0} \sum_{k\geq 1} cke^{(k+\alpha_2-1)x}.\end{aligned}$$

A similar derivation holds for the case of the VG studied by Schoutens and Damme [1]. In order to numerically implement the probability of survival, we need to truncate the expression of (13) which means a truncation of the Wiener–Hopf factors up to a finite product which in turns can be thought as the truncation of the above sum representation of the Lévy density and hence recovering the hyper-exponential jump-diffusion processes. Therefore we are not doing anything more sophisticated as the approximation proposed by Jeannin and Pistorius [15], but our approach do not need any exogenous assumptions on the choice of the intensities and weights. The completely general study of this methodology is out of the scope of this paper but, in view of the results in Schoutens and Damme [1] and the ones presented here, an investigation of how good meromorphic Lévy processes are as an approximating family is of great interest.

### *3.1. Rate of convergence*

As pointed out in the above section, when using the Wiener–Hopf factorization of meromorphic Lévy processes for numerical implementations we need to truncate the infinite products in (9). Assuming we are interested in the distribution of the infimum, $\underline{X}_{\tau(q)}$, we will be working with the approximate random variable $\underline{X}^N_{\tau(q)}$ whose moment generating function is

$$\mathbb{E}[e^{z\underline{X}^N_{\tau(q)}}] = \frac{1}{1+\frac{z}{\zeta_0^+}} \prod_{n\geq 1}^{N} \frac{1+\frac{z}{\rho_n}}{1+\frac{z}{\zeta_n}},$$

where we have avoided the dependence on $q$ to ease the notation. It is clear that the moment generating function of $\underline{X}^N_{\tau(q)}$ converge to the one of $\underline{X}_{\tau(q)}$ and hence we have convergence in distribution as $N \to \infty$. Now we derive the convergence in mean square. Notice that each factor of the moment generating function of $\underline{X}_{\tau(q)}$ in (9) is of the form

$$\frac{1+\frac{z}{\rho_n}}{1+\frac{z}{\zeta_n}} = \frac{\zeta_n}{\rho_n} + \left(1-\frac{\zeta_n}{\rho_n}\right)\frac{1}{1+\frac{z}{\zeta_n}}$$

and hence it can be thought as the moment generating function of the probability measure

$$\frac{\zeta_n}{\rho_n}\delta_0 + \left(1-\frac{\zeta_n}{\rho_n}\right)\tau(\zeta_n), \tag{18}$$

where $\delta_0$ is an atom at zero. This suggests that $\underline{X}_{\tau(q)}$ can be seen as the infinite sum of i.i.d. random variables with probabilities given by the above expression with the corresponding parameters. Therefore we can compute the moment generating function of the difference $\underline{X}_{\tau(q)} - \underline{X}^N_{\tau(q)}$ as

$$\mathbb{E}[e^{z(\underline{X}_{\tau(q)}-\underline{X}^N_{\tau(q)})}] = \prod_{n\geq N+1} \frac{1+\frac{z}{\rho_n}}{1+\frac{z}{\zeta_n}}, \tag{19}$$

where $z \in \mathbb{C}$ belongs to a neighbourhood of zero. The above expression is regular around zero and hence

$$\mathbb{E}[(\underline{X}_{\tau(q)} - \underline{X}^N_{\tau(q)})^2] = \frac{d^2}{dz^2}\mathbb{E}[e^{z(\underline{X}_{\tau(q)}-\underline{X}^N_{\tau(q)})}]\bigg|_{z=0}.$$

In the following we compute $\frac{d^2}{dz^2}\mathbb{E}[e^{z(\underline{X}_{\tau(q)}-\underline{X}^N_{\tau(q)})}]$:

$$\frac{d^2}{dz^2}\mathbb{E}[e^{z(\underline{X}_{\tau(q)}-\underline{X}^N_{\tau(q)})}] = \frac{d}{dz}\left(\sum_{n\geq N+1}^{\infty}\left(1-\frac{\zeta_n}{\rho_n}\right)\frac{-1}{\zeta_n}\left(1+\frac{z}{\zeta_n}\right)^{-2}\prod_{\substack{k\geq N+1\\k\neq n}}^{\infty}\frac{1+\frac{z}{\rho_k}}{1+\frac{z}{\zeta_k}}\right)$$
$$= \sum_{n\geq N+1}^{\infty}\left(1-\frac{\zeta_n}{\rho_n}\right)\frac{2}{\zeta_n^2}\left(1+\frac{z}{\zeta_n}\right)^{-3}\prod_{\substack{k\geq N+1\\k\neq n}}^{\infty}\frac{1+\frac{z}{\rho_k}}{1+\frac{z}{\zeta_k}}$$
$$+ \sum_{n\geq N+1}^{\infty}\left(1-\frac{\zeta_n}{\rho_n}\right)\frac{-1}{\zeta_n}\left(1+\frac{z}{\zeta_n}\right)^{-2}\sum_{\substack{k\geq N+1\\k\neq n}}^{\infty}\left(1-\frac{\zeta_k}{\rho_k}\right)\frac{-1}{\zeta_k}\left(1+\frac{z}{\zeta_k}\right)^{-2}\prod_{\substack{r\geq N+1\\r\neq n,k}}^{\infty}\frac{1+\frac{z}{\rho_r}}{1+\frac{z}{\zeta_r}}.$$

Finally

$$\frac{d^2}{dz^2}\mathbb{E}[e^{z(\underline{X}_{\tau(q)}-\underline{X}^N_{\tau(q)})}]\bigg|_{z=0} = \sum_{n\geq N+1}^{\infty}\left(\frac{1}{\zeta_n}-\frac{1}{\rho_n}\right)\frac{2}{\zeta_n}+\sum_{n\geq N+1}^{\infty}\left(\frac{1}{\zeta_n}-\frac{1}{\rho_n}\right)\sum_{\substack{k\geq N+1\\k\neq n}}^{\infty}\left(\frac{1}{\zeta_k}-\frac{1}{\rho_k}\right),$$

and now we observe that $0 < \zeta_n^{-1}-\rho_n^{-1} < \zeta_n^{-1}-\zeta_{n+1}^{-1}$ and hence

$$\frac{d^2}{dz^2}\mathbb{E}[e^{z(\underline{X}_{\tau(q)}-\underline{X}^N_{\tau(q)})}]\bigg|_{z=0} \leq \sum_{n\geq N+1}^{\infty}\left(\frac{1}{\zeta_n}-\frac{1}{\zeta_{n+1}}\right)\frac{2}{\zeta_n}+\sum_{n\geq N+1}^{\infty}\left(\frac{1}{\zeta_n}-\frac{1}{\zeta_{n+1}}\right)\sum_{\substack{k\geq N+1\\k\neq n}}^{\infty}\left(\frac{1}{\zeta_k}-\frac{1}{\zeta_{n+1}}\right)\leq\frac{3}{\zeta_{N+1}^2}.$$

This means that $\mathbb{E}[(\underline{X}_{\tau(q)}-\underline{X}^N_{\tau(q)})^2] = \mathrm{O}(\zeta_{N+1}^{-2}(q))$. Since the above derivation used only the interlacing property, it is clear that the result holds for general $M$-processes.

Unfortunately the distribution of $\underline{X}^N_{\tau(q)}$ is not the one we are interested in. There are different methodologies to invert the time change to compute $\underline{X}^N_t$ for a deterministic $t$. Schoutens and Damme [1] use the Gaver–Stehfest algorithm to invert the relation (15) while the approach in Kuznetsov et al. [25] uses a Monte Carlo algorithm and samples directly from the distribution of $\underline{X}^N_{\tau(q)}$. In the following section we will follow the algorithm proposed by Schoutens and Damme [1]. Different methodologies introduce different errors, nevertheless the above result justifies the approach used and can be taken as a benchmark to compare the performance of the implementation.

**Remark 3.1.** Note that the right hand side of (19) is a meromorphic function, in particular the infinite product converges uniformly on every compact set not containing the points $\{\zeta_n\}_{n\geq N+1}$ (cf. Levin [26, Section 27.2]). From a probabilistic point of view this means that 0 belongs to the interior of the domain of the moment generating function and thus the associated random variable – $(\underline{X}_{\tau(q)}-\underline{X}^N_{\tau(q)})$ – has moments of all orders which can be computed by evaluating the derivative of the infinite product at the origin.

## 4. Numerical applications

We are going to compare the performance of the Meixner and the $\boldsymbol{\beta}$-$M$ process by pricing a surface of call options and a CDS curve. In both cases we are going to follow an exponential Lévy process. We will assume that the underlying stock in the call options follow the equation

$$S_t = S_0e^{(r-d+\omega)t+X_t},$$

where $S_0$ is the spot at time 0, $r$ is the risk free rate, $d$ is the dividend yield, $\omega$ is the mean correcting drift to ensure that the discounted prices are martingales and $X_t$ is a Lévy process — here this will be either the Meixner or the $\boldsymbol{\beta}$-$M$ process. A key function in the following will be the characteristic function of the $\log(S_t)$. This can be derived as

$$\varphi_{\log(S_t)}(u) = e^{iu(\log(S_0)+(r-d+\omega)t)}\varphi_{X_t}(u)$$
$$= e^{iu(\log(S_0)+(r-d+\omega)t)-t\Psi_{X_1}(u)},$$

where $\omega = \Psi_{X_1}(-i) = -\log\varphi_{X_1}(-i)$, $\varphi_{X_t}$ is the characteristic function and $\Psi_{X_1}$ is the Lévy exponent of the process.

In the credit risk setting we will follow a firm value approach. Therefore the total aggregate asset value of a firm, $V_t$, follows the dynamics given by

$$V_t = V_0e^{(r-d+\omega)t+X_t}.$$

In such approximation the default of a company will be consider to occur when the process $V_t$ reaches a certain barrier, $B$, for the first time, i.e. at time

$$\tau_B = \inf\{0\leq t\leq T \mid V_t\leq B\}.$$

The barrier $B$ is typically consider to be $B = RV_0$ for a certain recovery rate $R\in(0,1)$.

**Table 1**
Calibration on the vanilla surface.

| | $\boldsymbol{\beta}$-$M$ model $(c, \alpha_1, \alpha_2)$ | Meixner model $(a, b, d)$ | RT model $(\hat{c}, \hat{\alpha}_1, \hat{\alpha}_2)$ |
|---|---|---|---|
| Optimal parameters | (0.0538, 7.9017, 1.7344) | (0.4764, −1.4723, 0.2581) | – |
| RMSE | 3.1612 | 3.3506 | 21.4620 |
| CPU (s) | 120.24 | 42.93 | – |

It is worth remarking here that both models have the same number of parameters. Essentially the Meixner model is a three parameter model, since it has a given drift for a given surface of data and it is a pure jump process. The $\boldsymbol{\beta}$-$M$ process was defined as a pure jump process and the drift is also given so that the discounted prices are martingales, therefore the $\boldsymbol{\beta}$-$M$ process is also a three parameter model.

One way of pricing call options is through the characteristic function of the process by the Carr and Madan [16] formula, the main advantage of the formula is the possibility of using the fast Fourier transform to invert the transformation. The price of a call option with strike $K$ and maturity $T$ is

$$C(K, T) = e^{-rT}\mathbb{E}[\max((S_T - K), 0)] = \frac{e^{-rT}}{\pi}\int_0^\infty e^{-iuk}\rho(u)du,$$

where

$$\rho(u) = \frac{e^{-rT}\varphi_{\log(S_T)}(u - i(\alpha + 1))}{\alpha^2 + \alpha - u^2 + i(2\alpha + 1)u}$$

and $\alpha > 0$ is a damping factor.

Recall that quoting CDS spreads is very similar to price digital down and out barrier options (DDOB) – or the probability of survival – as showed by the well known relation

$$c(B, T) = (1 - R)\left[\frac{1 - e^{rT}\mathbb{P}[\underline{V}_T > B]}{\int_0^T e^{-rt}\mathbb{P}[\underline{V}_t > B]dt} - r\right], \tag{20}$$

for a CDS spread at maturity $T$, barrier $B$ and recovery rate $R$ (cf. Cariboni and Schoutens [3, Chapter 3]). The price of a DDOB is just the discounted price of the probability of survival of the underlying. Therefore, we only need to price DDOB since the integral part in the above formula can be approximated by multi-step trapezoid rule.

### *4.1. Calibration*

The data set for the vanilla surface will be the one proposed in Schoutens [17, p. 6]. Since we already have a calibration of the Meixner model under this surface of call options (see [17, p. 81]). For such data the risk free interest rate is $r = 1.20\%$, the dividend yield is $d = 1.90\%$ and $S_0 = 1124.47$. This data set was taken at the close of the market on 18/04/2002. The CDS spreads are taken from Cariboni and Schoutens [3, p. 70]. In the credit risk setting we take $r = 2.24\%$, $d = 0$ and the recovery rate $R = 0.5$. This data was taken on 26/10/2004. All computations were carried out in a Intel(R) Core(TM)2 CPU 6300 at 1.86 GHz with Octave. The calibration in both cases is going to be with respect to the mean square error:

$$\text{RMSE} = \sqrt{\sum_{\text{options}} \frac{(\text{market price} - \text{model price})^2}{\text{number of options}}}.$$

#### *4.1.1. Vanilla options*

The optimal parameters for the calibration of the Meixner model and the $\boldsymbol{\beta}$-$M$ model are summarized in Table 1. On Figs. 2 and 3 we depicted the performance of such optimal parameters against the market data. Essentially the two models fail and succeed on the same regions although the calibration of the $\boldsymbol{\beta}$-$M$ model is better with respect to the RMSE error. The results are very similar to the ones in Schoutens and Damme [1]. If we apply the rule of thumb described in Section 2 we obtain the following parameters for the $\boldsymbol{\beta}$-$M$ model — denoted by RT parameters:

$$\hat{c} = 0.0391, \qquad \hat{\alpha}_1 = 9.6849, \qquad \hat{\alpha}_2 = 3.5039.$$

In Fig. 1 we compare the density function for the Meixner model and the $\boldsymbol{\beta}$-$M$ model with the optimal parameters (OP) given in Table 1 against the density of the $\boldsymbol{\beta}$-$M$ model with parameters computed with the rule of thumb from the OP parameters of the Meixner model. The density computed through the rule of thumb seem to perform very good on the right tail while reasonably good on the left one. On the other hand both models under the optimal parameters are almost indistinguishable.

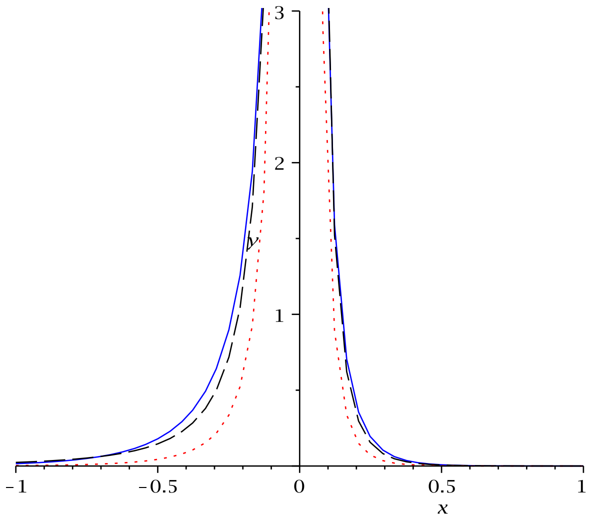


**Fig. 1.** Solid line: Meixner density with OP; dashed line: $\boldsymbol{\beta}$-$M$ with OP; dotted line: $\boldsymbol{\beta}$-$M$ with RT.

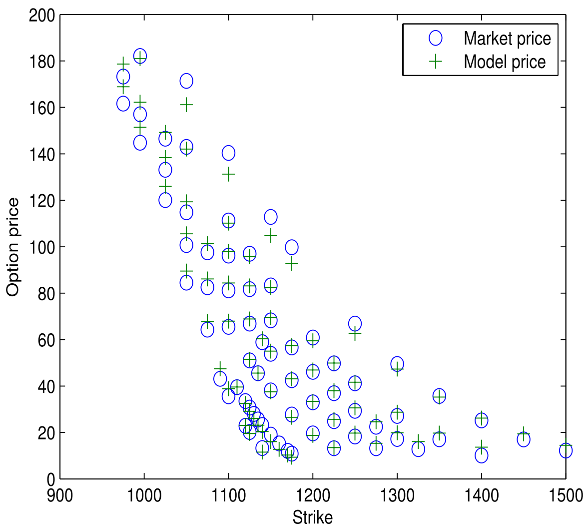


**Fig. 2.** Meixner calibration on the vanilla surface.

### *4.1.2. Credit default swaps*

The first thing we need to compute a CDS spread under the $\boldsymbol{\beta}$-$M$ process is to be able to compute an approximation of the probability of survival at an exponential time described in (13). For computing the coefficients $c_0^+(q)$, $\zeta_0^+(q)$, $c_n(q)$ and $\zeta_n(q)$ of Eq. (13) we have computed 100 roots of the equation $\Psi_{X_1}(i\zeta) + q = 0$ and used them to compute 75 coefficients $c_n(q)$, therefore we have discretized (13) by a sum of 75 terms. Finally the integral (15) was discretized following a Gaver–Stehfest algorithm by a sum of 8 terms while the integral in (20) was discretized by the trapezoid rule with 360 steps. Under the Meixner model we follow the COS algorithm described in Fang et al. in [4]. Table 2 shows the spreads of both models in comparison with market data. In Table 3 we summarize the resulting coefficients. The results are again very similar to the ones presented in Schoutens and Damme [1]. It turns out that the approximation with the $\boldsymbol{\beta}$-$M$ performs better than the Meixner model. The time of computation though is much more greater (see Table 3). As commented in Schoutens and Damme [1], the Wiener–Hopf approach algorithm spends most of the time in computing the roots $\zeta_n(q)$. Because these are localized – i.e. the 100 roots belong to disjoint (and known) sub-intervals – and the computation for a single root is fast, allowing a parallel computing implementation of the same algorithm would speed the process by a factor of 100, and therefore outperforming the COS method.

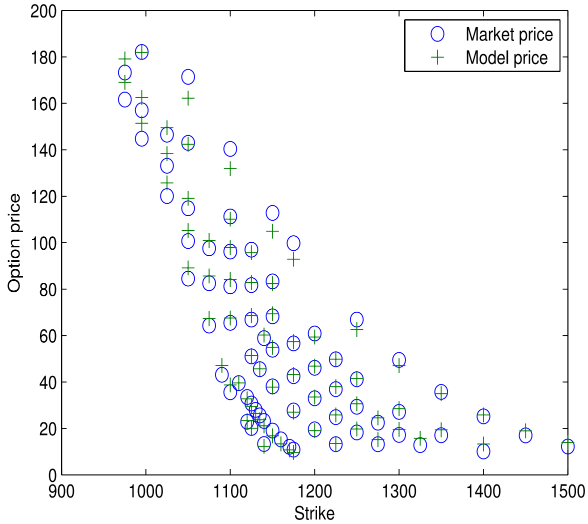


**Fig. 3.** $\boldsymbol{\beta}$-$M$ calibration on the vanilla surface.

**Table 2**
Calibration of the $\boldsymbol{\beta}$-$M$ and Meixner processes on CDS spreads.

| Company | | 1 year | 3 years | 5 years | 7 years | 10 years |
|---|---|---|---|---|---|---|
| General Elec. | Market | 5 | 14 | 25 | 29 | 36 |
| | $\boldsymbol{\beta}$-$M$ | 6 | 13 | 25 | 29 | 35 |
| | COS | 5 | 15 | 24 | 30 | 36 |
| General Motors | Market | 86 | 157 | 207 | 229 | 242 |
| | $\boldsymbol{\beta}$-$M$ | 88 | 162 | 206 | 230 | 239 |
| | COS | 80 | 159 | 208 | 229 | 238 |
| Whirlpool | Market | 16 | 36 | 66 | 73 | 86 |
| | $\boldsymbol{\beta}$-$M$ | 17 | 35 | 67 | 75 | 83 |
| | COS | 14 | 40 | 62 | 76 | 85 |
| Walt Disney | Market | 6 | 21 | 36 | 45 | 56 |
| | $\boldsymbol{\beta}$-$M$ | 6 | 24 | 37 | 47 | 55 |
| | COS | 6 | 21 | 36 | 46 | 55 |
| Eastman Kodak | Market | 54 | 86 | 127 | 143 | 157 |
| | $\boldsymbol{\beta}$-$M$ | 50 | 87 | 126 | 142 | 157 |
| | COS | 44 | 92 | 126 | 143 | 153 |

**Table 3**
Coefficients, RMSE error and computation time of the calibrated $\boldsymbol{\beta}$-$M$ and Meixner processes.

| Company | $\boldsymbol{\beta}$-$M$<br>COS | $c$<br>$a$ | $\alpha_1$<br>$b$ | $\alpha_2$<br>$d$ | RMSE (bps) | CPU (s) |
|---|---|---|---|---|---|---|
| General Elec. | $\boldsymbol{\beta}$-$M$ | 0.0673 | 12.1249 | 6.2399 | 0.5161 | 8240.8 |
| | COS | 0.2983 | −0.4972 | 0.4299 | 0.8406 | 629.2 |
| General Motors | $\boldsymbol{\beta}$-$M$ | 0.1356 | 8.0528 | 4.0011 | 2.8248 | 8660.7 |
| | COS | 0.9106 | 0.2355 | 0.1737 | 3.2221 | 633.9 |
| Whirlpool | $\boldsymbol{\beta}$-$M$ | 0.0728 | 5.6308 | 5.5544 | 1.9191 | 8152.3 |
| | COS | 0.4392 | 0.0318 | 0.3507 | 2.9893 | 640.1 |
| Walt Disney | $\boldsymbol{\beta}$-$M$ | 0.0695 | 6.4666 | 6.2615 | 1.6712 | 8149.5 |
| | COS | 0.3597 | 0.0127 | 0.4087 | 0.7459 | 681.8 |
| Eastman Kodak | $\boldsymbol{\beta}$-$M$ | 0.1421 | 12.2455 | 5.4404 | 2.1331 | 8669.7 |
| | COS | 0.7093 | 0.1401 | 0.2046 | 5.4497 | 684.9 |

## 5. Conclusion

We have showed that the $\boldsymbol{\beta}$-$M$ is a good approximation for the Meixner model and derived a fast – up to a parallel implementation – and accurate algorithm to price CDS based on the Wiener–Hopf factorization of the process. We have showed that the approximations of Schoutens and Damme [1] to the variance gamma process and the one made here for the Meixner model are particular cases of the more general framework of hyper-exponential jump-diffusion processes. Together, the results suggest that the Wiener–Hopf approach perform better than the general methodologies for pricing CDS options. Despite that the two results are based on members of the $\boldsymbol{\beta}$-family, what really makes the Wiener–Hopf approach possible is the fact that the $\boldsymbol{\beta}$-family belongs to the more general family of meromorphic Lévy processes.

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