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Journal of Functional Analysis 262 (2012) 210–233

JOURNAL OF Functional Analysis

www.elsevier.com/locate/jfa

Towards a q-analogue of the Kibble–Slepian formula in 3 dimensions

Paweł J. Szabłowski [1]

Department of Mathematics and Information Sciences, Warsaw University of Technology, pl. Politechniki 1, 00-661 Warsaw, Poland

Received 1 April 2011; accepted 13 September 2011

Available online 1 October 2011

Communicated by D. Voiculescu

Abstract

We study a generalization of the Kibble–Slepian (KS) expansion formula in 3 dimensions. The generalization is obtained by replacing the Hermite polynomials by the q-Hermite ones. If such a replacement would lead to non-negativity for all allowed values of parameters and for all values of variables ranging over certain Cartesian product of compact intervals then we would deal with a generalization of the 3-dimensional Normal distribution. We show that this is not the case. Nevertheless we indicate other applications of so-generalized KS formula. Namely we use it to sum certain kernels built of the Al-Salam–Chihara polynomials for the cases that were not considered by other authors. One of such kernels sums up to the Askey–Wilson density disclosing its new, interesting properties. In particular we are able to obtain a generalization of the 2-dimensional Poisson–Mehler formula. We also pose several open questions.

Keywords: Kibble–Slepian formula; Askey–Wilson density; Orthogonal polynomials; q-Hermite; Al-Salam–Chihara polynomials; Poisson–Mehler kernel; Positive kernels

1. Introduction

In 1945 W.F. Kibble [18] and later, independently D. Slepian [19] have extended the Poisson–Mehler formula to higher dimensions, expanding ratio of the standardized multidimensional

E-mail address: pawel.szablowski@gmail.com.

[1] The author is grateful to the unknown referee for indicating positions of the literature where somewhat similar but much lengthier results were obtained.

doi:10.1016/j.jfa.2011.09.007

Gaussian density divided by the product of one-dimensional marginal densities in a multiple sum involving only the powers of constants (correlation coefficients) and the Hermite polynomials. The symmetry of this beautiful formula encourages further generalizations in the sense that the Hermite polynomials in the KS formula are substituted by their generalizations.

Nice generalization of the Hermite polynomials emerged more than hundred years ago but only recently was intensively studied. The generalized Hermite polynomials are called the q-Hermite polynomials and constitute a one parameter family of orthogonal polynomials that for $q = 1$ is exactly equal to the family of the classical Hermite polynomials.

We indicate that the function of 3 variables obtained with a help of so-generalized KS formula has many properties of a 3-dimensional density. Let us call it f_{3D}. We know its marginals which are nonnegative. We are able to calculate all moments of a supposed to exist random vector that would have this function as its joint density. In particular we can calculate the variance–covariance matrix of this "random vector". This matrix is exactly the same as in the Gaussian case. Thus the question remains if function f_{3D} is really nonnegative for some values of parameter q (say $-1 < q \leqslant 1$) and all values of correlation coefficients that make the variance–covariance matrix positive definite and almost all values of variables from certain product of compact intervals. It will turn out that it is not. We will indicate particular values of q, of correlation coefficients and a subset in $\mathbb{R}^3$ of positive measure such that in 3 dimensions so constructed generalization of the KS formula is negative on this set.

Nevertheless we point out that it is worth to study described above sums since we obtain a nice and simple tool for examining properties of different kernels involving the so-called Al-Salam–Chihara (ASC) polynomials. The kernels built of the ASC polynomials were studied by Askey, Rahman and Suslov in [5]. Such kernels have many applications particularly in connection with certain models of the so-called q-oscillators considered in quantum physics. See e.g. [13,14].

By studying the described above sums of the q-Hermite or ASC polynomials we obtain kernels that are different than those considered in [5]. Hence we obtain new results related to an important problem of summing kernels.

One of these new results is a generalization of Poisson–Mehler expansion formula in the sense that the q-Hermite polynomials are replaced by the ASC ones. Of course the sum is different. Instead of the density of measure that makes ASC polynomials orthogonal (classical case) we get the density of measure that makes the Askey–Wilson polynomials orthogonal. We also analyze other non-symmetric kernels built of the ASC polynomials and sum them. As a by-product we point out in Remark 6, below the possibility of an interesting decomposition of the Askey–Wilson polynomials.

The paper is organized as follows. In the next Section 2 we introduce all necessary auxiliary information concerning the so-called q-series theory. In particular we introduce the q-Hermite and ASC polynomials and present their basic properties. In the following Section 3 we present our main results while less interesting or longer proofs are shifted to the last Section 5. We also include special Section 4 with open problems since not all questions that appeared when studying this beautiful object we were able to answer.

2. Notation and auxiliary results

Let us introduce notation traditionally used in the q-series theory. q is a parameter. It can be real or complex, usually if complex, then $|q| < 1$. We will assume however throughout the paper that $-1 < q \leqslant 1$. Having q we define $[0]_q = 0$, $[n]_q = 1 + q + \cdots + q^{n-1}$, $[n]_q! = \prod_{i=1}^{n} [i]_q$, with $[0]_q! = 1$,

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} \frac{[n]_q!}{[n-k]_q![k]_q!}, & n \geqslant k \geqslant 0, \\ 0, & \text{otherwise.} \end{cases}$$

It is useful to use the so-called q-Pochhammer symbol for $n \geqslant 1$: $(a;q)_n = \prod_{i=0}^{n-1}(1-aq^i)$, with $(a;q)_0 = 1$, $(a_1, a_2, \ldots, a_k; q)_n = \prod_{i=1}^{k}(a_i;q)_n$.

Often $(a;q)_n$ as well as $(a_1, a_2, \ldots, a_k; q)_n$ will be abbreviated to $(a)_n$ and $(a_1, a_2, \ldots, a_k)_n$, if it will not cause misunderstanding.

It is easy to notice that $(q)_n = (1-q)^n[n]_q!$ and that

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} \frac{(q)_n}{(q)_{n-k}(q)_k}, & n \geqslant k \geqslant 0, \\ 0, & \text{otherwise.} \end{cases}$$

Notice that for $n \geqslant k \geqslant 0$ we have: $[n]_1 = n$, $[n]_1! = n!$, $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_1 = \binom{n}{k}$ (binomial coefficient), $(a;1)_n = (1-a)^n$ and $[n]_0 = 1$ for $n \geqslant 1$, $[n]_0! = 1$, $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_0 = 1$, $(a;0)_n = \begin{cases} 1 & \text{if } n = 0, \\ 1-a & \text{if } n \geqslant 1. \end{cases}$

Let us also denote $I_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$

To define briefly and swiftly the one-dimensional distributions that later will be used to construct possible multidimensional generalizations of the Normal distributions, let us define the following sets:

$$S(q) = [-2/\sqrt{1-q}, 2/\sqrt{1-q}], \quad \text{for } |q| < 1 \quad \text{and} \quad S(1) = \mathbb{R}.$$

Further we define the following sets of polynomials:

– the q-Hermite polynomials defined by the relationship:

$$H_{n+1}(x|q) = xH_n(x|q) - [n]_q H_{n-1}(x|q), \tag{2.1}$$

for $n \geqslant 0$ with $H_{-1}(x|q) = 0$, $H_0(x|q) = 1$,

– the Al-Salam–Chihara (ASC) polynomials defined by the relationship:

$$P_{n+1}(x|y,\rho,q) = \left(x - \rho y q^n\right) P_n(x|y,\rho,q) - \left(1 - \rho^2 q^{n-1}\right)[n]_q P_{n-1}(x|y,\rho,q), \tag{2.2}$$

for $n \geqslant 0$ with $P_{-1}(x|y,\rho,q) = 0$, $P_0(x|y,\rho,q) = 1$,

– the Chebyshev polynomials of the second kind defined by the relationship:

$$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x),$$

for $n \geqslant 0$ with $U_{-1}(x) = 0$, $U_0(x) = 1$.

Notice that $H_n(x|1) = H_n(x)$, $H_n(x|0) = U_n(x/2)$ and $P_n(x|y,\rho,1) = H_n(\frac{x-\rho y}{\sqrt{1-\rho^2}})(1-\rho^2)^{n/2}$, where $\{H_n(x)\}$ denote the 'probabilistic' Hermite polynomials i.e. monic polynomials that are orthogonal with respect to the measure with the density $\exp(-x^2/2)/\sqrt{2\pi}$.

From Lemma 1 ii), below it follows that:

$$P_n(x|y,\rho,0) = U_n(x/2) - \rho y U_{n-1}(x/2) + \rho^2 U_{n-2}(x/2). \tag{2.3}$$

The polynomials (2.1) satisfy the following very useful identity originally formulated for the so-called continuous q-Hermite polynomials h_n (can be found in e.g. [15, Theorem 13.1.5]) related to the polynomials H_n by:

$$h_n(x|q) = (1-q)^{n/2} H_n\left(\frac{2x}{\sqrt{1-q}}\middle| q\right), \quad n \geqslant 1, \tag{2.4}$$

and here, below presented for the polynomials H_n: for $n, m \geqslant 0$ we have

$$H_n(x|q)H_m(x|q) = \sum_{j=0}^{\min(n,m)} \begin{bmatrix} m \\ j \end{bmatrix}_q \begin{bmatrix} n \\ j \end{bmatrix}_q [j]_q! H_{n+m-2j}(x|q). \tag{2.5}$$

One can also find in the literature (e.g. [10,12]) the following useful formula: for $m, n \geqslant 0$ we have

$$H_{n+m}(x) = \sum_{j=0}^{\min(n,m)} (-1)^j q^{\binom{j}{2}} \begin{bmatrix} m \\ j \end{bmatrix}_q \begin{bmatrix} n \\ j \end{bmatrix}_q [j]_q! H_{n-j}(x|q) H_{m-j}(x|q) \tag{2.6}$$

that was originally formulated for the so-called Rogers–Szegö polynomials defined by:

$$W_n(x|q) = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q x^j, \tag{2.7}$$

that are related to the continuous q-Hermite polynomials by:

$$h_n(x|q) = e^{in\theta} W_n\left(e^{-i2\theta}|q\right),$$

for $n \geqslant 0$ with $x = \cos\theta$ and i being imaginary unit. (See also [15].)

To simplify notation let us introduce the following auxiliary polynomials of order at most 2:

$$r_k(y|q) = \left(1+q^k\right)^2 - (1-q)yq^k, \tag{2.8}$$

$$v_0(x,y|\rho,q) = \left(1-\rho^2\right)^2 - (1-q)\rho\left(1+\rho^2\right)xy + (1-q)\rho^2\left(x^2+y^2\right), \tag{2.9}$$

$$v_k(x,y|\rho,q) = v_0\left(x,y|\rho q^k,q\right), \quad k \geqslant 1. \tag{2.10}$$

It is known (see e.g. [8]) that the q-Hermite polynomials are monic and orthogonal with respect to the measure that has density given by:

$$f_N(x|q) = \frac{(q)_\infty \sqrt{1-q}}{2\pi\sqrt{r_0(x^2|q)}} \prod_{k=0}^{\infty} r_k\left(x^2|q\right) I_{S(q)}(x), \tag{2.11}$$

defined for $|q| < 1$, $x \in \mathbb{R}$ and

$$f_N(x|1) = \frac{1}{\sqrt{2\pi}} \exp\bigl(-x^2/2\bigr). \tag{2.12}$$

Similarly it is known (e.g. from [9]) that the ASC polynomials are monic and orthogonal with respect to the measures that for $q \in (-1, 1]$ have densities. These densities are given for $|q| < 1$ by:

$$f_{CN}(x|y, \rho, q) = \frac{\sqrt{1-q}(q)_\infty(\rho^2)_\infty}{2\pi\sqrt{r_0(x^2|q)}} \prod_{k=0}^{\infty} \frac{r_k(x^2|q)}{v_k(x, y|\rho, q)} I_{S(q)}(x), \tag{2.13}$$

for $|\rho| < 1$, $x \in \mathbb{R}$, $y \in S(q)$ and for $q = 1$ by:

$$f_{CN}(x|y, \rho, 1) = \frac{1}{\sqrt{2\pi(1-\rho^2)}} \exp\biggl(-\frac{(x-\rho y)^2}{2(1-\rho^2)}\biggr).$$

It is known (see e.g. [15, formula 13.1.10]) that for $|q| < 1$:

$$\sup_{x \in S(q)} \bigl|H_n(x|q)\bigr| \leqslant W_n(1|q)(1-q)^{-n/2}, \tag{2.14}$$

where W_n is given by (2.7).

We will also need auxiliary polynomials $\{B_n(x|q)\}_{n \geqslant -1}$ defined by the following 3-term recurrence:

$$B_{n+1}(y|q) = -q^n y B_n(y|q) + q^{n-1}[n]_q B_{n-1}(y|q); \quad n \geqslant 0, \tag{2.15}$$

with $B_{-1}(y|q) = 0$, $B_0(y|q) = 1$.

In fact the polynomials $\{B_n(y|q)\}_{n \geqslant -1}$ are equal to the polynomials $\{h_n(y|q^{-1})\}$ scaled and normalized in a certain way. The polynomials $\{h_n(y|q^{-1})\}$ have been known for a long time and were studied in [4] and [3]. However with the present scaling and normalization they were introduced in [9] where some of their basic properties were presented and their auxiliary rôle in finding connection coefficients between ASC and q-Hermite polynomials was shown. One can show (see e.g. [9]) that $B_n(x|1) = i^n H_n(ix)$ and

$$B_n(x|0) = \begin{cases} 1 & \text{if } n = 0 \vee 2, \\ -x & \text{if } n = 1, \\ 0 & \text{if } n > 2. \end{cases}$$

Further properties of these polynomials including their relationship to the q-Hermite polynomials are presented in [21].

Facts concerning the q-Hermite, ASC and polynomials B_n, necessary for the derivation of the results of the paper, are collected in the following lemma.

Lemma 1. *Assume that* $0 < q \leqslant 1$, $|\rho|, |\rho_1|, |\rho_2| < 1$, $n, m \geqslant 0$, $x, y, z \in S(q)$, *then*

i)

$$H_n(x|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \rho^{n-k} H_{n-k}(y|q) P_k(x|y, \rho, q),$$

ii) $\forall n \geqslant 1$:

$$P_n(x|y, \rho, q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \rho^{n-k} B_{n-k}(y|q) H_k(x|q),$$

$$\sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} B_{n-j}(x|q) H_j(x|q) = 0,$$

iii)

$$\int_{S(q)} H_n(x|q) H_m(x|q) f_N(x|q)\, dx = \begin{cases} 0 & \text{when } n \neq m, \\ [n]_q! & \text{when } n = m, \end{cases}$$

iv)

$$\int_{S(q)} P_n(x|y, \rho, q) P_m(x|y, \rho, q) f_{CN}(x|y, \rho, q)\, dx = \begin{cases} 0 & \text{when } n \neq m, \\ (\rho^2)_n [n]_q! & \text{when } n = m, \end{cases}$$

v)

$$\int_{S(q)} H_n(x|q) f_{CN}(x|y, \rho, q)\, dx = \rho^n H_n(y|q),$$

vi)

$$\int_{S(q)} f_{CN}(x|y, \rho_1, q) f_{CN}(y|z, \rho_2, q)\, dy = f_{CN}(x|z, \rho_1\rho_2, q),$$

vii)

$$\sum_{n=0}^{\infty} \frac{\rho^n}{[n]_q!} H_n(x|q) H_n(y|q) = f_{CN}(x|y, \rho, q) / f_N(x|q),$$

viii) *for* $|t|, |q| < 1$:

$$\sum_{i=0}^{\infty} \frac{W_i(1|q)t^i}{(q)_i} = \frac{1}{(t)_\infty^2}, \qquad \sum_{i=0}^{\infty} \frac{W_i^2(1|q)t^i}{(q)_i} = \frac{(t^2)_\infty}{(t)_\infty^4},$$

convergence is absolute, where $W_i(x|q)$ *is defined by* (2.7)*,*

ix)

$$\forall x, y \in S(q): \quad 0 < C(y, \rho, q) \leqslant \frac{f_{CN}(x|y, \rho, q)}{f_N(x|q)} \leqslant \frac{(\rho^2)_\infty}{(\rho)_\infty^4}.$$

Proof. ii) was proved in [9]. i) is proved in [17, formula 4.7] for the polynomials h_n and $Q_n(x|a, b, q)$ related to the polynomials P_n by the relationship $Q_n(x|a, b, q) = (1-q)^{n/2} P_n(\frac{2x}{\sqrt{1-q}}|\frac{2a}{\sqrt{(1-q)b}}, \sqrt{b}, q)$. However one can also derive it easily from ii). iii) and iv) are known (see e.g. [15]) for polynomials h_n and Q_n. Thus here they are presented after necessary adaptation to polynomials H_n and P_n. v) and vi) can be found in [8], but their particular cases in different form were also shown in [6]. vii) is in fact the famous Poisson–Mehler formula which has many proofs. One of them is in [15] the others e.g. in [7] or [20]. viii) See Exercise 12.2(b) and 12.2(c) of [15]. ix) was proved in [21, Proposition 1 vii)]. □

Let us remark, following [21], that

$$f_{AW}(x|y, \rho_1, z, \rho_2, q) = \frac{f_{CN}(y|x, \rho_1, q) f_{CN}(x|z, \rho_2, q)}{f_{CN}(y|z, \rho_1\rho_2, q)} \tag{2.16}$$

is the density of the measure that makes the re-scaled Askey–Wilson (AW) polynomials orthogonal. The AW polynomials mentioned in the paper are considered for certain complex valued parameters related to y, ρ_1, z, ρ_2. For details see formula (2.5) of [21]. We will call the function defined by (2.16) the AW density.

As mentioned in the Introduction the main object of this paper is a generalization of Kibble–Slepian formula. Traditional KS formula has a form of an expansion of the ratio of the non-degenerated n-dimensional Gaussian density divided by the product of its one-dimensional marginals in the multiple series (in fact involving $n(n-1)/2$ fold sum) of the Hermite polynomials in one variable with coefficients that are powers of off-diagonal elements of variance–covariance matrix.

We will analyze only the generalization of its three-dimensional version. It is simple to express and general enough to expose interesting properties and applications.

Namely we will analyze the following function given by:

$$f_{3D}(x_1, x_2, x_3|\rho_{12}, \rho_{13}, \rho_{23}, q)$$

$$= \prod_{i=1}^{3} f_N(x_i|q) \tag{2.17}$$

$$\times \sum_{i,j,k\geqslant 0} \frac{\rho_{12}^j \rho_{23}^k \rho_{13}^i}{[i]_q![j]_q![k]_q!} H_{i+j}(x_1|q) H_{j+k}(x_2|q) H_{i+k}(x_3|q), \tag{2.18}$$

where $|\rho_{12}|, |\rho_{13}|, |\rho_{23}| < 1$. Let us denote

$$\boldsymbol{\rho} = \begin{bmatrix} 1 & \rho_{12} & \rho_{13} \\ \rho_{12} & 1 & \rho_{23} \\ \rho_{13} & \rho_{23} & 1 \end{bmatrix}, \qquad \Delta = \det \boldsymbol{\rho}.$$

We will assume that parameters $\rho_{12}, \rho_{13}, \rho_{23}$ are such that

$$\Delta = 1 + 2\rho_{12}\rho_{13}\rho_{23} - \rho_{12}^2 - \rho_{13}^2 - \rho_{23}^2 \geqslant 0.$$

Notice that parameter Δ can be presented in the following form:

$$\Delta = \left(1 - \rho_{12}^2\right)\left(1 - \rho_{23}^2\right) - (\rho_{13} - \rho_{12}\rho_{23})^2, \tag{2.19}$$

and similarly for other pairs of indices $(1,2)$ and $(2,3)$ since Δ is obviously symmetric in ρ_{12}, ρ_{13}, ρ_{23}.

The above mentioned assumptions concerning the parameters $\rho_{12}, \rho_{13}, \rho_{23}$ will be assumed throughout the remaining part of the paper. Similarly we will assume that $x_i \in S(q)$, $i = 1, 2, 3$, unless otherwise stated. Besides from Lemma 1 viii), ix) it follows that all considered in this paper series for $|q| < 1$ are absolutely convergent, hence we will not repeat this statement unless it will be necessary.

Remark 1. Let us immediately observe that when $q = 1$, that is when $H_n(x|q)$ is substituted by $H_n(x)$ and $f_N(x_i|q)$ by $\exp(-x_i^2/2)/\sqrt{2\pi}$, then by Kibble–Slepian formula (see e.g. [19, Example 2)]) for $n = 3$, $f_{3D}(x_1, x_2, x_3|\rho_{12}, \rho_{13}, \rho_{23}, 1)$ is the density of the Gaussian distribution $N(\mathbf{0}, \boldsymbol{\rho})$.

Besides let us notice that the function f_{3D} has the same symmetry with respect to variables $x_1, x_2, x_3, \rho_{12}, \rho_{13}, \rho_{23}$ as the density of the Normal distribution $N(\mathbf{0}, \boldsymbol{\rho})$.

Remark 2. In the literature functions somewhat similar to (2.17)–(2.18) were considered. In [16, formula 4.13] sums of the form $\sum_{k\geqslant 0} \frac{t^k}{[k]_q!} H_{n+k}(x|q) H_k(y|q) H_k(z|q)$ for all n's while in [1, formula 2.5] sum of the form $\sum_{m,n,k} \frac{x^m y^n z^k}{(q)_m (q)_n (q)_k} W_{m+k}(a|q) W_{n+k}(b|q)$ (for W_n see (2.7)) were analyzed. Both these results refer to and are based on the beautiful results of Carlitz [11]. The obtained sums are very complicated, expressed through the basic hypergeometric function ${}_6\phi_5$ or ${}_3\phi_2$. These results show that summing products of more than 2 q-Hermite polynomials is a very subtle and complicated task. On the other hand the results of these paper dealing also with sums of products of 3 q-Hermite polynomials follow a subtle path in this area leading to results of relatively simple form. By the way our results also refer to the above mentioned results of Carlitz.

Let us present some immediate remarks concerning the function f_{3D}.

Remark 3. i)

$$\int_{S(q)\times S(q)\times S(q)} f_{3D}(x_1, x_2, x_3|\rho_{12}, \rho_{13}, \rho_{23}, q)\, dx_1\, dx_2\, dx_3 = 1,$$

ii)

$$\int\limits_{S(q)\times S(q)} f_{3D}(x_1,x_2,x_3|\rho_{12},\rho_{13},\rho_{23},q)\,dx_1\,dx_2 = f_N(x_3|q),$$

$$\int\limits_{S(q)\times S(q)\times S(q)} H_n(x_1|q)H_m(x_2|q)f_{3D}(x_1,x_2,x_3|\rho_{12},\rho_{13},\rho_{23},q)\,dx_3\,dx_1\,dx_2$$
$$= \begin{cases} 0 & \text{if } n \neq m, \\ \rho_{12}^n[n]_q! & \text{if } m = n, \end{cases}$$

and similarly for other pairs $(1,3)$ and $(2,3)$.

In particular we have:

$$\int\limits_{S(q)\times S(q)\times S(q)} x_1 f_{3D}(x_1,x_2,x_3|\rho_{12},\rho_{13},\rho_{23},q)\,dx_1\,dx_2\,dx_3 = 0,$$
$$\int\limits_{S(q)\times S(q)\times S(q)} x_1^2 f_{3D}(x_1,x_2,x_3|\rho_{12},\rho_{13},\rho_{23},q)\,dx_1\,dx_2\,dx_3 = 1$$

and again similarly for the remaining indices 2 and 3.

iii)

$$\int\limits_{S(q)} f_{3D}(x_1,x_2,x_3|\rho_{12},\rho_{13},\rho_{23},q)\,dx_3 = f_{CN}(x_1|x_2,\rho_{12},q)f_N(x_2|q)$$

and similarly for the remaining integrating variables x_1 and x_2.

iv) If $\rho_{13} = \rho_{23} = 0$, then

$$f_{3D}(x_1,x_2,x_3|\rho_{12},\rho_{13},\rho_{23},q) = f_{CN}(x_1|x_2,\rho_{12},q)f_N(x_2|q)f_N(x_3|q)$$

and similarly for other pairs $(1,3)$ and $(2,3)$.

Proof. In all assertions we apply Lemma 1 iii), the fact that $H_1(x|q) = x$ and $H_2(x|q) = x^2 - 1$ and also formulas (2.5) and Lemma 1 v). iv) follows directly from Lemma 1 iiv). □

As it follows from the above mentioned Remark that f_{3D} is a serious candidate for the 3-dimensional density. It has nonnegative marginal densities equal to the densities f_N and $f_N f_{CN}$. Moreover if f_{3D} was a joint density of certain 3-dimensional random vector it would follow from Remark 3 ii) that the variance–covariance matrix of this vector would be equal to $\begin{bmatrix} 1 & \rho_{12} & \rho_{13} \\ \rho_{12} & 1 & \rho_{23} \\ \rho_{13} & \rho_{23} & 1 \end{bmatrix}$.

To analyze its properties deeper we will need the following lemma.

Lemma 2. *Let us denote* $\gamma_{m,k}(x,y|\rho,q)=\sum_{i=0}^{\infty}\frac{\rho^i}{[i]_q!}H_{i+m}(x|q)H_{i+k}(y|q)$.

i) *Then*

$$\gamma_{m,k}(x,y|\rho,q)=\gamma_{0,0}(x,y|\rho,q)Q_{m,k}(x,y|\rho,q), \tag{2.20}$$

where $Q_{m,k}$ *is a polynomial in* x *and* y *of order at most* $m+k$.

Further denote $C_n(x,y|\rho_1,\rho_2,\rho_3,q)=\sum_{i=0}^{n}\begin{bmatrix}n\\ i\end{bmatrix}_q\rho_1^{n-i}\rho_2^iQ_{n-i,i}(x,y|\rho_{3},q)$.
Then we have in particular:

ii)

$$\begin{aligned}Q_{k,m}(y,x|\rho,q)&=Q_{m,k}(x,y|\rho,q)\\&=\sum_{s=0}^{k}(-1)^sq^{\binom{s}{2}}\begin{bmatrix}k\\ s\end{bmatrix}_q\rho^sH_{k-s}(y|q)P_{m+s}(x|y,\rho,q)/\left(\rho^2\right)_{m+s},\end{aligned}$$

for all $x,y\in S(q)$ *and* $q\in(-1,1]$,

iii)

$$C_n(x,y|\rho_1,\rho_2,\rho_3,q)=\sum_{s=0}^{n}\begin{bmatrix}n\\ s\end{bmatrix}_qH_{n-s}(y|q)P_s(x|y,\rho_3,q)\rho_1^{n-s}\rho_2^s(\rho_1\rho_3/\rho_2)_s/\left(\rho_3^2\right)_s.$$

Proof. Assertions i) and ii) are proved in [21]. Thus we will prove only iii). We have

$$\begin{aligned}&C_n(x,y|\rho_1,\rho_2,\rho_3,q)\\&=\sum_{i=0}^{n}\begin{bmatrix}n\\ i\end{bmatrix}_q\rho_1^{n-i}\rho_2^i\sum_{j=0}^{n-i}(-1)^j\begin{bmatrix}n-i\\ j\end{bmatrix}_qq^{\binom{j}{2}}\rho_3^jH_{n-i-j}(y|q)P_{i+j}(x|y,\rho_3,q)/\left(\rho_3^2\right)_{i+j}\\&=\sum_{s=0}^{n}\begin{bmatrix}n\\ s\end{bmatrix}_qH_{n-s}(y|q)P_s(x|y,\rho_3,q)/\left(\rho_3^2\right)_s\sum_{j=0}^{s}\begin{bmatrix}s\\ j\end{bmatrix}_q(-1)^jq^{\binom{j}{2}}\rho_1^{n-s+j}\rho_2^{s-j}\rho_3^j\\&=\sum_{s=0}^{n}\begin{bmatrix}n\\ s\end{bmatrix}_qH_{n-s}(y|q)P_s(x|y,\rho_3,q)\rho_1^{n-s}\rho_2^s/\left(\rho_3^2\right)_s\sum_{j=0}^{s}\begin{bmatrix}s\\ j\end{bmatrix}_q(-1)^jq^{\binom{j}{2}}(\rho_1\rho_3/\rho_2)^j\\&=\sum_{s=0}^{n}\begin{bmatrix}n\\ s\end{bmatrix}_qH_{n-s}(y|q)P_s(x|y,\rho_3,q)\rho_1^{n-s}\rho_2^s(\rho_1\rho_3/\rho_2)_s/\left(\rho_3^2\right)_s.\end{aligned}$$

On the way we have used the following identity $(a)_n=\sum_{i=0}^{n}\begin{bmatrix}n\\ i\end{bmatrix}_q(-1)^iq^{\binom{i}{2}}a^j$. □

We get immediate observations:

Corollary 1. *For all $n \geqslant 1$ we have*:

i)

$$P_n(x|y,\rho,q) = \left(\rho^2\right)_n \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_q (-1)^i q^{\binom{i}{2}} \rho^i H_{n-i}(x|q) P_i(y|x,\rho,q) / \left(\rho^2\right)_i,$$

ii)

$$\int_{S(q)} P_n(x|y,\rho,q) f_{CN}(y|x,\rho,q)\, dy = \left(\rho^2\right)_n H_n(x|q),$$

iii)

$$C_n(x,y|\rho_2\rho_3,\rho_2,\rho_3,q) = \rho_2^n H_n(x|q),$$

iv)

$$C_n(x,y|\rho_1,\rho_1\rho_3,\rho_3,q) = \rho_1^n H_n(y|q),$$

v)

$$C_n(x,y|\rho_1,\rho_2,0,q) = \sum_{s=0}^{n} \begin{bmatrix} n \\ s \end{bmatrix}_q \rho_1^{n-s} \rho_2^s H_{n-s}(y|q) H_s(x|q),$$
$$C_n(x,y|0,\rho_2,\rho_3,q) = \rho_2^n P_n(x|y,\rho_3,q) / \left(\rho_3^2\right)_n,$$
$$C_n(x,y|\rho_1,0,\rho_3,q) = \rho_1^n P_n(y|x,\rho_3,q) / \left(\rho_3^2\right)_n,$$

vi)

$$C_n(x,y|\rho_1,\rho_2,\rho_3,1) = \left(\frac{\rho_1^2+\rho_2^2-2\rho_1\rho_2\rho_3}{1-\rho_3^2}\right)^{n/2} H_n\left(\frac{x(\rho_2-\rho_1\rho_3)+y(\rho_1-\rho_2\rho_3)}{\sqrt{(1-\rho_3^2)(\rho_1^2+\rho_2^2-2\rho_1\rho_2\rho_3)}}\right).$$

Proof. i) See Corollary 2 of [21]. ii) We use previous assertion and Lemma 1 iv). iii) We have $(\rho_2\rho_3^2/\rho_2)_s = (\rho_3^2)_s$, hence $C_n(x,y|\rho_2\rho_3,\rho_2,\rho_3,q) = \rho_2^n \sum_{s=0}^{n} \begin{bmatrix} n \\ s \end{bmatrix}_q H_{n-s}(y|q) \times P_s(x|y,\rho_3,q)\rho_3^{n-s} = \rho_2^n H_n(x|q)$ by Lemma 1 i). iv) $(\rho_1\rho_3/\rho_1\rho_3)_s = \begin{cases} 1 & \text{if } s=0, \\ 0 & \text{if } s>0. \end{cases}$ v) First two statements are direct consequence of the assumptions that $\rho_3 = 0$ or $\rho_1 = 0$. The third one follows the fact that $\rho_2^s(\rho_1\rho_3/\rho_2)_s = \prod_{i=1}^{s}(\rho_2 - q^{i-1}\rho_1\rho_3)$, which for $\rho_2 = 0$ gives $(-1)^s q^{\binom{s}{2}} \rho_1^s \rho_3^s$. Then we apply assertion i) of this corollary.

To get vi) first we notice that

$$P_n(x|y,\rho_3,1)/\left(1-\rho_3^2\right)^n = H_n\left(\frac{x-\rho_3 y}{\sqrt{1-\rho_3^2}}\right)/\left(1-\rho_3^2\right)^{n/2}.$$

Then we apply the well-known (see e.g. [2]) formula for addition of Hermite polynomials:

$$\sum_{i=0}^{n}\binom{n}{i}a^{n-i}b^{i}H_{n-i}(\xi)H_i(\theta)=\left(a^2+b^2\right)^{n/2}H_n\left(\frac{a\xi+b\theta}{\sqrt{a^2+b^2}}\right)$$

with $a=\rho_1$, $b=\frac{(\rho_2-\rho_1\rho_3)}{\sqrt{1-\rho_3^2}}$, $\xi=y$ and $\theta=\frac{x-\rho_3 y}{\sqrt{1-\rho_3^2}}$. Next we observe that $a^2+b^2=\frac{\rho_1^2+\rho_2^2-2\rho_1\rho_2\rho_3}{1-\rho_3^2}$ and $a\xi+b\theta=\frac{x(\rho_2-\rho_1\rho_3)+y(\rho_1-\rho_2\rho_3)}{1-\rho_3^2}$. □

3. Main results

Applying Lemma 2 to the function f_{3D} we have the following proposition:

Proposition 1.

i)

$$f_{3D}(x_1,x_2,x_3|\rho_{12},\rho_{13},\rho_{23},q)=f_{CN}(x_3|x_1,\rho_{13},q)f_N(x_1|q)f_N(x_2|q)\times\sum_{s\geqslant 0}\frac{1}{[s]_q!}H_s(x_2|q)C_s(x_1,x_3|\rho_{12},\rho_{23},\rho_{13},q),\quad(3.1)$$

similarly for other pairs $(1,3)$ *and* $(2,3)$*,*

ii)

$$\begin{aligned}f_{3D}&(x_1,x_2,x_3|\rho_{12},\rho_{13},\rho_{23},q)\\&=f_{CN}(x_1|x_3,\rho_{13},q)f_{CN}(x_3|x_2,\rho_{23},q)f_N(x_2|q)\\&\quad\times\sum_{s\geqslant 0}\frac{\rho_{12}^s(\rho_{13}\rho_{23}/\rho_{12})_s}{[s]_q!(\rho_{13}^2,\rho_{23}^2)_s}P_s(x_1|x_3,\rho_{13},q)P_s(x_2|x_3,\rho_{23},q),\end{aligned}\quad(3.2)$$

similarly for other pairs $(1,3)$ *and* $(2,3)$.

Proof. Lengthy proof is shifted to Section 5. □

As an immediate corollary we have the following formula:

Corollary 2.

$$\begin{aligned}&\sum_{s\geqslant 0}\frac{\rho_{12}^s(\rho_{13}\rho_{23}/\rho_{12})_s}{[s]_q!(\rho_{13}^2,\rho_{23}^2)_s}P_s(x_1|x_3,\rho_{13},q)P_s(x_2|x_3,\rho_{23},q)\\&\quad=\frac{f_{CN}(x_1|x_2,\rho_{12},q)}{f_{CN}(x_1|x_3,\rho_{13},q)}\sum_{k\geqslant 0}\frac{\rho_{13}^k(\rho_{12}\rho_{23}/\rho_{13})_k}{[k]_q!(\rho_{12}^2,\rho_{23}^2)_s}P_k(x_1|x_2,\rho_{12},q)P_k(x_3|x_2,\rho_{23},q).\end{aligned}$$

Proof. From Proposition 1 ii) we get

$$\sum_{s\geqslant 0}\frac{\rho_{12}^{s}(\rho_{13}\rho_{23}/\rho_{12})_s}{[s]_q!(\rho_{13}^2,\rho_{23}^2)_s}P_s(x_1|x_3,\rho_{13},q)P_s(x_2|x_3,\rho_{23},q)$$
$$=\frac{f_{CN}(x_1|x_2,\rho_{12},q)f_{CN}(x_2|x_3,\rho_{23},q)f_N(x_3|q)}{f_{CN}(x_1|x_3,\rho_{13},q)f_{CN}(x_3|x_2,\rho_{23},q)f_N(x_2|q)}$$
$$\times\sum_{k\geqslant 0}\frac{\rho_{13}^{k}(\rho_{12}\rho_{23}/\rho_{13})_k}{[k]_q!(\rho_{12}^2,\rho_{23}^2)_s}P_k(x_1|x_2,\rho_{12},q)P_k(x_3|x_2,\rho_{23},q).$$

Now we use the fact that: $f_{CN}(x_2|x_3,\rho_{23},q)f_N(x_3|q)=f_{CN}(x_3|x_2,\rho_{23},q)f_N(x_2|q)$. □

Corollary 3. *If* $\rho_{12}=\rho_{13}\rho_{23}$ *then*

i)

$$f_{3D}(x_1,x_2,x_3|\rho_{13}\rho_{23},\rho_{13},\rho_{23},q)=f_{CN}(x_1|x_3,\rho_{13},q)f_{CN}(x_3|x_2,\rho_{23},q)f_N(x_2|q),$$

ii)

$$\sum_{s\geqslant 0}\frac{\rho_{13}^{3}}{[s]_q!(\rho_{13}^2\rho_{23}^2)_s}P_s(x_1|x_2,\rho_{13}\rho_{23},q)P_s(x_3|x_2,\rho_{23},q)=\frac{f_{CN}(x_1|x_3,\rho_{13},q)}{f_{CN}(x_1|x_2,\rho_{13}\rho_{23},q)}. \quad (3.3)$$

If $\rho_{12}=0$ *then*

iii)

$$\sum_{s\geqslant 0}\frac{(-1)^s q^{\binom{s}{2}}\rho_{13}^{s}\rho_{23}^{s}}{[s]_q!(\rho_{13}^2,\rho_{23}^2)_s}P_s(x_1|x_3,\rho_{13},q)P_s(x_2|x_3,\rho_{23},q)$$
$$=\frac{f_N(x_1|q)}{f_{CN}(x_1|x_3,\rho_{13},q)}\sum_{k\geqslant 0}\frac{\rho_{13}^{k}}{[k]_q!(\rho_{23}^2)_k}H_k(x_1|q)P_k(x_3|x_2,\rho_{23},q). \quad (3.4)$$

Proof. i) We use Corollary 1 iii) and deduce that $\sum_{s\geqslant 0}\frac{1}{[s]_q!}H_s(x_2|q)C_s(x_1,x_3|\rho_{12},\rho_{23},\rho_{13},q)=\sum_{s\geqslant 0}\frac{\rho_{23}^{s}}{[s]_q!}H_s(x_2|q)H_s(x_3|q)=f_{CN}(x_3|x_2,\rho_{23},q)/f_N(x_3|q)$ (by Poisson–Mehler formula) or we notice that $(\rho_{13}\rho_{23}/\rho_{12})_s$ when $\rho_{12}=\rho_{13}\rho_{23}$ is equal to 0 for $s\geqslant 1$ and use (3.2).

ii) We use equivalent form of (3.2) that is

$$f_{3D}(x_1,x_2,x_3|\rho_{12},\rho_{13},\rho_{23},q)$$
$$=f_{CN}(x_1|x_2,\rho_{12},q)f_{CN}(x_2|x_3,\rho_{23},q)f_N(x_3|q)$$
$$\times\sum_{s=0}^{\infty}\frac{\rho_{13}^{s}(\rho_{12}\rho_{23}/\rho_{13})_s}{[s]_q!(\rho_{12}^2,\rho_{23}^2)_s}P_s(x_1|x_2,\rho_{13}\rho_{23},q)P_s(x_3|x_2,\rho_{23},q),$$

apply assumption, observing that then $(\rho_{12}\rho_{23}/\rho_{13})_s = (\rho_{23}^2)_s$ and using the assertion i) of this corollary.

iii) We use the fact that $\rho_{12}^s(\rho_{13}\rho_{23}/\rho_{12})_s = \prod_{i=1}^s(\rho_{12} - q^{i-1}\rho_{13}\rho_{23})$ which for $\rho_{12} = 0$ equals to $(-1)^s q^{1+\cdots+s-1}\rho_{13}^s\rho_{23}^s$ and the fact that $P_s(x|y,0,q) = H_s(x|q)$. □

We have also the following remark concerning the ordinary, probabilistic Hermite polynomials

Remark 4. We have

$$\sqrt{\frac{(1-\rho_{13}^2)(1-\rho_{23}^2)-(\rho_{12}-\rho_{13}\rho_{23})^2}{(1-\rho_{13}^2)(1-\rho_{23}^2)}} \times \sum_{s=0}^{\infty}\frac{(\rho_{12}-\rho_{13}\rho_{23})^s}{s!(1-\rho_{13}^2)^{s/2}(1-\rho_{23}^2)^{s/2}} H_s\left(\frac{x_1-\rho_{13}x_3}{\sqrt{1-\rho_{13}^2}}\right) H_s\left(\frac{x_2-\rho_{23}x_3}{\sqrt{1-\rho_{23}^2}}\right)$$
$$= \exp\left(-\frac{1}{2}[x_1 \quad x_2 \quad x_3]\left[\begin{bmatrix}1 & \rho_{12} & \rho_{13}\\ \rho_{12} & 1 & \rho_{23}\\ \rho_{13} & \rho_{23} & 1\end{bmatrix}^{-1} - \begin{bmatrix}1 & \rho_{13}\rho_{23} & \rho_{13}\\ \rho_{13}\rho_{23} & 1 & \rho_{23}\\ \rho_{13} & \rho_{23} & 1\end{bmatrix}^{-1}\right]\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}\right). \tag{3.5}$$

Proof. We use the fact that $P_n(x|y,\rho,1)/(\rho^2|1)_n = H_n(\frac{x-\rho y}{\sqrt{1-\rho^2}})/(1-\rho^2)^{n/2}$. Then $\sum_{s=0}^{\infty}\frac{(\rho_{12}-\rho_{13}\rho_{23})^s}{s!(1-\rho_{13}^2)^{s/2}(1-\rho_{23}^2)^{s/2}}H_s(\frac{x_1-\rho_{13}x_3}{\sqrt{1-\rho_{13}^2}})H_s(\frac{x_2-\rho_{23}x_3}{\sqrt{1-\rho_{23}^2}})$ can be summed by the classical Poisson–Mehler formula yielding

$$\frac{1}{\sqrt{1-r^2}}\exp\left(-\frac{r^2(\xi^2+\zeta^2)-2r\xi\zeta}{2(1-r^2)}\right), \tag{3.6}$$

with $r = \frac{(\rho_{12}-\rho_{13}\rho_{23})}{\sqrt{(1-\rho_{13}^2)(1-\rho_{23}^2)}}$, $\xi = \frac{x_1-\rho_{13}x_3}{\sqrt{1-\rho_{13}^2}}$ and $\zeta = \frac{x_2-\rho_{23}x_3}{\sqrt{1-\rho_{23}^2}}$. It is a matter of elementary calculus to check that the exponent of (3.6) equals to the exponent of the right hand side of (3.5). □

Remark 5. Notice that we get a new proof of the classical KS formula in 3 dimensions, since the left hand side of (3.2) is given by (2.17)–(2.18) while the right hand side can be, in view of Remark 4, written as

$$\frac{1}{\sqrt{2\pi(1-\rho_{13}^2)(1-\rho_{23}^2)}}\times\exp\left(-\frac{1}{2}[x_1 \quad x_2 \quad x_3]\begin{bmatrix}1 & \rho_{13}\rho_{23} & \rho_{13}\\ \rho_{13}\rho_{23} & 1 & \rho_{23}\\ \rho_{13} & \rho_{23} & 1\end{bmatrix}^{-1}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}\right)$$
$$\times\frac{1}{\sqrt{\frac{(1-\rho_{13}^2)(1-\rho_{23}^2)-(\rho_{12}-\rho_{13}\rho_{23})^2}{(1-\rho_{13}^2)(1-\rho_{23}^2)}}}$$
$$\times\exp\left(-\frac{1}{2}[x_1 \quad x_2 \quad x_3]\left[\begin{bmatrix}1 & \rho_{12} & \rho_{13}\\ \rho_{12} & 1 & \rho_{23}\\ \rho_{13} & \rho_{23} & 1\end{bmatrix}^{-1} - \begin{bmatrix}1 & \rho_{13}\rho_{23} & \rho_{13}\\ \rho_{13}\rho_{23} & 1 & \rho_{23}\\ \rho_{13} & \rho_{23} & 1\end{bmatrix}^{-1}\right]\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}\right)$$

$$\times \frac{1}{\sqrt{2\pi(1-\rho_{13}^2)(1-\rho_{23}^2)-(\rho_{12}-\rho_{13}\rho_{23})^2}}$$
$$\times \exp\left(-\frac{1}{2}\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}\begin{bmatrix} 1 & \rho_{12} & \rho_{13} \\ \rho_{12} & 1 & \rho_{23} \\ \rho_{13} & \rho_{23} & 1 \end{bmatrix}^{-1}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right),$$

which is the density of $N(\mathbf{0}, \boldsymbol{\rho})$ distribution. Thus (3.2) is really a generalization of 3-dimensional KS formula.

We have also the following observation concerning the AW density.

Remark 6. Following observation (2.16) and Lemma 1 vii) we deduce that

$$\frac{f_{CN}(x_1|x_3,\rho_{13},q)f_{CN}(x_3|x_2,\rho_{23},q)}{f_{CN}(x_1|x_2,\rho_{13}\rho_{23},q)} = f_{AW}(x_3|x_1,\rho_{13},x_2,\rho_{23},q).$$

Hence we have the following expansion of the AW density:

$$\begin{aligned} &f_{AW}(x_3|x_1,\rho_{13},x_2,\rho_{23}) \\ &\quad = f_{CN}(x_3|x_2,\rho_{23},q)\sum_{s\geqslant 0}\frac{\rho_{13}^s}{[s]_q!(\rho_{13}^2\rho_{23}^2)_s}P_s(x_1|x_2,\rho_{13}\rho_{23},q)P_s(x_3|x_2,\rho_{23},q), \end{aligned} \tag{3.7}$$

which is an analogue of the Poisson–Mehler expansion formula interpreted as the one-dimensional expansion given in Lemma 1 vii). This result has been obtained by other methods in [22]. Besides from (3.7) it follows that:

$$\begin{aligned} \forall n\geqslant 1\colon\quad &\int_{S(q)}\left(P_n(x_3|x_2,\rho_{23},q)-\frac{\rho_{13}^n(\rho_{23})_n}{(\rho_{13}^2\rho_{23}^2)_n}P_n(x_1|x_2,\rho_{13}\rho_{23},q)\right) \\ &\qquad \times f_{AW}(x_3|x_1,\rho_{13},x_2,\rho_{23})\,dx_3 = 0. \end{aligned}$$

Thus there must exist functions $F_{i,n}(x_1,x_2|\rho_{13},\rho_{23},q)$ such that:

$$\begin{aligned} &\forall j\geqslant 1\colon\quad A_j(x_3|x_1,\rho_{13},x_2,\rho_{23},q) \\ &\quad = \sum_{n=0}^{j}F_{j,n}(x_1,x_2|\rho_{13},\rho_{23},q)\left(P_n(x_3|x_2,\rho_{23},q)-\frac{\rho_{13}^n(\rho_{23})_n}{(\rho_{13}^2\rho_{23}^2)_n}P_n(x_1|x_2,\rho_{13}\rho_{23},q)\right), \end{aligned}$$

where $\{A_j\}_{j\geqslant 0}$ denote the Askey–Wilson polynomials with complex parameters related to x_1, ρ_{13}, x_2, ρ_{23} by the formulae (2.17)–(2.20) of [21].

As far as general properties of the function f_{3D} are concerned we have the following formula that expresses function C_n in terms of q-Hermite polynomials.

Proposition 2. $\forall n \geqslant 1$:

$$\begin{aligned} &C_n(x_1, x_3|\rho_{12}, \rho_{23}, \rho_{13}, q) \\ &= \frac{1}{(\rho_{13}^2)_n} \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k q^{\binom{k}{2}} \begin{bmatrix} n \\ 2k \end{bmatrix}_q \begin{bmatrix} 2k \\ k \end{bmatrix}_q [k]_q! \rho_{12}^k \rho_{13}^k \rho_{23}^k \left(\frac{\rho_{12}\rho_{13}}{\rho_{23}}, \frac{\rho_{13}\rho_{23}}{\rho_{12}} \right)_k \\ &\quad \times \sum_{i=0}^{n-2k} \begin{bmatrix} n-2k \\ i \end{bmatrix}_q \rho_{23}^i \left(\frac{\rho_{12}\rho_{13}}{\rho_{23}} q^k \right)_i \rho_{12}^{n-i-2k} \left(\frac{\rho_{13}\rho_{23}}{\rho_{12}} q^k \right)_{n-2k-i} H_i(x_1|q) H_{n-2k-i}(x_3|q). \end{aligned}$$

Proof. Long proof is shifted to Section 5. □

Let us remark that when $\rho_{13} = \rho_{12}\rho_{23}$ then

$$\begin{aligned} &C_n(x_1, x_3|\rho_{12}, \rho_{23}, \rho_{12}\rho_{23}, q) \\ &= \frac{1}{(\rho_{12}^2\rho_{23}^2)_n} \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k q^{\binom{k}{2}} \begin{bmatrix} n \\ 2k \end{bmatrix}_q \begin{bmatrix} 2k \\ k \end{bmatrix}_q [k]_q! \rho_{12}^{2k} \rho_{23}^{2k} (\rho_{12}^2, \rho_{23}^2)_k \\ &\quad \times \sum_{i=0}^{n-2k} \begin{bmatrix} n-2k \\ i \end{bmatrix}_q \rho_{23}^i (\rho_{12}^2 q^k)_i \rho_{12}^{n-i-2k} (\rho_{23}^2 q^k)_{n-2k-i} H_i(x_1|q) H_{n-2k-i}(x_3|q), \end{aligned}$$

the formula obtained in [21, 3.2–3.3] in the context of Askey–Wilson polynomials.

As stated in the Introduction the problem of non-negativity of the function f_{3D} for all allowed values of q and ρ's has negative solution. Above we indicated that if $\rho_{12} = \rho_{13}\rho_{23}$ (or similarly for some other pair of indices) then f_{3D} is positive for all $-1 < q \leqslant 1$ and $x_1, x_2, x_3 \in S(q)$. Now we will indicate some another relationship between ρ's and q so that for some values of $x_1, x_2, x_3 \in S(q)$ the function f_{3D} is negative.

We have the following theorem.

Theorem 1. *Let $|q| < 1$ and $\rho_{12} = q\rho_{13}\rho_{23}$, then there exists a set $S \subset S(q) \times S(q) \times S(q)$ of positive Lebesgue measure such that for all $(x_1, x_2, x_3) \in S$, f_{3D} is negative.*

Proof. Let us take $\rho_{12} = q\rho_{13}\rho_{23}$ and consider (3.2), then

$$\rho_{12}^s (\rho_{13}\rho_{23}/\rho_{12})_s = \begin{cases} 1 & \text{if } s = 0, \\ -(1-q)\rho_{13}\rho_{23} & \text{if } s = 1, \\ 0 & \text{if } s > 1. \end{cases}$$

Hence

$$\begin{aligned} &f_{3D}(x_1, x_2, x_3|q\rho_{13}\rho_{23}, \rho_{13}, \rho_{23}, q) \\ &\quad = f_{CN}(x_1|x_3, \rho_{13}, q) f_{CN}(x_3|x_2, \rho_{23}, q) f_N(x_2|q) \\ &\qquad \times \left(1 - \frac{(1-q)\rho_{13}\rho_{23}(x_1 - \rho_{13}x_3)(x_2 - \rho_{23}x_3)}{(1-\rho_{13}^2)(1-\rho_{23}^2)} \right). \end{aligned}$$

Thus the sign of f_{3D} is the same as the sign of

$$1-\frac{(1-q)\rho_{13}\rho_{23}(x_1-\rho_{13}x_3)(x_2-\rho_{23}x_3)}{(1-\rho_{13}^2)(1-\rho_{23}^2)}.$$

Equivalently f_{3D} would be positive if for all $x_1, x_2, x_3 \in S(q)$, $|\rho_{13}|, |\rho_{23}| < 1$ and

$$\left(1-\rho_{13}^2\right)\left(1-\rho_{23}^2\right) \geqslant (1-q)^2\rho_{13}^2\rho_{23}^2$$

which comes from the positivity in (2.19) we would have

$$\left(1-\rho_{13}^2\right)\left(1-\rho_{23}^2\right) \geqslant (1-q)\rho_{13}\rho_{23}(x_1-\rho_{13}x_3)(x_2-\rho_{23}x_3).$$

Since $y_i \stackrel{df}{=} \sqrt{1-q}x_i \in [-2, 2]$ the last inequality reduces to the following:

$$\left(1-\rho_{13}^2\right)\left(1-\rho_{23}^2\right) \geqslant \rho_{13}\rho_{23}(y_1-\rho_{13}y_3)(y_2-\rho_{23}y_3). \tag{3.8}$$

Let us select $y_1 = 2\rho_{13}$ and $y_2 = -2\rho_{23}$. The inequality now takes a form

$$\left(1-\rho_{13}^2\right)\left(1-\rho_{23}^2\right) \geqslant \rho_{13}^2\rho_{23}^2\left(4-y_3^2\right).$$

Now it suffices to take $\rho_{13}^2, \rho_{23}^2 = 0.6$, $1 > q > 1/3$ and $y_3^2 < 4 - \frac{4}{9}$. On one hand one gets

$$\left(1-\rho_{13}^2\right)\left(1-\rho_{23}^2\right) = .16 > \left(1-\frac{1}{3}\right)^2 .6^2$$

while on the other we have

$$\left(1-\rho_{13}^2\right)\left(1-\rho_{23}^2\right) = (1-.6)^2 \leqslant .6^2\left(4-\left(4-\frac{4}{9}\right)\right) \leqslant \rho_{13}^2\rho_{23}^2\left(4-y_3^2\right).$$

Since the function f_{3D} is continuous in x_1, x_2, x_3 hence there exists a neighborhood of points $(2\rho_{13}/\sqrt{1-q}, 2\rho_{23}/\sqrt{1-q}, 2y_3/\sqrt{1-q})$ of positive Lebesgue measure such that this function is negative on this neighborhood. □

As a corollary we get the following fact.

Corollary 4.

$$\begin{aligned}&\frac{f_{CN}(x_1|x_3,\rho_{13},q)}{f_{CN}(x_1|x_2,q\rho_{13}\rho_{23},q)}\left(1-\frac{(1-q)\rho_{13}\rho_{23}(x_1-\rho_{13}x_3)(x_2-\rho_{23}x_3)}{(1-\rho_{13}^2)(1-\rho_{23}^2)}\right)\\&\quad=\sum_{s=0}^{\infty}\frac{\rho_{13}^s(1-q^s\rho_{23}^2)}{[s]_q!(q^2\rho_{13}^2\rho_{23}^2)_s(1-\rho_{23}^2)}P_s(x_1|x_2,q\rho_{13}\rho_{23},q)P_s(x_3|x_2,\rho_{23},q).\end{aligned} \tag{3.9}$$

Proof. It follows the (3.2) and the fact that for $\rho_{12} = q\rho_{13}\rho_{23}$, $\rho_{13}^s(\rho_{12}\rho_{23}/\rho_{13})_s = \rho_{13}^s(q\rho_{23}^2)_s$ and further that $\frac{(q\rho_{23}^2)_s}{(\rho_{23}^2)_s} = \frac{1-q^s\rho_{23}^2}{1-\rho_{23}^2}$. $\square$

Remark 7. Notice that we can take $q\rho_{13}\rho_{23}$ instead of $\rho_{13}\rho_{23}$ in (3.3) getting $\frac{f_{CN}(x_1|x_3,\rho_{13},q)}{f_{CN}(x_1|x_2,q\rho_{13}\rho_{23},q)} = \sum_{s=0}^{\infty} \frac{\rho_{13}^s}{[s]_q!(q^2\rho_{13}^2\rho_{23}^2)_s} P_s(x_1|x_2,q\rho_{13}\rho_{23},q)P_s(x_3|x_2,\rho_{23},q)$. Hence the positivity conditions for kernels of the form $\sum_{s=0}^{\infty} a_s(\rho_1,\rho_2)P_s(x_1|x_2,\rho_1,q)P_s(x_3|x_2,\rho_2,q)$ are quite subtle. Besides comparison of (3.3) and (3.9) can be the source of many interesting kernels involving Al-Salam–Chihara polynomials. In particular for $\rho_{12} = q^k\rho_{13}\rho_{23}$ we have

$$\frac{f_{CN}(x_1|x_3,\rho_{13},q)}{f_{CN}(x_1|x_2,q^k\rho_{13}\rho_{23},q)} \sum_{j=0}^{k} \frac{q^{kj}\rho_{13}^j\rho_{23}^j(q^{-k})_j}{[j]_q!(\rho_{13}^2)_j(\rho_{23}^2)_j} P_j(x_1|x_3,\rho_{13},q)P_j(x_2|x_3,\rho_{23},q)$$
$$= \sum_{s=0}^{\infty} \frac{\rho_{13}^s(q^k\rho_{23}^2)_s}{[s]_q!(q^{2k}\rho_{13}^2\rho_{23}^2)_s(\rho_{23}^2)_s} P_s\big(x_1\big|x_2,q^k\rho_{13}\rho_{23},q\big)P_s(x_3|x_2,\rho_{23},q).$$

4. Open problems

(1) Consider the KS formulae for higher (than 3) dimensions. Can it be nonnegative for all allowed values of the parameters ρ and all values of the variables form $S(q)$?

(2) Since for $q = 0$ both densities f_{CN} and f_N and ASC polynomials (compare (2.3)) are very simple one can hope to obtain exact formula for function f_{3D}. One can deduce that then f_{3D} divided by the product of the one-dimensional marginals should have the form of the ratio of quadratic forms in 3 variables and the product of three quadratic functions of every pair of variables. Thus one can hope to find a set Θ in $(-1,1)^3$ such that if $(\rho_{12},\rho_{13},\rho_{23}) \in \Theta$ then f_{3D} is a 3-dimensional density.

(3) What about the cases $\rho_{12} = q^k\rho_{13}\rho_{23}$ for $k > 1$. Do we get non-positivity for all $k > 1$ or are there some k's for which it is nonnegative?

(4) Numerical simulations suggest that for say $\rho_{12} = 0$ one can find $q, \rho_{13}, \rho_{23}, x_1, x_2, x_3 \in S(q)$ such that one of the kernels (and consequently both) given in (3.4) is negative. What would be the proof of this fact? More precisely what would be range of parameters q, ρ_{13}, ρ_{23} and subset $A \subset S(q)^3$ of positive Lebesgue measure such that for $x_1, x_2, x_3 \in A$ the kernel is negative.

(5) What about the cases $|\rho_{12}| > \rho_{13}\rho_{23}$?

5. Proofs

Proof of Proposition 1. First of all let us notice that due to inequality (2.14) and assertion viii) of Lemma 1 series defining function f_{3D} is absolutely convergent. Now to prove i) we apply Lemma 2 i) to (2.17) and (2.18) and remembering that $\gamma_{0,0}(x,y|\rho,q) = f_{CN}(x|y,\rho,q)/f_N(x|q)$ we get:

$$f_{3D}(x_1,x_2,x_3|\rho_{12},\rho_{13},\rho_{23},q)$$
$$= \prod_{i=1}^{3} f_N(x_i|q) \sum_{j,k\geqslant 0} \frac{\rho_{12}^j\rho_{23}^k}{[j]_q![k]_q!} H_{j+k}(x_2|q) \sum_{i\geqslant 0} \frac{\rho_{13}^i}{[i]_q!} H_{i+j}(x_1|q)H_{i+k}(x_3|q)$$

$$= f_{CN}(x_1|x_3,\rho_{13},q)f_N(x_3|q)f_N(x_2|q)\sum_{j,k\geqslant 0}\frac{\rho_{12}^j\rho_{23}^k}{[j]_q![k]_q!}H_{j+k}(x_2|q)Q_{j,k}(x_1,x_3|\rho_{13},q).$$

Now changing order of summation and substituting $j+k\to s$, $j\to k$ we get

$$f_{3D}(x_1,x_2,x_3|\rho_{12},\rho_{13},\rho_{23},q) = f_{CN}(x_3|x_1,\rho_{13},q)f_N(x_1|q)f_N(x_2|q)\times\sum_{s\geqslant 0}\frac{1}{[s]_q!}H_s(x_2|q)C_s(x_1,x_3|\rho_{12},\rho_{23},\rho_{13},q).$$

ii) We apply (2.6). Then

$$\begin{aligned}
&f_{3D}(x_1,x_2,x_3|\rho_{12},\rho_{13},\rho_{23},q)\\
&\quad=\prod_{i=1}^{3}f_N(x_i|q)\sum_{i,j,k\geqslant 0}\frac{\rho_{12}^j\rho_{23}^k\rho_{13}^i}{[i]_q![j]_q![k]_q!}H_{i+j}(x_1|q)H_{j+k}(x_2|q)\\
&\qquad\times\sum_{n\geqslant 0}\begin{bmatrix} i\\ n\end{bmatrix}_q\begin{bmatrix} k\\ n\end{bmatrix}_q[n]_q!(-1)^n q^{\binom{n}{2}}H_{i-n}(x_3|q)H_{k-n}(x_3|q)\\
&\quad=\prod_{i=1}^{3}f_N(x_i|q)\sum_{n=0}^{\infty}(-1)^n q^{\binom{n}{2}}\frac{\rho_{23}^n\rho_{13}^n}{[n]_q!}\sum_{j=0}^{\infty}\frac{\rho_{12}^j}{[j]_q!}\\
&\qquad\times\sum_{i=n}^{\infty}\sum_{k=n}^{\infty}\frac{\rho_{23}^{k-n}\rho_{13}^{i-n}}{[i-n]_q![k-n]_q!}H_{i-n}(x_3|q)H_{k-n}(x_3|q)H_{i+j}(x_1|q)H_{j+k}(x_2|q)\\
&\quad=\prod_{i=1}^{3}f_N(x_i|q)\sum_{n=0}^{\infty}(-1)^n q^{\binom{n}{2}}\frac{\rho_{23}^n\rho_{13}^n}{[n]_q!}\sum_{j=0}^{\infty}\frac{\rho_{12}^j}{[j]_q!}\\
&\qquad\times\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\frac{\rho_{23}^{k}\rho_{13}^{i}}{[i]_q![k]_q!}H_{i}(x_3|q)H_{i+n+j}(x_1|q)H_{k}(x_3|q)H_{j+n+k}(x_2|q).
\end{aligned}$$

Now we use quantities: $\gamma_{m,k}(x,y|\rho,q)=\sum_{i=0}^{\infty}\frac{\rho^i}{[i]_q!}H_{i+m}(x|q)H_{i+k}(y|q)$ and apply (2.20) and Lemma 2 iii). We get then

$$\begin{aligned}
&f_{3D}(x_1,x_2,x_3|\rho_{12},\rho_{13},\rho_{23},q)\\
&\quad=\prod_{i=1}^{3}f_N(x_i|q)\sum_{n=0}^{\infty}(-1)^n q^{\binom{n}{2}}\frac{\rho_{23}^n\rho_{13}^n}{[n]_q!}\\
&\qquad\times\sum_{j=0}^{\infty}\frac{\rho_{12}^j}{[j]_q!}\gamma_{0,n+j}(x_3,x_1|\rho_{13},q)\gamma_{0,n+j}(x_3,x_2|\rho_{23},q)\\
&\quad=f_{CN}(x_1|x_3,\rho_{13},q)f_{CN}(x_2|x_3,\rho_{23},q)f_N(x_3|q)
\end{aligned}$$

$$\times\sum_{n=0}^{\infty}(-1)^n q^{\binom{n}{2}}\frac{\rho_{23}^n\rho_{13}^n}{[n]_q!}\sum_{j=0}^{\infty}\frac{\rho_{12}^j}{[j]_q!(\rho_{13}^2)_{n+j}(\rho_{23}^2)_{n+j}}P_{n+j}(x_1|x_3,\rho_{13},q)P_{n+j}(x_2|x_3,\rho_{23},q)$$

$$= f_{CN}(x_1|x_3,\rho_{13},q)f_{CN}(x_2|x_3,\rho_{23},q)f_N(x_3|q)$$

$$\times\sum_{s=0}^{\infty}\frac{1}{[s]_q!(\rho_{13}^2)_s(\rho_{23}^2)_s}P_s(x_1|x_3,\rho_{13},q)P_s(x_2|x_3,\rho_{23},q)$$

$$\times\sum_{n=0}^{s}\begin{bmatrix}s\\n\end{bmatrix}_q(-1)q^{\binom{n}{2}}(\rho_{13}\rho_{23})^n\rho_{12}^{s-n}$$

$$= f_{CN}(x_1|x_3,\rho_{13},q)f_{CN}(x_2|x_3,\rho_{23},q)f_N(x_3|q)$$

$$\times\sum_{s=0}^{\infty}\frac{\rho_{12}^s(\rho_{13}\rho_{23}/\rho_{12})_s}{[s]_q!(\rho_{13}^2)_s(\rho_{23}^2)_s}P_s(x_1|x_3,\rho_{13},q)P_s(x_2|x_3,\rho_{23},q). \qquad\square$$

Proof of Proposition 2. We start with $C_n(x,y|\rho_1,\rho_2,\rho_3,q) = \sum_{i=0}^{n}\begin{bmatrix}n\\i\end{bmatrix}_q\rho_1^{n-i}\rho_2^i \times Q_{n-i,i}(x,y|\rho_{3,}q) = \sum_{s=0}^{n}\begin{bmatrix}n\\s\end{bmatrix}_q H_{n-s}(y|q)P_s(x|y,\rho_3,q)\rho_1^{n-s}\rho_2^s(\rho_1\rho_3/\rho_2)_s/(\rho_3^2)_s$ and apply Lemma 1 ii).

$$C_n(x_1,x_3|\rho_{12},\rho_{23},\rho_{13},q)$$

$$=\sum_{s=0}^{n}\begin{bmatrix}n\\s\end{bmatrix}_q\rho_{12}^{n-s}H_{n-s}(x_3|q)\rho_{23}^s\left(\frac{\rho_{12}\rho_{13}}{\rho_{23}}\right)_s P_s(x_1|x_3,\rho_{13},q)/\left(\rho_{13}^2\right)_s$$

$$=\frac{1}{(\rho_{13}^2)_n}\sum_{s=0}^{n}\begin{bmatrix}n\\s\end{bmatrix}_q\rho_{12}^{n-s}H_{n-s}(x_3|q)\rho_{23}^s\left(\frac{\rho_{12}\rho_{13}}{\rho_{23}}\right)_s P_s(x_1|x_3,\rho_{13},q)\left(\rho_{13}^2q^s\right)_{n-s}$$

$$=\frac{1}{(\rho_{13}^2)_n}\sum_{s=0}^{n}\begin{bmatrix}n\\s\end{bmatrix}_q\rho_{12}^{n-s}H_{n-s}(x_3|q)\rho_{23}^s\left(\frac{\rho_{12}\rho_{13}}{\rho_{23}}\right)_s\left(\rho_{12}^2q^s\right)_{n-s}$$

$$\times\sum_{i=0}^{s}\begin{bmatrix}s\\i\end{bmatrix}_q\rho_{13}^{s-i}B_{s-i}(x_3|q)H_i(x_1|q)$$

$$=\frac{1}{(\rho_{13}^2)_n}\sum_{i=0}^{n}\begin{bmatrix}n\\i\end{bmatrix}_q H_i(x_1|q)$$

$$\times\sum_{s=i}^{n}\begin{bmatrix}n-i\\s-i\end{bmatrix}_q\rho_{12}^{n-s}H_{n-s}(x_3|q)\rho_{23}^s\left(\frac{\rho_{12}\rho_{13}}{\rho_{23}}\right)_s\left(\rho_{13}^2q^s\right)_{n-s}\rho_{13}^{s-i}B_{s-i}(x_3|q)$$

$$=\frac{1}{(\rho_{13}^2)_n}\sum_{i=0}^{n}\begin{bmatrix}n\\i\end{bmatrix}_q H_i(x_1|q)$$

$$\times\sum_{j=0}^{n-i}\begin{bmatrix}n-i\\j\end{bmatrix}_q\rho_{12}^{n-i-j}H_{n-i-j}(x_3|q)\rho_{23}^{i+j}\left(\frac{\rho_{12}\rho_{13}}{\rho_{23}}\right)_{i+j}\left(\rho_{13}^2q^{i+j}\right)_{n-i-j}\rho_{13}^jB_j(x_3|q).$$

And further using formula

$$H_m(x|q)B_n(x|q) = (-1)^n q^{\binom{n}{2}} \sum_{i=0}^{\lfloor (n+m)/2\rfloor} \begin{bmatrix} n \\ i \end{bmatrix}_q \begin{bmatrix} n+m-i \\ i \end{bmatrix}_q [i]_q! q^{-i(n-i)} H_{n+m-2i}(x|q),$$

proved in [21, Lemma 2 i)] we get

$$\begin{aligned} &C_n(x_1, x_3|\rho_{12}, \rho_{23}, \rho_{13}, q) \\ &\quad = \frac{1}{(\rho_{13}^2)_n} \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_q H_i(x_1|q) \\ &\quad\quad \times \sum_{j=0}^{n-i} \begin{bmatrix} n-i \\ j \end{bmatrix}_q \rho_{12}^{n-i-j} \rho_{23}^{i+j} \left(\frac{\rho_{12}\rho_{13}}{\rho_{23}}\right)_{i+j} \left(\rho_{13}^2 q^{i+j}\right)_{n-i-j} \rho_{13}^j (-1)^j q^{\binom{j}{2}} \\ &\quad\quad \times \sum_{k=0}^{\lfloor (n-i)/2\rfloor} \begin{bmatrix} j \\ k \end{bmatrix}_q \begin{bmatrix} n-i-k \\ k \end{bmatrix}_q [k]_q! q^{-k(j-k)} H_{n-i-2k}(x_3|q). \end{aligned}$$

Now keeping in mind that $\begin{bmatrix} j \\ k \end{bmatrix}_q = 0$ for $j < k$ we split first internal sum into two sums: the first one from 0 to $\lfloor (n-i)/2\rfloor$ and the second one from $\lfloor (n-i)/2\rfloor + 1$ to $(n-i)$.

$$\begin{aligned} &= \frac{1}{(\rho_{13}^2)_n} \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_q H_i(x_1|q) \\ &\quad \times \Bigg(\sum_{j=0}^{\lfloor (n-i)/2\rfloor} \begin{bmatrix} n-i \\ j \end{bmatrix}_q \rho_{12}^{n-i-j} \rho_{23}^{i+j} \left(\frac{\rho_{12}\rho_{13}}{\rho_{23}}\right)_{i+j} \left(\rho_{13}^2 q^{i+j}\right)_{n-i-j} \rho_{13}^j (-1)^j q^{\binom{j}{2}} \\ &\quad \times \sum_{k=0}^{j} \begin{bmatrix} j \\ k \end{bmatrix}_q \begin{bmatrix} n-i-k \\ k \end{bmatrix}_q [k]_q! q^{-k(j-k)} H_{n-i-2k}(x_3|q) \\ &\quad + \sum_{j=\lfloor (n-i)/2\rfloor+1}^{(n-i)} \begin{bmatrix} n-i \\ j \end{bmatrix}_q \rho_{12}^{n-i-j} \rho_{23}^{i+j} \left(\frac{\rho_{12}\rho_{13}}{\rho_{23}}\right)_{i+j} \left(\rho_{13}^2 q^{i+j}\right)_{n-i-j} \rho_{13}^j (-1)^j q^{\binom{j}{2}} \\ &\quad \times \sum_{k=0}^{\lfloor (n-i)/2\rfloor} \begin{bmatrix} j \\ k \end{bmatrix}_q \begin{bmatrix} n-i-k \\ k \end{bmatrix}_q [k]_q! q^{-k(j-k)} H_{n-i-2k}(x_3|q) \Bigg). \end{aligned}$$

We have further after changing the order of summation.

$$\begin{aligned} &C_n(x_1, x_3|\rho_{12}, \rho_{23}, \rho_{13}, q) \\ &\quad = \frac{1}{(\rho_{13}^2)_n} \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_q H_i(x_1|q) \end{aligned}$$

$$\times\left(\sum_{k=0}^{\lfloor (n-i)/2\rfloor}\frac{[n-i]_q!}{[k]_q![n-i-2k]_q!}H_{n-i-2k}(x_3|q)\right.$$

$$\times\sum_{j=k}^{\lfloor (n-i)/2\rfloor}q^{-k(j-k)}\rho_{13}^{j}(-1)^{j}q^{\binom{j}{2}}\frac{[n-i-k]_q!}{[j-k]_q![n-i-j]_q!}\rho_{12}^{n-i-j}\rho_{23}^{i+j}$$

$$\times\left(\frac{\rho_{12}\rho_{13}}{\rho_{23}}\right)_{i+j}\left(\rho_{13}^2q^{i+j}\right)_{n-i-j}$$

$$+\sum_{k=0}^{\lfloor (n-i)/2\rfloor}\frac{[n-i]_q!}{[k]_q![n-i-2k]_q!}H_{n-i-2k}(x_3|q)$$

$$\times\sum_{j=\lfloor (n-i)/2\rfloor+1}^{(n-i)}\frac{[n-i-k]_q!}{[j-k]_q![n-i-j]_q!}(-1)^{j}q^{-k(j-k)}q^{\binom{j}{2}}\rho_{13}^{j}\rho_{12}^{n-i-j}\rho_{23}^{i+j}$$

$$\left.\times\left(\frac{\rho_{12}\rho_{13}}{\rho_{23}}\right)_{i+j}\left(\rho_{13}^2q^{i+j}\right)_{n-i-j}\right)$$

$$=\frac{1}{(\rho_{13}^2)_n}\sum_{i=0}^{n}\begin{bmatrix}n\\i\end{bmatrix}_q H_i(x_1|q)$$

$$\times\left(\sum_{k=0}^{\lfloor (n-i)/2\rfloor}\frac{[n-i]_q!}{[k]_q![n-i-2k]_q!}H_{n-i-2k}(x_3|q)\right.$$

$$\left.\times\sum_{j=k}^{(n-i)}\begin{bmatrix}n-i-k\\j-k\end{bmatrix}_q(-1)^{j}q^{-k(j-k)}q^{\binom{j}{2}}\rho_{13}^{j}\rho_{12}^{n-i-j}\rho_{23}^{i+j}\left(\frac{\rho_{12}\rho_{13}}{\rho_{23}}\right)_{i+j}\left(\rho_{13}^2q^{i+j}\right)_{n-i-j}\right).$$

Hence

$$C_n(x_1,x_3|\rho_{12},\rho_{23},\rho_{13},q)$$

$$=\frac{1}{(\rho_{13}^2)_n}\sum_{i=0}^{n}\begin{bmatrix}n\\i\end{bmatrix}_q H_i(x_1|q)$$

$$\times\sum_{k=0}^{\lfloor (n-i)/2\rfloor}(-1)^{k}q^{\binom{k}{2}}\frac{[n-i]_q!}{[k]_q![n-i-2k]_q!}\rho_{13}^{k}H_{n-i-2k}(x_3|q)$$

$$\times\sum_{s=0}^{n-i-k}\begin{bmatrix}n-i-k\\s\end{bmatrix}_q(-1)^{s}q^{\binom{s}{2}}\rho_{13}^{s}\rho_{12}^{n-i-k-s}\rho_{23}^{i+k+s}\left(\frac{\rho_{12}\rho_{13}}{\rho_{23}}\right)_{i+k+s}\left(\rho_{13}^2q^{i+k+s}\right)_{n-i-k-s}.$$

So

$$\begin{aligned}
&C_n(x_1,x_3|\rho_{12},\rho_{13},\rho_{23},q)\\
&\quad=\frac{1}{(\rho_{13}^2)_n}\sum_{i=0}^{n}\begin{bmatrix}n\\i\end{bmatrix}_q H_i(x_1|q)\\
&\qquad\times\sum_{k=0}^{\lfloor(n-i)/2\rfloor}(-1)^k q^{\binom{k}{2}}\frac{[n-i]_q!}{[k]_q![n-i-2k]_q!}\rho_{13}^k\rho_{23}^{k+i}H_{n-i-2k}(x_3|q)\left(\frac{\rho_{12}\rho_{13}}{\rho_{23}}\right)_{k+i}\\
&\qquad\times\sum_{s=0}^{n-i-k}\begin{bmatrix}n-i-k\\s\end{bmatrix}_q(-1)^s q^{\binom{s}{2}}\rho_{13}^s\rho_{12}^{n-i-k-s}\rho_{23}^s\left(\frac{\rho_{12}\rho_{13}}{\rho_{23}}q^{i+k}\right)_s\left(\rho_{13}^2q^{i+k+s}\right)_{n-i-k-s}.
\end{aligned}$$

Now we use formula $\sum_{i=0}^{n}(-1)^i q^{\binom{i}{2}}\begin{bmatrix}n\\i\end{bmatrix}_q(a)_i b^i(abq^i)_{n-i}=(b)_n$ proved in [21, Lemma 1 ii)] with $a=\frac{\rho_{12}\rho_{13}}{\rho_{23}}q^{i+k}$, $b=\rho_{13}\rho_{23}/\rho_{12}$ getting

$$\begin{aligned}
&C_n(x_1,x_3|\rho_{12},\rho_{13},\rho_{23},q)\\
&\quad=\frac{1}{(\rho_{13}^2)_n}\sum_{i=0}^{n}\begin{bmatrix}n\\i\end{bmatrix}_q H_i(x_1|q)\\
&\qquad\times\sum_{k=0}^{\lfloor(n-i)/2\rfloor}(-1)^k q^{\binom{k}{2}}\frac{[n-i]_q!}{[k]_q![n-i-2k]_q!}\rho_{13}^k\rho_{23}^{k+i}H_{n-i-2k}(x_3|q)\left(\frac{\rho_{12}\rho_{13}}{\rho_{23}}\right)_{k+i}\rho_{12}^{n-i-k}\\
&\qquad\times\left(\frac{\rho_{12}\rho_{23}}{\rho_{12}}\right)_{n-k-i}.
\end{aligned}$$

Finally change the order of summation using on the way an obvious property of $(a)_n=(a)_j(aq^j)_{n-j}$ for every $0\leqslant j\leqslant n$

$$\begin{aligned}
&C_n(x_1,x_3|\rho_{12},\rho_{13},\rho_{23},q)\\
&\quad=\frac{1}{(\rho_{13}^2)_n}\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k q^{\binom{k}{2}}\begin{bmatrix}n\\2k\end{bmatrix}_q\begin{bmatrix}2k\\k\end{bmatrix}_q[k]_q!\rho_{12}^k\rho_{23}^k\rho_{13}^k\left(\frac{\rho_{12}\rho_{13}}{\rho_{23}}\right)_k\left(\frac{\rho_{12}\rho_{23}}{\rho_{12}}\right)_k\\
&\qquad\times\sum_{i=0}^{n-2k}\frac{[n-2k]_q!}{[n-i-2k]_q![i]_q!}\rho_{23}^i\left(\frac{\rho_{12}\rho_{13}}{\rho_{23}}q^k\right)_i\rho_{12}^{n-i-2k}\\
&\qquad\times\left(\frac{\rho_{13}\rho_{23}}{\rho_{12}}q^k\right)_{n-i-2k}H_i(x_1|q)H_{n-2k-i}(x_3|q).\qquad\square
\end{aligned}$$

References

[1] W.A. Al-Salam, Mourad E.H. Ismail, q-Beta integrals and the q-Hermite polynomials, Pacific J. Math. 135 (2) (1988) 209–221. MR0968609 (90c:33001).

[2] George E. Andrews, Richard Askey, Ranjan Roy, Special Functions, Cambridge Univ. Press, 1999.

[3] Richard Askey, Continuous q-Hermite polynomials when $q > 1$, in: q-Series and Partitions, Minneapolis, MN, 1988, in: IMA Vol. Math. Appl., vol. 18, Springer, New York, 1989, pp. 151–158. MR1019849 (90h:33019).
[4] Richard Askey, Mourad Ismail, Recurrence relations, continued fractions, and orthogonal polynomials, Mem. Amer. Math. Soc. 49 (300) (1984), iv+108 pp. MR0743545 (85g:33008).
[5] Richard A. Askey, Mizan Rahman, Sergeĭ K. Suslov, On a general q-Fourier transformation with nonsymmetric kernels, J. Comput. Appl. Math. 68 (1–2) (1996) 25–55. MR1418749 (98m:42033).
[6] Marek Bożejko, Burkhard Kümmerer, Roland Speicher, q-Gaussian processes: non-commutative and classical aspects, Comm. Math. Phys. 185 (1) (1997) 129–154. MR1463036 (98h:81053).
[7] D.M. Bressoud, A simple proof of Mehler's formula for q-Hermite polynomials, Indiana Univ. Math. J. 29 (4) (1980) 577–580. MR0578207 (81f:33009).
[8] Włodzimierz Bryc, Stationary random fields with linear regressions, Ann. Probab. 29 (1) (2001) 504–519. MR1825162 (2002d:60014).
[9] Włodzimierz Bryc, Wojciech Matysiak, Paweł J. Szabłowski, Probabilistic aspects of Al-Salam–Chihara polynomials, Proc. Amer. Math. Soc. 133 (4) (2005) 1127–1134 (electronic). MR2117214 (2005m:33033).
[10] L. Carlitz, Some polynomials related to theta functions, Ann. Mat. Pura Appl. (4) 41 (1956) 359–373. MR0078510 (17,1205e).
[11] L. Carlitz, Generating functions for certain Q-orthogonal polynomials, Collect. Math. 23 (1972) 91–104. MR0316773 (47 #5321).
[12] William Y.C. Chen, Husam L. Saad, Lisa H. Sun, The bivariate Rogers–Szegő polynomials, J. Phys. A 40 (23) (2007) 6071–6084. MR2343510 (2008k:33064).
[13] Roberto Floreanini, Jean LeTourneux, Luc Vinet, More on the q-oscillator algebra and q-orthogonal polynomials, J. Phys. A 28 (10) (1995) L287–L293. MR1343867 (96e:33043).
[14] Roberto Floreanini, Jean LeTourneux, Luc Vinet, Symmetry techniques for the Al-Salam–Chihara polynomials, J. Phys. A 30 (9) (1997) 3107–3114. MR1456902 (98k:33036).
[15] Mourad E.H. Ismail, Classical and Quantum Orthogonal Polynomials in One Variable, Encyclopedia Math. Appl., vol. 98, Cambridge University Press, Cambridge, 2005, xviii+706 pp. ISBN 978-0-521-78201-2; 0-521-78201-5. With two chapters by Walter Van Assche. With a foreword by Richard A. Askey. MR2191786 (2007f:33001).
[16] Mourad E.H. Ismail, Dennis Stanton, q-Integral and moment representations for q-orthogonal polynomials, Canad. J. Math. 54 (4) (2002) 709–735. MR1913916 (2003k:33024).
[17] Mourad E.H. Ismail, Mizan Rahman, Stanton Dennis, Quadratic q-exponentials and connection coefficient problems, Proc. Amer. Math. Soc. 127 (10) (1999) 2931–2941. MR1621949 (2000a:33027).
[18] W.F. Kibble, An extension of a theorem of Mehler's on Hermite polynomials, Proc. Cambridge Philos. Soc. 41 (1945) 12–15. MR0012728 (7,65f).
[19] David Slepian, On the symmetrized Kronecker power of a matrix and extensions of Mehler's formula for Hermite polynomials, SIAM J. Math. Anal. 3 (1972) 606–616. MR0315173 (47 #3722).
[20] Paweł J. Szabłowski, Expansions of one density via polynomials orthogonal with respect to the other, J. Math. Anal. Appl. 383 (2011) 35–54, http://arxiv.org/abs/1011.1492.
[21] Paweł J. Szabłowski, On the structure and probabilistic interpretation of Askey–Wilson densities and polynomials with complex parameters, J. Funct. Anal. 262 (2011) 635–659, MR2799574, http://arxiv.org/abs/1011.1541.
[22] P.J. Szabłowski, On summable form of Poisson–Mehler kernel for big q-Hermite and Al-Salam–Chihara polynomials, submitted for publication, http://arxiv.org/abs/1011.1848.