



Probing the $f(R)$ formalism through gravitational wave polarizations

M.E.S. Alves*, O.D. Miranda, J.C.N. de Araujo

INPE-Instituto Nacional de Pesquisas Espaciais – Divisão de Astrofísica, Av.dos Astronautas 1758, São José dos Campos, 12227-010 SP, Brazil

ARTICLE INFO

Article history:

Received 4 May 2009

Received in revised form 31 July 2009

Accepted 5 August 2009

Available online 8 August 2009

Editor: S. Dodelson

PACS:

04.30.-w

04.50.Kd

Keywords:

Gravitational wave polarizations

Modified theories of gravity

ABSTRACT

The direct observation of gravitational waves (GW) in the near future, and the corresponding determination of the number of independent polarizations, is a powerful tool to test general relativity and alternative theories of gravity. In the present work we use the Newman–Penrose formalism to characterize GWs in quadratic gravity and in a particular class of $f(R)$ Lagrangians. We find that both quadratic gravity and the $f(R)$ theory belong to the most general invariant class of GWs, i.e., they can present up to six independent polarizations of GWs. For a particular combination of the parameters, we find that quadratic gravity can present up to five polarizations states. On the other hand, if we use the Palatini approach for $f(R)$ theories, GWs present only the usual two transverse-traceless polarizations such as in general relativity. Thus, we conclude that the observation of GWs can strongly constrain the suitable formalism for these theories.

© 2009 Elsevier B.V. Open access under [CC BY license](#).

1. Introduction

Modifications of Einstein's gravity have been considered in several approaches. The Lagrangians which consider higher orders of the Ricci scalar R and the Ricci tensor $R_{\mu\nu}$ have been proposed as extensions of general relativity [1]. The semiclassical theory considers the backreaction of quantum fields in a classical geometric background [2,3]. Such Lagrangians predict field equations with four orders derivatives of the metric, rather than the two orders derivatives in the general relativity theory [2,4]. Quadratic Lagrangians have been used also to yield renormalizable theories of gravity coupled to matter [5]. Moreover, higher-derivative theories arise as a low energy limit of string theories [6,7].

Starobinsky [8] argues that the higher order terms could mimic a cosmological constant. Recently, this idea has been largely studied as a potential way to address the dark energy problem. In this context, several different forms of the modified Lagrangians have been recently considered constituting a class of theories, the so-called $f(R)$ theories (see e.g. [9] and references therein).

In the context of gravitational waves (GWs), it was shown that quadratic gravity¹ presents a frequency-dependent shift in the wave amplitude with respect to the wave amplitude obtained

from general relativity [10]. For the $f(R)$ theories, it was found that GWs can have a massive-like scalar mode besides the usual transverse-traceless modes [11].

A powerful tool to study the properties of GWs in any metric theory of gravity was developed by Eardley et al. [12]. The basic idea is to analyze all the physically relevant components of the Riemann tensor $R_{\lambda\mu\kappa\nu}$, which cause relative acceleration between test particles. The GWs in a metric theory involves the metric field $g_{\mu\nu}$ and any auxiliary gravitational fields that could exist. But the resultant Riemann tensor is the only measurable field.

In their work, Eardley et al. used a null-tetrad basis in order to calculate the Newman–Penrose [13] quantities in terms of the irreducible parts of $R_{\lambda\mu\kappa\nu}$, namely, the Weyl tensor, the traceless Ricci tensor and the Ricci scalar. This analysis showed that there are six possible modes of polarization of GWs in the most general case, which can be completely resolved by feasible experiments. Thus, it is possible to classify a given theory by the non-null Newman–Penrose quantities [12].

The aim of the present work is to characterize GWs for a particular class of $f(R)$ gravity and for quadratic gravity making use of the Newman–Penrose formalism. Since the field equations derived from such Lagrangians in the metric formalism yield dynamical equations for R and $R_{\mu\nu}$, the analysis consists basically to find the resultant expressions for R and $R_{\mu\nu}$ in the weak field limit, without the need to write explicitly the expressions in terms of the metric perturbations. This makes the classification easier and clear. We also mention how the use of the Palatini approach in deriving the field equations for $f(R)$ gravity affect the number of independent polarizations of GWs. This comes

* Corresponding author.

E-mail addresses: alvesmes@das.inpe.br (M.E.S. Alves), oswaldo@das.inpe.br (O.D. Miranda), jcarlos@das.inpe.br (J.C.N. de Araujo).

¹ For the sake of definiteness we will call quadratic gravity that theory which takes into account not only R^2 but also $R_{\mu\nu}R^{\mu\nu}$ in the Lagrangian.

from the fact that the classification of a given $f(R)$ is formalism-dependent.

It is argued that the observations of the GWs in the near future (for the current status of GWs detectors see e.g. [14–18]), and the corresponding determination of all possible states of polarization, is a very powerful test for the present studied alternative theories of gravity. Particularly, we show that GWs experiments can be decisive for quadratic gravity and in the determination of the suitable formalism for $f(R)$ theories, i.e., the use of the metric or the Palatini approaches, since the number of polarizations of GWs depends on the formalism used.²

2. The Newman–Penrose formalism – an overview

Throughout this Letter we consider GWs propagating in the $+z$ direction. So, all the quantities are functions only of t and z .

At any point P , the null complex tetrad $(\mathbf{k}, \mathbf{l}, \mathbf{m}, \bar{\mathbf{m}})$ is related to the Cartesian tetrad $(\mathbf{e}_t, \mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ by:

$$\mathbf{k} = \frac{1}{\sqrt{2}}(\mathbf{e}_t + \mathbf{e}_z), \quad (1)$$

$$\mathbf{l} = \frac{1}{\sqrt{2}}(\mathbf{e}_t - \mathbf{e}_z), \quad (2)$$

$$\mathbf{m} = \frac{1}{\sqrt{2}}(\mathbf{e}_x + i\mathbf{e}_y), \quad (3)$$

$$\bar{\mathbf{m}} = \frac{1}{\sqrt{2}}(\mathbf{e}_x - i\mathbf{e}_y). \quad (4)$$

It is easy to verify that the tetrad vectors obey the relations:

$$-\mathbf{k} \cdot \mathbf{l} = \mathbf{m} \cdot \bar{\mathbf{m}} = 1, \quad (5)$$

$$\mathbf{k} \cdot \mathbf{m} = \mathbf{k} \cdot \bar{\mathbf{m}} = \mathbf{l} \cdot \mathbf{m} = \mathbf{l} \cdot \bar{\mathbf{m}} = 0. \quad (6)$$

The null-tetrad components of a tensor \mathbf{T} are written according to the notation:

$$T_{abc\dots} = T_{\mu\nu\lambda\dots} a^\mu b^\nu c^\lambda \dots, \quad (7)$$

where (a, b, c, \dots) run over $(\mathbf{k}, \mathbf{l}, \mathbf{m}, \bar{\mathbf{m}})$ and $(\mu, \nu, \lambda, \dots)$ run over (t, x, y, z) since we are working in Cartesian coordinates.

In general, the Newman–Penrose quantities, namely, the ten Ψ 's, nine Φ 's, and Λ , which represent the irreducible parts of the Riemann tensor $R_{\lambda\mu\kappa\nu}$, are all algebraically independent. When we restrict ourselves to nearly plane waves, however, we find that the differential and symmetry properties of $R_{\lambda\mu\kappa\nu}$ reduce the number of independent, nonvanishing components, to six. Thus, we shall choose the set $\{\Psi_2, \Psi_3, \Psi_4, \Phi_{22}\}$ to describe, in a given null frame, the six independent components of a wave in the generic metric theory. In the tetrad basis, the Newman–Penrose quantities of the Riemann tensor are, therefore, given by:

$$\Psi_2 = -\frac{1}{6} R_{lklk}, \quad (8)$$

$$\Psi_3 = -\frac{1}{2} R_{lkl\bar{m}}, \quad (9)$$

$$\Psi_4 = -R_{l\bar{m}l\bar{m}}, \quad (10)$$

$$\Phi_{22} = -R_{lml\bar{m}}. \quad (11)$$

Note that, Ψ_3 and Ψ_4 are complex, thus each one represents two independent polarizations. One polarization for the real part and

Gravitational–Wave Polarization

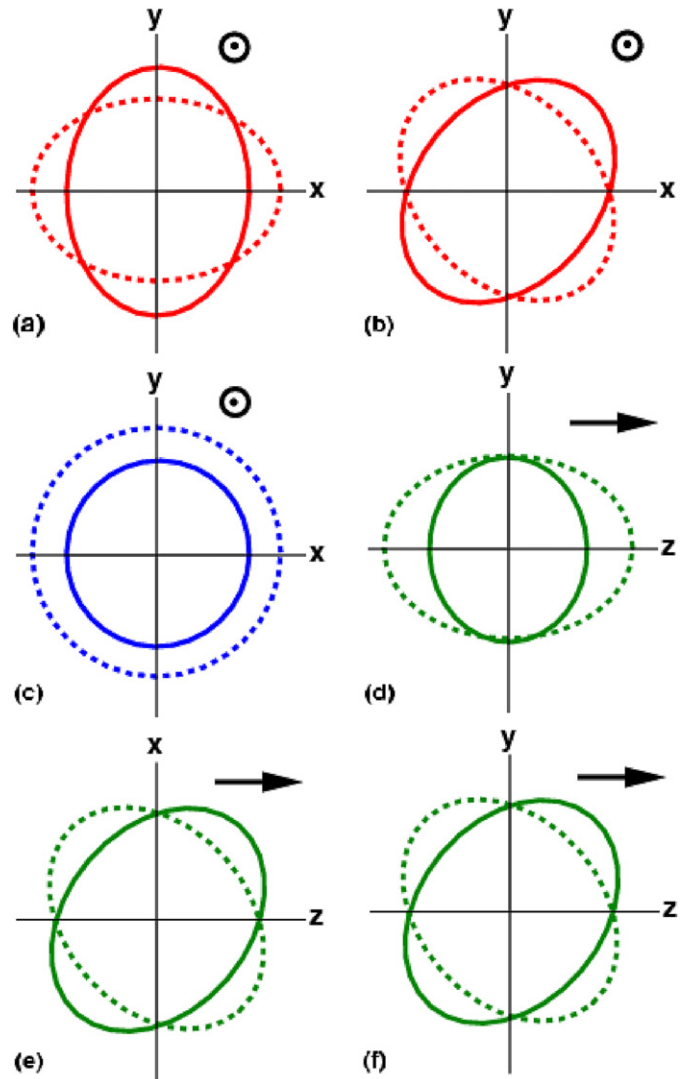


Fig. 1. The six polarization modes of weak, plane, null GW permitted in any metric theory of gravity. Also shown is the displacement that each mode induces on a sphere of test particles. The wave propagates out of the plane in (a), (b) and (c), and it propagates in the plane in (d), (e) and (f). The displacement induced on the sphere of test particles corresponds to the following Newman–Penrose quantities: $\text{Re } \Psi_4$ (a), $\text{Im } \Psi_4$ (b), Φ_{22} (c), Ψ_2 (d), $\text{Re } \Psi_3$ (e), $\text{Im } \Psi_3$ (f).

one for the imaginary part, totaling six components (see Fig. 1 which was taken from [20]).

Analyzing the behavior of the set $\{\Psi_2, \Psi_3, \Psi_4, \Phi_{22}\}$ under rotations, we see that they have the respective helicity values $s = \{0, \pm 1, \pm 2, 0\}$.

We also have the very useful relations for the Ricci tensor:

$$R_{lk} = R_{lklk}, \quad (12)$$

$$R_{ll} = 2R_{lml\bar{m}}, \quad (13)$$

$$R_{lm} = R_{lkml}, \quad (14)$$

$$R_{l\bar{m}} = R_{lk\bar{m}}, \quad (15)$$

and for the Ricci scalar:

$$R = -2R_{lk} = -2R_{lklk}. \quad (16)$$

² See, in particular, [19] for an application of the Newman–Penrose formalism to determine the GW polarizations in massive gravity.

The six amplitudes $\{\Psi_2, \Psi_3, \Psi_4, \Phi_{22}\}$ of a wave are generally observer-dependent [12] (for more details see Appendix A). However, there are certain invariant statements about them that are true for all standard observers if they are true for any one. These statements characterize invariant $E(2)$ classes of waves. The name of each class is composed of the Petrov type of its nonvanishing Weyl tensor and the maximum number of nonvanishing amplitudes $\{\Psi_2, \Psi_3, \Psi_4, \Phi_{22}\}$ (the dimension of representation) as seen by any observer. Both the Petrov type and the dimension of representation are independent of observer. Considering standard observers such that: (a) each observer sees the wave travelling in the $+z$ direction, and (b) each observer measures the same frequency for a monochromatic wave, then the $E(2)$ classes in order of decreasing generality are:

- Class II_6 : $\Psi_2 \neq 0$. All standard observers measure the same non-zero amplitude in the Ψ_2 mode. But the presence or absence of all other modes is observer-dependent;
- Class III_5 : $\Psi_2 = 0, \Psi_3 \neq 0$. All standard observers measure the absence of Ψ_2 and the presence of Ψ_3 . But the presence or absence of Ψ_4 and Φ_{22} is observer-dependent;
- Class N_3 : $\Psi_2 = \Psi_3 = 0, \Psi_4 \neq 0, \Phi_{22} \neq 0$. Presence or absence of all modes is observer-independent;
- Class N_2 : $\Psi_2 = \Psi_3 = \Phi_{22} = 0; \Psi_4 \neq 0$. Observer-independent;
- Class O_1 : $\Psi_2 = \Psi_3 = \Psi_4 = 0; \Phi_{22} \neq 0$. Observer-independent;
- Class O_0 : $\Psi_2 = \Psi_3 = \Psi_4 = \Phi_{22} = 0$. Observer-independent. All standard observers measure no wave.

3. Polarization modes of gravitational waves in $f(R)$ theories

3.1. The metric formalism

Let the gravitational action be an arbitrary function of the Ricci scalar:

$$I = \int d^4x \sqrt{-g} f(R). \quad (17)$$

By varying this action with respect to the metric $g_{\mu\nu}$ we have the following vacuum field equations:

$$f' R_{\mu\nu} - \frac{1}{2} f g_{\mu\nu} - \nabla_\mu \nabla_\nu f' + g_{\mu\nu} \square f' = 0, \quad (18)$$

where the prime represents derivatives with respect to R .

In the sequence we restrict ourselves to the following class of the $f(R)$ theories:

$$f(R) = R - \alpha R^{-\beta}. \quad (19)$$

Substituting (19) in the field equations (18) we obtain the following relations between $R_{\mu\nu}$ and the Ricci scalar R :

$$R_{\mu\nu} = \frac{(R - \alpha R^{-\beta}) g_{\mu\nu} + 2\alpha\beta \nabla_\mu \nabla_\nu R^{-(1+\beta)} - 2\alpha\beta g_{\mu\nu} \square R^{-(1+\beta)}}{2[1 + \alpha\beta R^{-(1+\beta)}]}. \quad (20)$$

Contracting this expression we have a dynamical equation for R :

$$R = \alpha [(\beta + 2)R^{-\beta} + 3\beta \square R^{-(1+\beta)}]. \quad (21)$$

The classification procedure involves examining the far-field, linearized, vacuum field equations of a theory. In what follows we examine different cases for the possible values of α and β . In each case, we first find R from Eq. (21) and then we can compute $R_{\mu\nu}$ from (20).

• Case $\alpha = 0$

This is the trivial case, which reduces to the general relativity theory. From Eqs. (21) and (20) we find $R = 0$ and $R_{\mu\nu} = 0$. Consequently, from the relations of Section 2 we deduce that:

$$R_{lklk} = R_{lm\bar{l}\bar{m}} = R_{lk\bar{l}\bar{m}} = R_{l\bar{l}k\bar{m}} = 0, \quad (22)$$

and so we have:

$$\Psi_2 = \Psi_3 = \Phi_{22} = 0. \quad (23)$$

And since we have no further constrains:

$$\Psi_4 \neq 0. \quad (24)$$

And, as expected, the $E(2)$ classification for general relativity is N_2 .

• Case $\alpha \neq 0, \beta \geq 1$

For $\alpha \neq 0$, Eq. (21) can be written as:

$$\square R^{-(1+\beta)} + \frac{\beta + 2}{3\beta} R^{-\beta} - \frac{1}{3\alpha\beta} R = 0. \quad (25)$$

Working in the weak field regime, if $\beta \geq 1$ we have $R^{-\beta} \gg R$ and Eq. (25) now reads:

$$\square \phi + \frac{\beta + 2}{3\beta} \phi^{\frac{\beta}{1+\beta}} = 0, \quad (26)$$

where we have renamed $\phi \equiv R^{-(1+\beta)}$. But this equation is of the form:

$$\square \phi - \frac{\partial U}{\partial \phi} = 0. \quad (27)$$

Comparing (26) and (27) we have the potential:

$$U(\phi) = - \left[\frac{(\beta + 2)(\beta + 1)}{3\beta(2\beta + 1)} \right] \phi^{\frac{2\beta+1}{\beta+1}}. \quad (28)$$

Since the field ϕ is Lorentz-invariant, we can solve Eq. (27) by a very known method (see, e.g., [21]). First, consider the static solution of (27), i.e., the solution of the equation:

$$\frac{d^2 \phi}{dz^2} = \frac{\partial U}{\partial \phi}, \quad (29)$$

which can be written as:

$$\frac{1}{2} \left(\frac{d\phi}{dz} \right)^2 = U(\phi). \quad (30)$$

Assuming the potential (28) in (30) and integrating we have:

$$\phi(z) = [i\xi(z - z_0) + \phi_0^{1/2(\beta+1)}]^{2(\beta+1)}, \quad (31)$$

where

$$\xi = \frac{1}{2(\beta + 1)} \left[\frac{2(\beta + 2)(\beta + 1)}{3\beta(2\beta + 1)} \right]^{1/2} \quad (32)$$

and $\phi_0 = \phi(z_0)$ is the value of the field ϕ in some initial position z_0 . Now, because the system is Lorentz-invariant, given the static solution (31), one can Lorentz-transform it to obtain the time dependent solution:

$$\phi(z, t) = \left[i\xi \frac{(z - z_0) - vt}{\sqrt{1 - v^2}} + \phi_0^{1/2(\beta+1)} \right]^{2(\beta+1)}. \quad (33)$$

And, therefore, the Ricci scalar reads:

$$R(z, t) = \left[i\xi \frac{(z - z_0) - vt}{\sqrt{1 - v^2}} + R_0^{-1/2} \right]^{-2}, \quad (34)$$

where v is the wave propagation velocity. This is the solution of Eq. (26) as can be verified by direct substitution.

Substituting the solution (34) in (20) we find the non-zero components of the Ricci tensor to first order in R :

$$R_{tt} = \frac{1}{6\beta} \left[(1 - 2\beta) - \frac{2(\beta + 2)v^2}{1 - v^2} \right] R, \quad (35)$$

$$R_{tz} = \frac{(\beta + 2)v}{3\beta(1 - v^2)} R, \quad (36)$$

$$R_{zz} = \frac{1}{6\beta} \left[(2\beta - 1) - \frac{2(\beta + 2)}{1 - v^2} \right] R. \quad (37)$$

Therefore, from the relations of Section 2 we find:

$$R_{iklm} = R_{iklm} = 0, \quad R_{iklk} \neq 0, \quad R_{lmli} \neq 0, \quad (38)$$

and so:

$$\Psi_2 \neq 0, \quad \Psi_3 = 0, \quad \Psi_4 \neq 0, \quad \text{and} \quad \Phi_{22} \neq 0, \quad (39)$$

and the $E(2)$ classification for this case is II_6 .

• Case $\alpha \neq 0, \beta < -2$

Considering now $\beta < -2$ we have $R^{-\beta} \ll R$ and Eq. (25) reads:

$$\square\phi - \frac{1}{3\alpha\beta} \phi^{-1/(1+\beta)} = 0, \quad (40)$$

where ϕ has the same definition presented above. However, now the potential is given by:

$$U(\phi) = \frac{1}{3\alpha\beta} \phi^{\frac{\beta}{\beta+1}}, \quad (41)$$

from which we obtain the static solution:

$$\phi(z) = \left[\zeta(z - z_0) + \phi_0^{\frac{\beta+2}{2(\beta+1)}} \right]^{\frac{2(\beta+1)}{\beta+2}}, \quad (42)$$

where

$$\zeta = \frac{\beta + 2}{(\beta + 1)\sqrt{6\alpha\beta}}. \quad (43)$$

And after a Lorentz transformation we find the full solution:

$$\phi(z, t) = \left[\zeta \frac{(z - z_0) - vt}{\sqrt{1 - v^2}} + \phi_0^{\frac{\beta+2}{2(\beta+1)}} \right]^{\frac{2(\beta+1)}{\beta+2}}. \quad (44)$$

Thus, the evolution of the Ricci scalar for this case is:

$$R(z, t) = \left[\zeta \frac{(z - z_0) - vt}{\sqrt{1 - v^2}} + R_0^{-\frac{\beta+2}{2}} \right]^{-\frac{2}{\beta+2}}. \quad (45)$$

The components of the Ricci tensor reads:

$$R_{tt} = \frac{1}{3} \left[\frac{\beta}{(\beta + 1)} \frac{v^2}{(1 - v^2)} - \frac{1}{2} \right] R, \quad (46)$$

$$R_{tz} = -\frac{\beta}{3(\beta + 1)} \frac{v}{(1 - v^2)} R, \quad (47)$$

$$R_{zz} = \frac{1}{3} \left[\frac{\beta}{(\beta + 1)} \frac{1}{(1 - v^2)} + \frac{1}{2} \right] R, \quad (48)$$

while all the others components are null.

Therefore, together with the formalism presented in Section 2, we can deduce that:

$$\Psi_2 \neq 0, \quad \Psi_3 = 0, \quad \Psi_4 \neq 0, \quad \text{and} \quad \Phi_{22} \neq 0. \quad (49)$$

And we are lead to classify the theories with $\beta < -2$ in the $E(2)$ class II_6 .

• Case $\alpha \neq 0, \beta = -2$

This is a particular case, where the behavior of the Ricci scalar and Ricci tensor are oscillatory. That is, if we use $\beta = -2$ in Eq. (25), we obtain:

$$\square R - \frac{1}{6\alpha} R = 0, \quad (50)$$

with the solution:

$$R = R_0 \exp(ik_\alpha x^\alpha), \quad k_\alpha k^\alpha = \frac{1}{6\alpha}. \quad (51)$$

Considering this solution in Eq. (20) with $R \ll 1$ we find the non-null components of the Ricci tensor:

$$R_{tt} = \frac{1}{2} (4\alpha k^2 - 1) R, \quad (52)$$

$$R_{tz} = -2\alpha k \sqrt{k^2 - \frac{1}{6\alpha}} R, \quad (53)$$

$$R_{zz} = \frac{1}{6} (12\alpha k^2 + 1) R. \quad (54)$$

So, again we have:

$$\Psi_2 \neq 0, \quad \Psi_3 = 0, \quad \Psi_4 \neq 0, \quad \text{and} \quad \Phi_{22} \neq 0, \quad (55)$$

and the $E(2)$ classification is II_6 .

As can be seen, for all the studied cases (except the case $\alpha = 0$), the theory given by Eq. (19) is classified in the class II_6 , i.e., the most general classification where all the six polarization can appear for some specific Lorentz observer, but the amplitude Ψ_2 is observer-independent.

3.2. The Palatini approach

In the Palatini approach, the metric g and the (usually torsionless) connection Γ are considered as independent variables, entering the definition of the Ricci tensor. The vacuum field equations, derived from the Palatini variational principle applied in the action (17) are:

$$f' R_{(\mu\nu)} - \frac{1}{2} f g_{\mu\nu} = 0, \quad (56)$$

$$\nabla_\alpha^\Gamma (\sqrt{-g} f' g^{\mu\nu}) = 0, \quad (57)$$

where ∇_α^Γ is the covariant derivative with respect to Γ . We shall use the standard notation denoting by $R_{(\mu\nu)}$ as the symmetric part of $R_{\mu\nu}$, i.e. $R_{(\mu\nu)} \equiv \frac{1}{2}(R_{\mu\nu} + R_{\nu\mu})$. It was shown that the vacuum field equations (56) leads to 'universal' equations for a wide range of functions $f(R)$ [22]. These universal equations are just Einstein equations with cosmological constant Λ .

Thus, the properties of vacuum GWs in the $f(R)$ gravity using the Palatini approach reduces to the problem of GWs in the Einstein equations considering the cosmological term. In a recent work, Náf et al. [23] have analyzed this case. They expanded the perturbations in a de Sitter and an anti-de Sitter background. Since the Minkowski metric is not a solution of the vacuum field equations this approach seems to be the most straightforward. Considering terms up to linear order in Λ they calculated the non-null

components of the Riemann tensor and found that Λ does not introduce additional polarization states for the GWs. Moreover, they shown that the cosmological term introduces tiny modifications in the amplitude of the wave which are well below the detectability of the present GWs detectors.

Therefore, we can conclude that GWs in the $f(R)$ gravity using the Palatini approach has only the two usual polarizations of general relativity, i.e., the polarization $+$ and \times .

4. Polarization modes of gravitational waves in quadratic gravity

For completeness we analyze the polarizations for the following Lagrangian of quadratic gravity:

$$I = \int d^4x \sqrt{-g} [R + \alpha R^2 + \gamma R_{\mu\nu} R^{\mu\nu}]. \quad (58)$$

Variation of this action in respect to the metric gives the field equations:

$$G_{\mu\nu} + \alpha H_{\mu\nu} + \gamma I_{\mu\nu} = 0, \quad (59)$$

where

$$H_{\mu\nu} = 2\nabla_\mu \nabla_\nu R - 2g_{\mu\nu} \square R + \frac{1}{2} g_{\mu\nu} R^2 - 2RR_{\mu\nu} \quad (60)$$

and

$$I_{\mu\nu} = \nabla_\mu \nabla_\nu R - \frac{1}{2} g_{\mu\nu} \square R - \square R_{\mu\nu} - 2R_{\mu}{}^\alpha R_{\alpha\nu} + \frac{1}{2} g_{\mu\nu} R_{\alpha\beta} R^{\alpha\beta}. \quad (61)$$

The field equations (59) were analyzed in the linearized regime by de Rey Neto et al. [10] using a perturbative approach developed from the concept of regular reduction of a system of partial differential equations. Working only with the transverse-traceless part of the metric perturbations h_{ij}^{TT} , they found a frequency-dependent correction in the gravitational wave amplitude due to the presence of the Ricci-squared term in the gravitational action.

Here, we do not write explicitly the field equations in terms of the perturbations. Instead, we consider the Ricci tensor and the Ricci scalar as first order functions of the general metric perturbations $h_{\mu\nu}$. Then, we find solutions for the dynamical equations to linear order in $R_{\mu\nu}$ and R . These solutions enable us to find the non-null Newman–Penrose quantities and classify the quadratic gravity in analogy with was made for $f(R)$ theories in the preceding section.

Considering Eq. (59) to linear order in R and $R_{\mu\nu}$ we have the following equation for the Ricci tensor:

$$\square R_{\mu\nu} - \frac{1}{\gamma} R_{\mu\nu} = \frac{1}{\gamma} S_{\mu\nu}, \quad (62)$$

where

$$S_{\mu\nu} = (2\alpha + \gamma) \left[\partial_\mu \partial_\nu R - \frac{\eta_{\mu\nu} R}{4(3\alpha + \gamma)} \right]. \quad (63)$$

Taking the trace of (62) we find:

$$2(3\alpha + \gamma) \square R + R = 0. \quad (64)$$

From this equation, we can find that a particular combination of parameters, namely $\gamma = -3\alpha$, leads to $R = 0$. For this case the solution of Eq. (62) reads:

$$R_{\mu\nu} = A_{\mu\nu} \exp(iq_\alpha x^\alpha), \quad q_\alpha q^\alpha = -\frac{1}{\gamma}. \quad (65)$$

Since there are no further constrains we find:

$$R_{llk} = 0, \quad R_{lm\bar{m}} \neq 0, \quad R_{lk\bar{m}} \neq 0, \quad R_{lk\bar{m}} \neq 0. \quad (66)$$

Therefore:

$$\Psi_2 = 0, \quad \Psi_3 \neq 0, \quad \Psi_4 \neq 0, \quad \text{and} \quad \Phi_{22} \neq 0, \quad (67)$$

and the correspond $E(2)$ class is III₅.

For the case $\gamma \neq -3\alpha$, Eq. (64) gives:

$$R = R_0 \exp(ik_\alpha^1 x^\alpha), \quad k_\alpha^1 k_1^\alpha = \frac{1}{2(3\alpha + \gamma)}. \quad (68)$$

With this expression in (62), the full solution for the Ricci tensor reads:

$$R_{\mu\nu} = A_{\mu\nu} e^{i(k_1 z - \omega t)} + B_{\mu\nu} e^{i(k_2 z - \omega t)} + \text{c.c.}, \quad (69)$$

where

$$A^{\mu\nu} = \frac{2}{3} R_0 (3\alpha + \gamma) \left[k_1^\mu k_1^\nu + \frac{\eta^{\mu\nu}}{4(3\alpha + \gamma)} \right], \quad (70)$$

and

$$k_1 = \sqrt{\omega^2 + \frac{1}{2(3\alpha + \gamma)}}, \quad k_2 = \sqrt{\omega^2 - \frac{1}{\gamma}}. \quad (71)$$

Now, from (69) we write explicitly all the components of $R_{\mu\nu}$:

$$R_{tt} = \frac{1}{6} [4(3\alpha + \gamma)\omega^2 - 1] R_0 e^{i(k_1 z - \omega t)} + B_{tt} e^{i(k_2 z - \omega t)} + \text{c.c.}, \quad (72)$$

$$R_{tz} = -\frac{2}{3} (3\alpha + \gamma) \omega \sqrt{\omega^2 + \frac{1}{2(3\alpha + \gamma)}} R_0 e^{i(k_1 z - \omega t)} + B_{tz} e^{i(k_2 z - \omega t)} + \text{c.c.}, \quad (73)$$

$$R_{zz} = \frac{1}{2} \left[1 + \frac{4}{3} (3\alpha + \gamma) \omega^2 \right] R_0 e^{i(k_1 z - \omega t)} + B_{zz} e^{i(k_2 z - \omega t)} + \text{c.c.}, \quad (74)$$

and all the other components satisfies:

$$R_{ij} = B_{ij} e^{i(k_2 z - \omega t)} + \text{c.c.}, \quad (75)$$

where $i, j = x, y, z$.

Therefore, since there are no further constrains on $B_{\mu\nu}$, all the components of the Riemann tensor in the tetrad basis are non-null:

$$R_{llk} \neq 0, \quad R_{lm\bar{m}} \neq 0, \quad R_{lk\bar{m}} \neq 0, \quad R_{lk\bar{m}} \neq 0, \quad (76)$$

and so, all the Newman–Penrose quantities are also non-null:

$$\Psi_2 \neq 0, \quad \Psi_3 \neq 0, \quad \Psi_4 \neq 0, \quad \text{and} \quad \Phi_{22} \neq 0. \quad (77)$$

Thus, the $E(2)$ classification for the quadratic gravity in the most general case is II₆.

5. Conclusions

Recently, working in the metric formalism, Capozziello et al. [11] showed that GWs in $f(R)$ gravity can have a massive-like scalar mode and a longitudinal force besides the two polarizations which appear in general relativity. This agrees with our result that the quantities Φ_{22} and Ψ_2 are non-null for the particular class $f(R) = R + \alpha R^{-\beta}$. The Φ_{22} and Ψ_2 amplitudes correspond, respectively, to a perpendicular scalar mode (breathing mode) and to a longitudinal scalar mode. However, it is worth emphasizing that since the Ψ_2 mode are non-null, the $E(2)$ classification is II_6 (see Section 2). So, for this class, it is always possible to find a Lorentz observer who measures all the six polarization states. On the other hand, GWs in the Palatini approach have only the two usual polarizations states such as general relativity.

Furthermore, the method we use is not only simple but very robust and we are able to obtain some important information regarding the theories considered here. The key observational GW amplitude is Ψ_2 . If the amplitude Ψ_2 would be detected, the $f(R)$ models in the metric formalism like the one considered here would be supported and so the quadratic gravity for $\gamma \neq -3\alpha$. In this case we need another method to distinguish the two theories, we could compare the wave form in both cases, for example. If the Ψ_3 mode would be detected, but not the amplitude Ψ_2 , only the quadratic gravity for $\gamma = -3\alpha$ would be supported.

Therefore, if we would be able to detect GWs, an important way to identify the theory of gravity could be established [24]. In the particular case of $f(R)$ gravity in the Palatini approach, we showed that the polarizations of GWs are the same of general relativity. However, it is worth stressing that other information contained in the GW signals, like waveform and phase of the signal, could be important to permit, together with the polarizations, a clear identification of the theory.

Acknowledgements

The authors thank Prof. Fábio C. Carvalho and Antônio C.A. Faria for helpful discussions and for clarifying some important points. M.E.S.A. would like to thank the Brazilian Agency FAPESP for support (grant 06/03158-0). O.D.M. and J.C.N.A. would like to thank the Brazilian Agency CNPq for partial support (grants 305456/2006-7 and 307424/2007-3, respectively).

Appendix A. Behavior of the NP amplitudes under Lorentz transformations

We can understand the $E(2)$ classification scheme by analyzing the behavior of the Newman–Penrose (NP) amplitudes $\{\Psi_2, \Psi_3, \Psi_4, \Phi_{22}\}$ under a Lorentz transformation of the complex tetrad basis. This point is well explained in Ref. [12]. Here we summarize the main idea.

Consider two standard observers O and O' with tetrads $(\mathbf{k}, \mathbf{l}, \mathbf{m}, \mathbf{n})$ and $(\mathbf{k}', \mathbf{l}', \mathbf{m}', \mathbf{n}')$ respectively. If we choose the \mathbf{k} as proportional to the wave vector (following the convention of Eardley et al.), so we have $\mathbf{k} = \mathbf{k}'$. The most general proper Lorentz transformation relating the tetrads that keep \mathbf{k} fixed is:

$$\mathbf{k}' = \mathbf{k}, \quad (78)$$

$$\mathbf{m}' = e^{i\varphi}(\mathbf{m} + \sigma \mathbf{k}), \quad (79)$$

$$\bar{\mathbf{m}}' = e^{-i\varphi}(\bar{\mathbf{m}} + \bar{\sigma} \mathbf{k}), \quad (80)$$

$$\mathbf{l}' = \mathbf{l} + \bar{\sigma} \mathbf{m} + \sigma \bar{\mathbf{m}} + \sigma \bar{\sigma} \mathbf{k}, \quad (81)$$

where σ is an arbitrary complex number which produces null rotations (particular combinations of boosts and rotations), while φ , which runs from 0 to 2π , is an arbitrary real phase that produces a rotation about \mathbf{e}_z .

The transformations induced on the amplitudes of a wave by (φ, α) is:

$$\Psi_2' = \Psi_2, \quad (82)$$

$$\Psi_3' = e^{-i\varphi}(\Psi_3 + 3\bar{\sigma}\Psi_2), \quad (83)$$

$$\Psi_4' = e^{-2i\varphi}(\Psi_4 + 4\bar{\sigma}\Psi_3 + 6\bar{\sigma}^2\Psi_2), \quad (84)$$

$$\Phi_{22}' = \Phi_{22} + 2\sigma\Psi_3 + 2\bar{\sigma}\bar{\Psi}_3 + 6\sigma\bar{\sigma}\Psi_2. \quad (85)$$

Now, it is evident from this set of equations that the amplitudes $\{\Psi_2, \Psi_3, \Psi_4, \Phi_{22}\}$ cannot be specified in an observer-independent manner. For example, suppose that the observer in O measure a wave having the only nonvanishing amplitude Ψ_2 ($s = 0$). The observer in O' , in relative motion with respect to O , will conclude that the wave has the nonvanishing amplitudes Ψ_2, Ψ_3, Ψ_4 and Φ_{22} ($s = 0, 1, 2, 0$). However, there is a set of invariant statements which define the $E(2)$ classification scheme. Thus, we classify waves in an $E(2)$ invariant manner by uncovering all representations of $E(2)$ embodied in Eqs. (82)–(85). Each such representation, in which some of the NP amplitudes vanish identically, is a distinct invariant class. The description of each class can be found in the main text.

References

- [1] H. Weyl, *Space–Time–Matter*, 4th edn., Dover, New York, 1922.
- [2] B.S. De Witt, *The Dynamical Theory of Groups and Fields*, Gordon and Breach, New York, 1965.
- [3] B.L. Hu, E. Verdaguer, *Living Rev. Relativ.* 7 (2004) 3.
- [4] N.D. Birrel, P.C.W. Davies, *Quantum Fields in Curved Space*, Cambridge University Press, Cambridge, 1982.
- [5] K.S. Stelle, *Phys. Rev. D* 16 (1977) 953.
- [6] B. Zwiebach, *Phys. Lett. B* 156 (1985) 315.
- [7] A. Tseytlin, *Phys. Lett. B* 176 (1986) 92.
- [8] A.A. Starobinsky, *Phys. Lett. B* 91 (1980) 99.
- [9] S. Capozziello, M. Francaviglia, *Gen. Relativ. Gravit.* 40 (2008) 357.
- [10] E.C. de Rey Neto, O.D. Aguiar, J.C.N. de Araujo, *Class. Quantum Grav.* 20 (2003) 2025.
- [11] C. Corda, *JCAP* 0704 (2007) 009;
C. Corda, *Int. J. Mod. Phys. A* 23 (10) (2008) 1521;
S. Capozziello, C. Corda, M.F. De Laurentis, *Phys. Lett. B* 669 (2008) 255.
- [12] D.M. Eardley, D.L. Lee, A.P. Lightman, *Phys. Rev. D* 8 (1973) 3308.
- [13] E. Newman, R. Penrose, *J. Math. Phys.* 3 (1962) 566;
E. Newman, R. Penrose, *J. Math. Phys.* 4 (1962) 998, Erratum.
- [14] F. Acernese, et al., *Virgo Collaboration*, *Class. Quantum Grav.* 24 (19) (2007) S381.
- [15] D. Sigg, for LIGO Collaboration, http://www.ligo.org/pdf_public/P050036.pdf.
- [16] B. Abbott, et al., LIGO Collaboration, *Phys. Rev. D* 72 (2005) 042002.
- [17] M. Ando, TAMA Collaboration, *Class. Quantum Grav.* 21 (5) (2002) 1615.
- [18] O.D. Aguiar, et al., *Class. Quantum Grav.* 25 (11) (2008) 114042.
- [19] W.L.S. de Paula, O.D. Miranda, R.M. Marinho, *Class. Quantum Grav.* 21 (2004) 4595.
- [20] C.M. Will, *Living Reviews in Relativity* (2006), <http://relativity.livingreviews.org/Articles/lrr-2006-3>.
- [21] R. Rajaraman, *Solitons and Instantons: An Introduction to Solitons and Instantons in Quantum Field Theory*, Elsevier Science Publishers, Amsterdam, The Netherlands, 1982.
- [22] M. Ferraris, M. Francaviglia, I. Volovich, *Class. Quantum Grav.* 11 (1994) 1505.
- [23] J. Näf, P. Jetzer, M. Sereno, *Phys. Rev. D* 79 (2009) 024014.
- [24] C. Corda, arXiv:0905.2502.