On a problem related to a non-squeezing theorem

Alessandro Fedeli a,∗, Attilio Le Donne b

a Dipartimento di Matematica, Università dell’Aquila, 67100 l’Aquila, Italy
b Dipartimento di Matematica, Università di Roma “La Sapienza”, 00100 Roma, Italy

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Abstract

In this paper we give a solution to a problem of Kulpa about the interior of the image of certain continuous maps \( f: X \to \mathbb{R}^n \) where \( X \) is a compact subset of \( \mathbb{R}^n \) with non-empty interior. Moreover we show that the image of every continuous map \( f: X \to \mathbb{R}^2 \) where \( X \) is a non-empty compact subset of \( \mathbb{R}^2 \) and \( \text{diam} f^{-1}(f(x)) < \sqrt{3}a_X \) for every \( x \in \text{Fr} X \), has non-empty interior.

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Let \( d \) be the euclidean metric on \( \mathbb{R}^n \). For each (non-empty) \( A \subset \mathbb{R}^n \), \( x \in \mathbb{R}^n \) and \( \varepsilon > 0 \) let us set \( d(x, A) = \inf \{d(x, a): a \in A\} \), \( \text{diam} A = \sup \{d(x, y): x, y \in A\} \), \( D(x, \varepsilon) = \{p \in \mathbb{R}^n: d(x, p) < \varepsilon\} \) and \( \overline{D}(x, \varepsilon) = \{p \in X: d(p, x) \leq \varepsilon\} \). The boundary of \( A \) will be denoted by \( \text{Fr} A \).

For each non-empty compact set \( X \subset \mathbb{R}^n \) let \( a_X = \sup \{d(x, \text{Fr} X): x \in X\} \).

The reader is referred to [3,4] for notations and terminology not explicitly given. In this note we will give a complete solution to the following problem of Kulpa.

**Problem** [6]. Let \( f: X \to \mathbb{R}^n \) be a continuous map from a compact subset \( X \subset \mathbb{R}^n \) with non-empty interior, such that \( \text{diam} f^{-1}(f(x)) < 2a_X \).

Is \( \text{Int} f(X) \neq \emptyset \)?
The problem above is related to various results of Kulpa [6,7], e.g., the following non-squeezing theorem (which is a strengthening of a classical result of Brouwer [1]):

Let $a > 0$ and let $I^n = [-a,a]^n$; if $f : I^n \to R^n$ is a continuous map such that
\[
\text{diam } f^{-1}(f(x)) < 2a
\]
for each $x \in \text{Fr } I^n$, then $\text{Int } f(I^n) \neq \emptyset$.

A mapping $f : X \to Y$ between topological spaces is called (linearly) Darboux if $f(C)$ is connected whenever $C$ is (pathwise) connected. Clearly every Darboux function (see, e.g., [2,5]) is linearly Darboux.

Now let us show that the problem above has a positive answer for $n = 1$ and a negative one for $n > 1$.

**Theorem 1.**

(i) Let $f : X \to R$ be a linearly Darboux function from a non-empty compact subset $X \subset R^n$ such that
\[
\text{diam } f^{-1}(f(x)) < 2a_X \text{ for every } x \in \text{Fr } X.
\]
Then $\text{Int } f(X) \neq \emptyset$.

(ii) For every $n > 1$ there is a continuous map $f : X \to R^n$ from a compact subset $X \subset R^n$ with non-empty interior such that $\text{Int } f(X) = \emptyset$ and $\text{diam } f^{-1}(f(x)) < 2a_X$ for every $x \in X$.

**Proof.** (i) Let $U = \text{Int } X$. Clearly $U \neq \emptyset$. Since the mapping $\psi : X \to R$ given by $\psi(x) = d(x, \text{Fr } X)$ for every $x \in X$ is continuous and $X$ is compact, we can take some $p \in X$ such that $\psi(p) = a_X$. Clearly such a $p$ belongs to $U$.

We claim that $D(p, a_X) \subset U$ Let $q \in D(p, a_X)$ and let $I_{pq}$ be the closed line segment joining $p$ and $q$. Since $a_X = d(p, \text{Fr } X)$, it follows that $I_{pq} \cap \text{Fr } X = \emptyset$.

So $I_{pq} = (I_{pq} \cap U) \cup (I_{pq} \cap (R^n \setminus X))$. Since $I_{pq}$ is connected, it follows that $I_{pq} \subset X$. Therefore $q \in I_{pq} \subset U$.

Now let $\phi : X \to R$ be the mapping given by $\phi(x) = d(p, x)$. By the continuity of $\phi$ and the compactness of Fr $X$, it follows that there is some $y \in \text{Fr } X$ such that $\phi(y) = \min \phi(\text{Fr } X)$, i.e., $d(p, y) = a_X$.

Let $g : \overline{D}(p, a_X) \to R$ be the restriction of $f$ to $\overline{D}(p, a_X)$. Clearly $g$ is a linearly Darboux function. Moreover $g$ is not constant, otherwise $\overline{D}(p, a_X) = g^{-1}(g(y)) \subset f^{-1}(f(y))$, so $2a_X = \text{diam } \overline{D}(p, a_X) \leq \text{diam } f^{-1}(f(y))$, a contradiction (recall that $y \in \text{Fr } X$).

So $g(\overline{D}(p, a_X))$ is a non-degenerate interval contained in $f(X)$.

Therefore $\text{Int } f(X) \neq \emptyset$.

(ii) Let $0 = (0, 0)$ and $X = \{p \in R^2 : d(p, 0) \leq 1\}$. For each $k \in \{0, \ldots, 5\}$ set $A_k = (\cos \frac{2k\pi}{6}, \sin \frac{2k\pi}{6})$, set also $A_6 = A_0$.

For every $k \in \{0, \ldots, 5\}$ let $D_k$ be the arc $A_kA_{k+1}$ and the two radii $0A_k$ and $0A_{k+1}$.

For every $k \in \{0, 2, 4\}$ let $g_k$ be the projection of $D_k$ onto the segment $0A_k$ along $0A_{k+1}$.

For every $k \in \{1, 3, 5\}$ let $g_k$ be the projection of $D_k$ onto the segment $0A_{k+1}$ along $0A_k$.

Clearly every mapping $g_k : D_k \to R^2$ is continuous.
Since \( \{D_0, \ldots, D_5\} \) is a finite closed cover of \( X \) and \( \{g_0, \ldots, g_5\} \) is a family of compatible mappings, it follows that the combination \( f = \bigvee g : X \to \mathbb{R}^2 \) is continuous.

Since \( f(X) = \overline{0A_0} \cup \overline{0A_3} \cup \overline{0A_4} \), it follows that \( \text{Int} f(X) = \emptyset \).

Since \( a_X = 1 \), it remains to show that \( \text{diam} f^{-1}(f(x)) < 2 \) for every \( x \in X \). Let \( P \in f(X) \). If \( P = \emptyset \), then \( f^{-1}(P) = \overline{0A_1} \cup \overline{0A_3} \cup \overline{0A_5} \) and \( \text{diam} f^{-1}(P) = d(A_1, A_3) = \sqrt{3} \).

If \( P \in \overline{0A_0} \setminus \{0\} \), then \( f^{-1}(P) = PQ \cup PQ' \) for some \( Q \in D_0 \) and \( Q' \in D_5 \) such that \( PQ \parallel \overline{0A_1} \) and \( PQ' \parallel \overline{0A_5} \).

So \( \text{diam} f^{-1}(P) = d(Q, Q') < d(A_1, A_5) = \sqrt{3} \) for every \( P \in (\overline{0A_2} \cup \overline{0A_4}) \setminus \{0\} \).

With suitable modifications one can show, in the same vein, that the answer is negative for every \( n \geq 2 \). □

The space \( X \) constructed in Theorem 1 is such that \( \text{diam} f^{-1}(f(x)) \leq \sqrt{3}a_X \) for every \( x \in \text{Fr} X \). The next theorem shows that this is, in the plane, the best possible bound for a counterexample.

**Theorem 2.** Let \( X \) be a non-empty compact subspace of \( \mathbb{R}^2 \) and \( f : X \to \mathbb{R}^2 \) a continuous map such that \( \text{diam} f^{-1}(f(x)) < \sqrt{3}a_X \) for every \( x \in \text{Fr} X \). Then \( \text{Int} f(X) \neq \emptyset \).

**Proof.** Clearly \( a_X > 0 \), so \( \text{Int} X \neq \emptyset \). Moreover we may assume that \( a_X = 1 \), \( B((0, 0), 1) \subset X \) and \( X \) is connected.

Let us suppose that \( \text{Int} f(X) = \emptyset \).

**Claim.** We may assume that \( X \) is the union of finitely many closed rectangles whose edges are parallel to the axis and do not contain the origin.

**Proof.** Let \( D_n = \{x \in X : d(x, \text{Fr} X) \leq \frac{1}{n}\} \) for every \( n \in \mathbb{N} \). Let us show that there exists some \( m \) such that \( \text{diam} f^{-1}(f(x)) < \sqrt{3} \) for every \( x \in D_m \). If not, there exist \( x_n, y_n \in D_n \) such that \( f(x_n) = f(y_n) \), \( d(x_n, y_n) \geq \sqrt{3} \) for every \( n \in \mathbb{N} \). So, by eventually considering a subsequence, we may assume that \( x_n \to x, y_n \to y \) for some \( x, y \in X \). Therefore \( d(x, y) \geq \sqrt{3} \) and \( x, y \in \text{Fr} X \).

So, by compactness of \( D_m \), \( b = \frac{1}{\sqrt{3}} \sup \{\text{diam} f^{-1}(f(x)) : x \in D_m\} < 1 \). Since \( \frac{b^{1+1}}{2} < 1 \), we may take some \( \tilde{n} > m \) such that \( B((0, 0), \frac{1}{\sqrt{3}}) \subset X \setminus D_{\tilde{n}} \). Since \( X \setminus D_{\tilde{n}} \subset \text{Int} X \), we may cover it by the union \( V \) of finitely many open squares contained in \( X \), whose edges are parallel to the axis and do not contain the origin. So \( X \) may be replaced with \( Y = V \), in fact \( Y \) is the union of finitely many closed rectangles, with edges parallel to the axis, and \( \text{diam} f^{-1}(f(x)) < \sqrt{3}b < \sqrt{3}a_Y \) for every \( x \in \text{Fr} Y \) (observe that \( \text{Fr} Y \subset D_{\tilde{n}} \) and \( b < \frac{b^{1+1}}{2} \leq a_Y \)).

So, by the claim, there is a grid \( \mathcal{G} \) (i.e., a union of finitely many lines in the plane, each parallel to the \( x \)-axis or the \( y \)-axis, see, e.g., [4, p. 85]) such that \( X \) is the union of some minimal compact rectangles \( R_1, \ldots, R_m \) for the grid \( \mathcal{G} \) and no line of the grid contains the origin. □
A grid $\mathcal{H}$ is said to be a refinement of a grid $\mathcal{G}$ if $\mathcal{G} \subset \mathcal{H}$. A node of a grid $\mathcal{H}$ is the intersection of two non-parallel lines of $\mathcal{H}$. $\mathcal{N}(\mathcal{G})$ will denote the set of all nodes of $\mathcal{G}$.

Moreover, for every non-empty set $A$, let us denote by $F(A)$ the free Abelian group on $A$.

So $FrX$ can be seen as the closed rectangular 1-chain $\partial X = \partial R_1 + \partial R_2 + \cdots + \partial R_m$ for $\mathcal{G}$ (for $R = [a, b] \times [c, d]$ with $a \leq b$ and $c \leq d$, the boundary $\partial R$ is the 1-chain $\gamma_1 + \gamma_2 - \gamma_3 - \gamma_4$ where $\gamma_1$ is the straight path from $(a, c)$ to $(b, c)$, $\gamma_2$ from $(b, c)$ to $(b, d)$, $\gamma_3$ from $(a, d)$ to $(b, d)$, $\gamma_4$ from $(a, c)$ to $(a, d)$), so $\partial X = f \circ \gamma_1 + f \circ \gamma_2 - f \circ \gamma_3 - f \circ \gamma_4$, see, e.g., [4, Chapter 6]). Since each $R_i$ is contained in $X$, it follows that $\partial X$ is a 1-boundary in $X$ (see, e.g., [4, p. 82]). Moreover, since $f : X \to f(X)$ is continuous, it follows also that $f_* \partial X = f_* \partial R_1 + f_* \partial R_2 + \cdots + f_* \partial R_m$ is a 1-boundary in $f(X)$ (see, [4, Chapter 6]).

Now let us take three half-lines $l_1$, $l_2$, $l_3$ starting from the origin, not parallel to the axis, and dividing the plane in three equal angles $A_1$, $A_2$, $A_3$, with $A_i \cap A_{i+1} = l_{i+1}$ for $i = 1, 2, 3$, $A_4 = A_1$ and $l_4 = l_1$.

Now set $S_i = A_i \cap FrX$ and $Y_i = f(S_i)$ for $i = 1, 2, 3$. Note that each $Y_i$ is compact. Since $\partial X = f^{-1} f(x) < \sqrt{3}$ for every $x \in FrX$ and $B((0, 0), 1) \subset X$, it follows that $Y_1 \cap Y_2 \cap Y_3 = \emptyset$.

Let us call $M(\mathcal{H})$ the family of all minimal compact rectangles for a grid $\mathcal{H}$.

Since $Y_1 \cap \text{Int} (K \cup S)$ is disjoint from $Y_2 \setminus \text{Int} (K \cup S)$, we can, up to a refinement of the grid $\mathcal{G}$, assume that the sets $T = \bigcup \{C \in M(\mathcal{G}^{'}) : C \cap (Y_1 \cap Y_2) \neq \emptyset\}$ and $U = \bigcup \{C \in M(\mathcal{G}^{'}) : C \cap (Y_2 \setminus (K \cup S)) \neq \emptyset\}$ are disjoint.

So we have $f(FrX) \subset \text{Int} (K \cup S \cup T \cup U)$ and $K \cap S = T \cup U = \emptyset$. Moreover observe that $\text{Int} K \cap \text{Int} T = \text{Int} T \cap \text{Int} S = \text{Int} U \cap \text{Int} K = \text{Int} U \cap \text{Int} S = \emptyset$.

Set $\mathcal{M}(\mathcal{G}^{'}) = \{Q_1, \ldots, Q_\kappa\}$ and put $M = \bigcup \{\text{Fr} Q_i : 1 \leq i \leq \kappa\}$.

Since we are assuming that $f(X)$ has empty interior we may pick some $q_i \in \text{Int} (Q_i)$ \ $f(X)$ for every $i$. Let $g : \bigcup \{Q_i \setminus \{q_i\} : 1 \leq i \leq \kappa\} \to M$ be the continuous map such that $g_{|Q_i \setminus \{q_i\}}$ is the radial projection from $q_i$ to $\text{Fr} Q_i$. Note that $g(K) \subset K$, $g(S) \subset S$, $g(T) \subset T$ and $g(U) \subset U$.

Set $h = g \circ f$. Then $h_* \partial X = g_* f_* \partial X$ is a 1-boundary (in $M$). Since we assumed that the half-lines $l_1$, $l_2$, $l_3$ are not parallel to the axis, it follows that $L = FrX \cap (l_1 \cup l_2 \cup l_3)$ is finite. So we may assume, up to a refinement of the grid $\mathcal{G}^{'},$ that the points of $L$ go under $f$ to some nodes of $G^{'},$ hence $g$ is the identity on $L$.

Let $S$ be the family of all straight paths $\sigma_{ab}$ from $a$ to $b$ (i.e., $\sigma_{ab}(t) = a + t(b - a)$), where $a, b$ is a pair of distinct adjacent nodes of the grid $G'$. $\partial \sigma_{ab} = b - a$ is an element of $F(N(\mathcal{G}^{'})$).

For every $\mu = \sum \{n_{\sigma} \sigma : \sigma \in S\} \in F(S)$, $n_{\sigma} \in Z$, let us put $\text{Im} \mu = \bigcup \{\text{Im} \sigma : n_{\sigma} \neq 0\}$, $\partial \mu = \sum \{n_{\sigma} \partial \sigma : \sigma \in S \}$ $\in F(N(\mathcal{G}^{'})$).

Now observe that $\partial X = \sum_{i=1}^{n} \gamma_i$, where $\gamma_i$ is either a path from $a_i$ to $b_i$ and $\text{Im}(\gamma_i) \cap L = \{a_i, b_i\}$ or it is a closed path whose image is disjoint from $L$.

Now for each $i \in \{1, \ldots, n\}$ let us fix a member $\mu_i$ of $F(S)$ homologous to $h_\gamma \gamma_i$ on $M$, so that $\text{Im} \mu_i \subset \text{Im} h_\gamma \gamma_i$ and $\partial \mu_i = \partial (h_\gamma \gamma_i) = \partial (f \circ \gamma_i) = f_{\mu_i} (\gamma_i) \equiv f \gamma_i = f(b_i) - f(a_i) \in F(N(\mathcal{G}^{'})$ (where the last equality holds whenever $\gamma_i = b_i - a_i$). Note that if $\gamma_i$ is a closed path then $\partial \mu_i = 0$. 

...
Observe that a member \( \mu = \sum \{ n_{\sigma} \sigma : \sigma \in \mathcal{S} \} \) is a 1-boundary in \( M \) (i.e., homologous to zero on \( M \)) if and only if \( n_{\sigma} = n_{\tau} \) whenever \( \partial \sigma = - \partial \tau \) (i.e., \( \sigma \) is the opposite straight path of \( \tau \)).

Moreover if \( \mu = \sum \{ n_{\sigma} \sigma : \sigma \in \mathcal{S} \} \) is a 1-boundary in \( M \), then \( \mu | B = \sum \{ n_{\sigma} \sigma : \text{Im} \sigma \subset B \} \) is a 1-boundary in \( M \) for every \( B \subset R^2 \).

Now let \( \sharp : F(\mathcal{N}(G')) \rightarrow Z \) be the group homomorphism given, for each \( x \in \mathcal{N}(G') \), by \( \sharp(x) = 1 \) whenever \( x \in K \), \( \sharp(x) = 0 \) otherwise.

Let us set \( \sharp \mu = \sharp \partial \mu \) for each \( \mu \in F(S) \) and \( \sharp \gamma_i = \sharp \mu_i \) for every \( i \in \{1, \ldots, n\} \). Observe that, if \( \partial \gamma_i = b_i - a_i \), then \( \sharp \gamma_i \) is equal to 1 whenever \( a_i \notin l_i \) and \( b_i \notin l_i \), it is equal to \(-1\) whenever \( a_i \subset l_i \) and \( b_i \notin l_i \), and it is equal to 0 otherwise. Observe that \( \sharp \gamma_i = \sharp f(b_i) - \sharp f(a_i) \) if \( \partial \gamma_i = b_i - a_i \), in fact \( f(x) \in K \) if and only if \( x \in l_i \) for every \( x \in l_i \).

Now let us show that \( \sum_{i=1}^n \mu_i | T \) is not a 1-boundary in \( M \), thus reaching a contradiction, in fact \( \sum_{i=1}^n \mu_i = \sum_{i=1}^n \gamma_i \) is homologous to \( \sum_{i=1}^n \text{Im} \gamma_i = h^* \partial X \) which is a 1-boundary.

Observe that if \( \mu \) is a 1-boundary on \( M \), then \( \partial \mu \) is the identity element of \( F(\mathcal{N}(G')) \), so \( \sharp \mu = 0 \).

We will show that \( \sharp(\sum_{i=1}^n \mu_i | T) \neq 0 \).

**Claim.** \( \sharp(\mu_i | T) \) is equal to \( \sharp \mu_i \) whenever \( \text{Im} \gamma_i \subset A_1 \), it is 0 otherwise.

**Proof.** First observe that \( \exists \gamma = 0 \) whenever \( \text{Im} \sigma \subset K \) or \( \text{Im} \sigma \cap K = \emptyset \). Hence \( \exists \gamma = 0 \). Therefore \( \exists(\mu_i | T) \cap K = \emptyset \), so \( \sharp(\mu_i | T) = 0 \).

If \( \text{Im} \gamma_i \subset A_2 \), then \( \mu_i = \sum \{ n_{\sigma} \sigma : \text{Im} \sigma \subset K \cap U \} + \sum \{ n_{\sigma} \sigma : \text{Im} \sigma \subset S \setminus U \} \).

So \( \mu_i | T = \sum \{ n_{\sigma} \sigma : \text{Im} \sigma \subset (K \setminus U) \cap T \} + \sum \{ n_{\sigma} \sigma : \text{Im} \sigma \subset (S \setminus U) \cap T \} \).

Hence \( \sharp(\mu_i | T) = 0 \).

If \( \text{Im} \gamma_i \subset A_1 \), then \( \mu_i = \sum \{ n_{\sigma} \sigma : \text{Im} \sigma \subset T \} + \sum \{ n_{\sigma} \sigma : \text{Im} \sigma \subset K \setminus T \} + \sum \{ n_{\sigma} \sigma : \text{Im} \sigma \subset S \setminus T \} \).

So \( \sharp \mu_i = \sharp(\mu_i | T) \). \( \square \)

Therefore we have \( \sharp(\sum_{i=1}^n \mu_i | T) = \sum_{i=1}^n \sharp \mu_i | T = \sum \{ \sharp \mu_i : \text{Im} \gamma_i \subset A_1 \} = \sum \{ \sharp \gamma_i : \text{Im} \gamma_i \subset A_1 \} = \sharp(\sum_{i=1}^n \gamma_i | A_1) \).

Now if \( R \) is a rectangle in \( R^2 \), we can write the 1-chain \( \partial R \) as a finite sum of paths \( \eta_1, \ldots, \eta_r \) whose images are contained in one of \( A_1, A_2 \) and \( A_3 \).

Then we set \( \partial R|_{A_i} = \sum \{ \eta_i : \text{Im} \eta_i \subset A_i \} \) (this definition is independent from the choice of the paths \( \eta_1, \ldots, \eta_r \)).

Now \( \sharp(\sum_{i=1}^n \gamma_i | A_1) = \sharp(\sum_{i=1}^n \gamma_i | A_1) = \sharp R|_{A_1} = \sum_{i=1}^m \partial R|_{A_1} = \sum_{i=1}^m \partial R|_{A_1} = \partial R_{|A_1} = 1 \), where \( k \) is the only index for which \( (0,0) \in R_k \).

This completes the proof. \( \square \)

We conclude the paper with the following observation. Let \( r \) be the greatest real number such that for each compact subset \( X \) of \( R^2 \) and for each \( x \in \text{Fr} X \), \( \text{diam} f^{-1}(f(x)) < r \alpha x \)
implies \( \text{Int} \ f(X) \neq \emptyset \). As pointed out by the referee, it should be interesting to investigate \( r \) and its dependence on the dimension \( n \).

References