



## An isomorphism between the $p$ -adic integers and a ring associated with a tiling of $N$ -space by permutohedra

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(Received 21 June 1991)

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### Abstract

The classical lattice  $A_n^*$ , whose Voronoi cells tile Euclidean  $n$ -space by permutohedra, can be given the *generalized balance ternary* ring structure  $\mathbf{GBT}_n$  in a natural way as a quotient ring of  $\mathbb{Z}[x]$ . The ring  $\mathbf{GBT}_n$  can also be considered as the set of all finite sequences  $s_0 s_1 \dots s_k$ , with  $s_i \in \mathbf{GBT}_n / \alpha \mathbf{GBT}_n$  for all  $i$ , where  $\alpha$  is an appropriately chosen element in  $\mathbf{GBT}_n$ . The *extended generalized balance ternary* ( $\mathbf{EGBT}_n$ ) ring consists of all such infinite sequences. A primary goal of this paper is to prove that if  $2^{n+1} - 1$  and  $n + 1$  are relatively prime, then  $\mathbf{EGBT}_n$  is isomorphic as a ring to the  $(2^{n+1} - 1)$ -adic integers.

*Key words:*  $p$ -adic integers; Hexagonal tilings; Isomorphism

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### 1. Introduction

In computer vision the set of pixel locations in an image can be thought of as a finite subset of the plane. While this subset is typically chosen to be a rectangular grid, Gibson and Lucas [3–7] have been able to exploit both the geometric and the computer software advantages of a hexagonal grid in their applications to automatic target recognition. The geometric advantage of the hexagonal grid is that a hexagon provides an efficient and reasonably accurate approximation of a circle. The computer software advantage is that high throughput rates can be achieved by converting the location of a particular hexagon to a string of zeros and ones. The centers of the hexagonal cells form a lattice in the plane. A natural generalization of the hexagonal lattice in  $n$ -dimensional Euclidean space  $\mathbb{R}^n$  is the classical lattice  $A_n^*$  (see Section 2) consisting of the centers of the cells in a tiling of  $\mathbb{R}^n$  by permutohedra. One of the intentions of this paper is to describe the interplay between the geometry of tiling  $\mathbb{R}^n$

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by permutohedral cells and the algebraic structure of the corresponding cell addresses. The address of a particular permutohedron in a tiling is a certain finite sequence  $s_0 s_1 s_2 \dots s_k$ ; a precise definition of addressing is formulated in Section 3.

Since any two addresses associated with cells in  $A_n^*$  can be added and multiplied by rules that can be formulated in terms of “remainders” and “carries”, addressing is closely related to radix (or positional) representations of the integers. Positional number systems have a long history tracing back to the Babylonians, where sexagesimal (radix 60) positional notation was highly developed at least as early as 1750 B.C. [10, p. 163]. Leibniz seems to have been the inventor of radix 2 arithmetic [10, p. 167]. While there are a multitude of interesting number systems, Knuth [10, pp. 173–174] remarks that “Perhaps the prettiest number system of all is the *balanced ternary* notation, which is a base 3 representation using the “trits”  $-1, 0, +1$  instead of  $0, 1, \text{ and } 2$ . If we use the symbol  $\bar{1}$  to stand for  $-1$ , we have the following properties:

(a) The negative of a number is obtained by interchanging  $1$  and  $\bar{1}$ .

(b) The sign of a number is given by its most significant nonzero “trit”, and more generally we can compare any two numbers by reading them from left to right and using lexicographic order, as in the decimal system.

(c) The operation of rounding to the nearest integer is identical to “truncation”. The balanced ternary also has the advantage that every integer, not just the positive integers, can be represented. There are no known necessary and sufficient conditions as to which bases and which digits allow unique representation of all integers [13].

The *generalized balanced ternary* ( $\mathbf{GBT}_n$ ) provides one method for addressing the cells in a tiling of  $\mathbb{R}^n$  by permutohedra using finite sequences of integers between  $0$  and  $2^{n+1} - 2$ . In particular, when  $n = 1$  a string of “trits” is the address of a 1-dimensional cell (unit line segment) in the tiling of the line associated with the integer lattice  $\mathbb{Z} = A_1^*$ . If  $n = 2$  a sequence  $s_1 s_2 \dots s_k$ , where  $0 \leq s_i \leq 6$  for  $i, \dots, k$ , is the address of a hexagon in the hexagonal tiling of the plane associated with  $A_2^*$ . In general, the addresses of the  $n$ -dimensional cells inherit a natural ring structure so that any two sequences can be added or multiplied. The addition of addresses corresponds to vector addition of the corresponding lattice points in  $A_n^*$ . Kitto and Wilson [9] provide tables giving the exotic carries and remainder rules for addition and multiplication in two and three dimensions. In Section 3 of this paper we formulate a mathematical framework for this type of addressing; in Section 5 we show how the arithmetic operations, as well as accessing and retrieval, follow from simple and natural binary operations.

The arithmetic of  $\mathbf{GBT}_n$  can be easily extended to strings of infinite length, and the resulting algebraic structure is referred to as the *extended generalized balanced ternary* ( $\mathbf{EGBT}_n$ ). Let  $\mathbb{Z}_{(p)}$  denote the ring of  $p$ -adic integers. The  $p$ -adic integers can be regarded as the set of all series  $a_1 + a_2 p + \dots + a_n p^{n-1} + \dots$ , where  $0 \leq a_k < p$  for all  $k \geq 1$ . The addition and multiplication operations are performed on  $\mathbb{Z}_{(p)}$  with the usual “carries” rules for numbers written in the base  $p$ . A comprehensive discussion of the  $p$ -adic integers can be found in the treatments by Fuchs [2, p. 62] or Jacobson [8, p. 74]. A main result in this paper is Theorem 4.4. If  $q = 2^{n+1} - 1$  and  $n + 1$  are

relatively prime, then there is ring isomorphism  $\mathbf{EGBT}_n \cong \mathbb{Z}_{(q)}$ . This theorem was proved by Kitto and Wilson [9] for the special cases  $n = 2$  and  $n = 3$ . (In particular,  $\mathbf{EGBT}_2$  is isomorphic to the 7-adic integers and  $\mathbf{EGBT}_3$  is isomorphic to the 15-adic integers.) This result is surprising because such an isomorphism does not necessarily exist between the subrings consisting of the respective finite sequences.

The cells of the tiling of  $n$ -space can be grouped into “aggregates” according to the length of the corresponding addresses. These aggregates form a nested sequence of tessellations of  $n$ -space. Properties of these tilings by aggregates are contained in Section 6.

### 2. Tiling space by permutohedra

An  $n$ -dimensional lattice is the set of all integer linear combinations of  $n$  linearly independent vectors in an  $n$ -dimensional Euclidean space. In this paper all tilings will be lattice tilings, so the cells will be represented by the lattice of centers of the cells. By abuse of language, we will often make no distinction between a point of the lattice and the associated cell containing it.

Let  $A_n^*$  denote the dual of the classical  $n$ -dimensional root lattice  $A_n = \{(x_0, x_1, \dots, x_n) \in \mathbb{Z}^{n+1} : x_0 + \dots + x_n = 0\}$  [1, p. 115]. Alternatively,  $A_n^*$  is the lattice in  $\mathbb{R}^n$  generated by the set of vertices  $\{v_0, \dots, v_n\}$  of a regular  $n$ -simplex with barycenter at the origin. Each vector  $v_i$  can be chosen as a unit vector and the angle between each pair of such vectors is  $\cos^{-1}(1/n)$ . For a particular choice of coordinates for the  $v_i$ , see [11]. The lattice  $A_n^*$  is simply the set of integer points on the line; the points of  $A_2^*$  are the centers of the tiling of the plane by hexagons, and the lattice points of  $A_3^*$  are the barycenters of the tiling of 3-space by “truncated octahedra” [12, p. 110]. In general, the points of  $A_n^*$  are the centers of a tiling of  $\mathbb{R}^n$  by permutohedra. A *permutohedron* [1] is the  $n$ -dimensional polytope whose vertices (in  $\mathbb{R}^{n+1}$ ) consist of the  $(n+1)!$  points obtained by permuting the coordinates of  $(\frac{1}{2}(-n), \frac{1}{2}(-n+2), \frac{1}{2}(-n+4), \dots, \frac{1}{2}(n-2), \frac{1}{2}n)$ .

### 3. Addressing

Intuitively, an addressing system for a tiling of space is a labeling of each cell with a finite string  $s_0 s_1 s_2 \dots s_m$  of elements from a designated finite set  $S$ . This assignment is done formally as follows. Suppose that there is a lattice  $A$  associated with the tiling, and  $A$  has the structure of a ring  $R$ . Let  $\alpha$  be an arbitrary element of  $R$  and consider the inverse system

$$R/\alpha R \xleftarrow{f_1} R/\alpha^2 R \xleftarrow{f_2} \dots \xleftarrow{f_{k-1}} R/\alpha^k R \xleftarrow{f_k} \dots, \tag{1}$$

where the ring homomorphisms  $f_k$  are defined so that  $f_k(\bar{\beta})$  is equal to the equivalence class of  $\beta \pmod{\alpha^k}$ . The *inverse limit*  $R_{(\alpha)}$  of this system consists of all sequences

$\{\bar{\beta}_0, \bar{\beta}_1, \dots\}$  such that  $f_k(\bar{\beta}_k) = \bar{\beta}_{k-1}$ . The definition and notation are analogous to that of the  $p$ -adic integers  $\mathbb{Z}_{(p)}$ . Addition and multiplication in  $\mathbf{R}_{(\alpha)}$  are defined in the usual manner for inverse systems. If  $\{\bar{\beta}_0, \bar{\beta}_1, \dots\}$  is an element in  $\mathbf{R}_{(\alpha)}$  and  $S$  is a set of coset representatives for  $R/\alpha R$ , then it follows from the definition of the homomorphisms  $f_k$  that there exists a unique sequences  $s_0, s_1, s_2, \dots$  of elements of  $S$  such that

$$\begin{aligned}\beta_0 &\equiv s_0 \pmod{\alpha} \\ \beta_1 &\equiv s_0 + s_1\alpha \pmod{\alpha^2} \\ &\vdots \\ \beta_k &\equiv s_0 + s_1\alpha + \dots + s_k\alpha^k \pmod{\alpha^{k+1}} \\ &\vdots\end{aligned}$$

The sum  $s_0 + s_1\alpha + s_2\alpha^2 + \dots$  is called the *canonical* representation of the element  $\{\bar{\beta}_0, \bar{\beta}_1, \dots\} \in \mathbf{R}_{(\alpha)}$  and will be abbreviated  $s_0 s_1 s_2 \dots$  when  $\alpha$  is understood.

There is a natural map

$$\Phi: A \rightarrow \mathbf{R}_{(\alpha)}$$

given by  $\Phi(\mathbf{x}) = (\bar{x}, \bar{x}, \dots)$ . The canonical representation  $s_0 s_1 s_2 \dots$  of  $\Phi(\mathbf{x})$  will be called the *address* of the “cell”  $\mathbf{x}$ . The assignment  $\Phi: A \rightarrow \mathbf{R}_{(\alpha)}$  is called the *addressing system* and  $\Phi$  is called the *addressing map*. Note that the address depends on the choice of the set  $S$  of coset representatives. Two particularly elegant and useful choices of  $S$  are discussed in Section 5. Since  $\Phi$  is a group homomorphism, the addition of two addresses in  $\mathbf{R}_{(\alpha)}$  corresponds to vector addition of points in  $A \subset \mathbb{R}^n$ .

Let  $n$  be a positive integer and consider the special case where  $R$  is the quotient ring,

$$R =: \mathbf{GBT}_n = \mathbb{Z}[x]/(f),$$

with  $f(x) = 1 + x + \dots + x^n$ . Let  $\omega = \bar{x}$ , where the bar denotes the coset containing  $x$ . Note that  $\omega^{n+1} = 1$ , so the ring  $\mathbf{GBT}_n$  appears somewhat like adjoining an  $(n+1)$ st root of unity to  $\mathbb{Z}$ . As a free abelian group,  $\mathbf{GBT}_n$  has basis  $\{1, \omega, \dots, \omega^{n-1}\}$ . For example,  $\mathbf{GBT}_1 \cong \mathbb{Z}$ . Let

$$\alpha = \bar{2} - \omega \in \mathbf{GBT}_n.$$

For  $n=2$  the location of the element  $\alpha$  is displayed in Fig. 2 as the center of the appropriate cell. Let  $\{v_0, v_1, \dots, v_n\}$  be the set of generators of  $A_n^*$  given in Section 2. As abelian groups  $A_n^*$  and  $\mathbf{GBT}_n$  are isomorphic with the isomorphism induced by  $v_i \mapsto \omega^i$  and extended by linearity, in this case denote the inverse limit  $\mathbf{R}_{(\alpha)}$  by  $\mathbf{EGBT}_n$ . The addressing system

$$\Phi: A_n^* \rightarrow \mathbf{EGBT}_n$$

is the main topic of this paper.

The particular choice of the rings  $\mathbf{GBT}_n$  and the element  $\alpha$  may seem somewhat arbitrary. Indeed, much of the theory can be generalized to arbitrary monic  $f(x) \in \mathbb{Z}$

and arbitrary  $\alpha$  [13]. However, the facts that  $\omega^{n+1} = 1$  and that  $\omega \equiv 2 \pmod{\alpha}$  result in an extremely simple base 2 implementation of the addressing system and a correspondingly aesthetic geometric interpretation. This is discussed in Section 5 and 6.

#### 4. The structure of $\mathbf{EGBT}_n$

For an integer  $q$ , the  $q$ -adic integers  $\mathbb{Z}_{(q)}$  are defined as the inverse limit of the inverse system

$$\mathbb{Z}/q\mathbb{Z} \xleftarrow{g_1} \mathbb{Z}/q^2\mathbb{Z} \xleftarrow{g_2} \dots \xleftarrow{g_{k-1}} \mathbb{Z}/q^k\mathbb{Z} \xleftarrow{g_k} \dots,$$

where the homomorphisms  $g_k$  take an integer  $j \pmod{q^{k+1}}$  to the integer  $j \pmod{q^k}$ . Three preliminary lemmas concerning the quotient rings  $\mathbf{GBT}_n/\alpha^k \mathbf{GBT}_n$ ,  $k = 1, 2, \dots$ , are used to show that  $\mathbf{EGBT}_n$  often has the structure of the  $q$ -adic integers. The following notation is used for subsets of  $\mathbf{GBT}_n$ :  $A = B \oplus C$  if each element  $a \in A$  can be expressed uniquely as a sum  $a = b + c$  where  $b \in B$  and  $c \in C$ .

**Lemma 4.1.** *If  $R$  is a ring,  $\alpha \in R$  and  $S \subset R$  is a set of coset representatives of  $R/\alpha R$ , then*

$$S \oplus \alpha S \oplus \dots \oplus \alpha^{k-1} S \subset R$$

*is a set of coset representatives of  $R/\alpha^k R$ .*

**Proof.** Since  $S$  is a set of coset representatives of  $R/\alpha R$ ,

$$\begin{aligned} R &= S \oplus \alpha R = S \oplus \alpha(S \oplus \alpha R) = S \oplus \alpha S \oplus \alpha^2 R \\ &\vdots \\ &= S \oplus \alpha S \oplus \alpha^2 S \oplus \dots \oplus \alpha^{k-1} S \oplus \alpha^k R. \quad \square \end{aligned}$$

**Lemma 4.2.** *If  $m$  is a positive integer, then  $\bar{m}$  is divisible by  $\alpha$  in  $\mathbf{GBT}_n$  if and only if  $m$  is divisible by  $2^{n+1} - 1$  in  $\mathbb{Z}$ .*

**Proof.** Let  $q = 2^{n+1} - 1$  and suppose  $m$  is divisible by  $q$ . Since  $f(x)$  is the polynomial of minimum degree over  $\mathbb{Z}$  satisfied by  $\omega$ ,  $g(x) = f(2 - x)$  is the polynomial of minimum degree over  $\mathbb{Z}$  satisfied by  $\alpha$ . Moreover,

$$0 = g(\alpha) = \sum_{i=0}^n (2 - \alpha)^i = (2^{n+1} - 1) - \alpha h(\alpha),$$

where  $h(x)$  is some polynomial in  $\mathbb{Z}[x]$ . The above equation implies that  $q$  is divisible by  $\alpha$ . Therefore, since  $m$  is divisible by  $q$ ,  $\bar{m}$  is divisible by  $\alpha$ .

Conversely, if  $\bar{m}$  is divisible by  $\alpha$ , then  $\bar{m} = \alpha h_1(\alpha)$  for some  $h_1(x) \in \mathbb{Z}[x]$ . Let  $k_1(x) = xh_1(x) - m$ . Let  $d$  be the greatest common divisor of the coefficients of  $k_1(x)$  and let  $k(x) = (1/d)k_1(x)$ . Since  $k(\alpha) = 0$  and  $g(x)$  are the polynomial of minimum

degree satisfied by  $\alpha$ , it must be the case that  $k(x) = g(x)q(x)$ , where  $q(x) \in \mathbb{Q}[x]$ . Since the greatest common divisor of the coefficients of  $k(x)$  (and thus also of  $g(x)$ ) is 1, it follows that  $q(x) \in \mathbb{Z}[x]$ . The constant terms in  $k(x)$  and  $g(x)$  are  $m/d$  and  $q$ , respectively. Therefore, since  $q$  divides  $m/d$ ,  $q$  divides  $m$ .  $\square$

**Lemma 4.3.** *If  $q = 2^{n+1} - 1$ , then  $|\mathbf{GBT}_n/\alpha^k\mathbf{GBT}_n| = q^k$ .*

**Proof.** Lemma 4.3 follows from Lemma 4.1 once it is shown that  $|\mathbf{GBT}_n/\alpha\mathbf{GBT}_n| = q$ . Since every element of  $\mathbf{GBT}_n$  can be represented by a polynomial in  $\omega$  with coefficients in  $\mathbb{Z}$ , every element of  $\mathbf{GBT}_n$  can be written as a polynomial in  $\alpha$  with coefficients in  $\mathbb{Z}$ . This last fact implies that every element of  $\mathbf{GBT}_n/\alpha\mathbf{GBT}_n$  can be represented as  $\bar{m}$  for some integer  $m$ . Now  $|\mathbf{GBT}_n/\alpha\mathbf{GBT}_n| = q$  follows from Lemma 4.2.  $\square$

**Theorem 4.4.** *Let  $q = 2^{n+1} - 1$ . If  $q$  and  $n + 1$  are relatively prime, then there is a ring isomorphism  $\mathbf{EGBT}_n \cong \mathbb{Z}_{(q)}$ .*

**Proof.** It is sufficient to find “vertical” isomorphisms that make the following diagram commute:

$$\begin{array}{ccccccc} \mathbf{GBT}/\alpha\mathbf{GBT} & \leftarrow & \mathbf{GBT}/\alpha^2\mathbf{GBT} & \leftarrow & \dots & \leftarrow & \mathbf{GBT}/\alpha^k\mathbf{GBT} & \leftarrow & \dots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \mathbb{Z}/q\mathbb{Z} & \leftarrow & \mathbb{Z}/q^2\mathbb{Z} & \leftarrow & \dots & \leftarrow & \mathbb{Z}/q^k\mathbb{Z} & \leftarrow & \dots \end{array}$$

Since each vertical map is to be a ring isomorphism, each of these vertical maps must take the multiplicative identity 1 in  $\mathbf{GBT}/\alpha^k\mathbf{GBT}$  to the 1 in  $\mathbb{Z}/q^k\mathbb{Z}$ . By Lemma 4.3 the order of the additive group  $\mathbf{GBT}/\alpha^k\mathbf{GBT}$  is  $q^k$ . Therefore, these isomorphisms exist if and only if the additive order of the element 1 in  $\mathbf{GBT}/\alpha^k\mathbf{GBT}$  is  $q^k$ . This fact will be proved by induction on  $k$ . The case  $k = 1$  is exactly Lemma 4.2. By way of induction, assume that the order of 1 in  $\mathbf{GBT}/\alpha^{k-1}\mathbf{GBT}$  is  $q^{k-1}$ . With the polynomial  $g(x)$  defined exactly as it is in the proof of Lemma 4.2,

$$0 = g(\alpha) = \sum_{i=0}^n (2 - \alpha)^i = [2^{n+1} - 1] - [(n - 1)2^n + 1]\alpha + \alpha^2 h(\alpha),$$

where  $h(x) \in \mathbb{Z}[x]$ . Letting  $a = (n - 1)2^n + 1$  implies that

$$q^{k-1} = a^{k-1}\alpha^{k-1} + \alpha^k l(\alpha), \tag{2}$$

where  $l(x) \in \mathbb{Z}[x]$ . Note that  $(n - 1)2^n + 1 \equiv (n + 1)2^n \pmod{q}$  which implies that  $\gcd(a, q) = \gcd((n + 1)2^n, q) = \gcd(n + 1, q)$ . By the induction assumption the order of 1 in  $\mathbf{GBT}/\alpha^k\mathbf{GBT}$  must be a product of the form  $cq^{k-1}$ . It now suffices to show that  $q$  is the least positive integer  $c$  such that  $cq^{k-1}$  is divisible by  $\alpha^k$ . From (2),

$$cq^{k-1} = ca^{k-1}\alpha^{k-1} + c\alpha^k l(\alpha).$$

This equation implies that  $cq^{k-1}$  is divisible by  $\alpha^k$  if and only if  $ca^{k-1}$  is divisible by  $\alpha$ . By Lemma 4.2 this holds if and only if  $ca^{k-1}$  is divisible by  $q$ . The least such  $c$  must be

$q$  if and only if  $\gcd(a, q) = 1$ . Since  $\gcd(a, q) = \gcd(n + 1, q) = 1$ , the induction is true for the integer  $k$ .  $\square$

**Corollary 4.5.** *If  $n + 1$  is prime, then  $\mathbf{EGBT}_n$  is ring isomorphic to  $\mathbb{Z}_{(q)}$ , where  $q = 2^{n+1} - 1$ .*

**Proof.** If  $p = n + 1$  is prime, then by Fermat’s theorem  $2^{n+1} \equiv 2 \pmod{p}$ . Thus,  $n + 1$  does not divide  $2^{n+1} - 1$ .  $\square$

### 5. The standard and canonical address

Three topics are discussed in this section: the standard and canonical address of a cell, access and retrieval from cells, and the algebraic operations of addition and multiplication on cell addresses. Using standard addressing, access and retrieval and the arithmetic operations have efficient implementations in terms of binary bit string operations. These are illustrated in Examples 5.4 and 5.6.

**Lemma 5.1.** *Each of the following subsets of  $\mathbf{GBT}_n$  comprise a set of coset representatives for the ring  $\mathbf{GBT}_n/\alpha\mathbf{GBT}_n$ :*

- (1)  $S = \{\bar{0}, \bar{1}, \dots, \overline{2^{n+1} - 2}\}$ .
- (2)  $T = \{\varepsilon_0 + \varepsilon_1\omega + \dots + \varepsilon_n\omega^n : \varepsilon_i \in \{0, 1\}, \text{ not all } \varepsilon_i = 1\}$ .

**Proof.** The content of Lemma 4.2 is that  $S$  is a set of coset representatives. Since  $\omega \equiv 2 \pmod{\alpha}$ ,

$$\varepsilon_0 + \varepsilon_1\omega + \dots + \varepsilon_n\omega^n \equiv \sum_{i=0}^n \varepsilon_i 2^i \pmod{\alpha}.$$

Therefore, the elements of  $T$  represent exactly the same cosets as the elements of  $S$ .  $\square$

Recall that the address of a lattice point depends on the choice of coset representatives of  $\mathbf{GBT}/\alpha\mathbf{GBT}$ . The address with respect to  $S$  and  $T$  of a lattice point  $x \in \mathcal{A}_n^*$  will be called the *canonical address* and *standard address* of  $x$ , respectively. Each term  $t_i$  in the standard address is of the form  $\varepsilon_0 + \varepsilon_1\omega + \dots + \varepsilon_n\omega^n$ , which will be abbreviated  $\varepsilon_0\varepsilon_1 \dots \varepsilon_n$ . This string consists of  $n + 1$  zeros and ones. The canonical address can be readily obtained from the standard address by regarding each term  $t_i$  of the standard address as a binary number; for example, with  $n = 2$  if the standard address is  $t_0t_1t_2$  where  $t_0 = (110)$ ,  $t_1 = (011)$ ,  $t_2 = (010)$ , then the canonical address is 362. This conversion rule follows directly from  $\omega \equiv 2 \pmod{\alpha}$ .

The standard or canonical address is called *finite* if  $t_i = 0$  for all  $i$  sufficiently large. The following theorem was proved by Vince [13].

**Theorem 5.2.** *Every lattice point in  $A_n^*$  has a unique finite standard (canonical) address.*

The canonical addresses in dimension 2 are shown in Fig. 3. Theorem 5.2 implies that the addressing map

$$\Phi: A_n^* \rightarrow \mathbf{EGBT}_n$$

is injective, which gives the following corollary.

**Corollary 5.3.** *The ring  $\mathbf{GBT}_n$  is isomorphic to the subring of  $\mathbf{EGBT}_n$  consisting of those elements with finite standard (or canonical) address.*

In the same way that usual addition or multiplication of two digits base 10 have a remainder and carry term, addition or multiplication of two addresses can also be thought of as having both a “remainder” and a “carry” component. While these components seem foreign when canonical addressing is used, they have an extremely rapid implementation from a computer science point of view when standard addressing is used. Although the derivation of these rules is somewhat tedious, they are based on the two observations  $1 + 1 = \omega + \alpha$  and  $\omega^{n+1} = 1$ . The implementation of one such operation is illustrated in the following example.

**Example 5.4.** In 3-space the sum of the addresses  $6 = 0110$  and  $14 = 0111$  is  $(5)(14)$  which, in standard form, is  $t_0 t_1$  where  $t_0 = 1010$  and  $t_1 = 0111$ . The “remainder”  $5 = 1010$  is obtained as the circular binary sum of  $0110$  and  $0111$ . (That is, the last digit carried on the right is added to the first digit on the left.) This is equivalent to adding  $6$  and  $14 \pmod{15}$ . The “carry”  $14 = 0111$  is obtained by first taking the “exclusive or” (binary addition without carries) of the strings  $0110$ ,  $0111$ , and remainder  $1010$  and then “shifting” each digit of the result one place to the left. (The leftmost digit is moved to the far right position.)

The  $\mathbf{EGBT}$  addressing system also allows for extremely fast access to and retrieval from cells. The first part of Proposition 5.5 provides an algorithm to access a given cell from its address. Conversely, the second part provides an algorithm to retrieve the address of a cell. Let  $V$  be the  $n \times (n + 1)$  matrix whose columns are the vectors  $\{v_0, v_1, \dots, v_n\}$  that generate the lattice  $A_n^*$ . Define the  $(n + 1) \times (n + 1)$  circulant matrix

$$B = \begin{pmatrix} 2 & 0 & \cdots & 0 & -1 \\ -1 & 2 & \cdots & 0 & 0 \\ 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & 2 \end{pmatrix}.$$

Note that the inverse of  $B$  is the circulant matrix given by the formula  $B^{-1} = (1/q)(b_{ij})$ , where  $b_{ij} = 2^{n+j-i}$  for  $i \geq j$ ,  $b_{ij} = 2^{j-i-1}$  for  $i < j$ , and  $q = 2^{n+1} - 1$ .



Each term  $t = \varepsilon_0 \varepsilon_1 \dots \varepsilon_n$  in the standard address of a cell is understood to be written as a bit string of length  $n + 1$  and this bit string is considered to be the column vector  $\mathbf{t} = (\varepsilon_0, \varepsilon_1, \dots, \varepsilon_n)^T$ . If  $q = 2^{n+1} - 1$  and  $\mathbf{b} = (b_0, b_1, \dots, b_n)$  is any vector with integer coordinates, then since  $2^{n+1} \equiv 1 \pmod{q}$ , there is a uniquely determined vector  $\varepsilon = (\varepsilon_0, \varepsilon_1, \dots, \varepsilon_n)$  with  $\varepsilon_i \in \{0, 1\}$  such that

$$\sum_{i=0}^n b_i 2^i \equiv \sum_{i=0}^n \varepsilon_i 2^i \pmod{q}.$$

In fact, if the  $b_i$  are written base 2, then  $\varepsilon$  can be obtained from  $\mathbf{b}$  by repeated shifts and adds. Let  $\Theta$  denote the map that takes  $\mathbf{b}$  to  $\varepsilon$ .

**Proposition 5.5.** (1) *The cell  $\mathbf{x} \in A_n^*$  with standard address  $t_0 t_1 \dots t_m$  is given by*

$$\mathbf{x} = V(\mathbf{t}_0 + B\mathbf{t}_1 + B^2\mathbf{t}_2 + \dots + B^m\mathbf{t}_m).$$

(2) *If  $\mathbf{x} = \sum_{i=0}^n a_i \mathbf{v}_i$  is a cell center, let  $\mathbf{a}_0 = (a_0, a_1, \dots, a_n)$ . The standard address  $t_0 t_1 t_2 \dots$  of  $\mathbf{x}$  is generated by the following algorithm for  $i = 0, 1, \dots$*

$$\mathbf{t}_i = \Theta(\mathbf{a}_i),$$

$$\mathbf{a}_{i+1} = B^{-1}(\mathbf{a}_i - \mathbf{t}_i).$$

**Proof.** Concerning statement (1), the standard address  $t_0 t_1 \dots t_m$  represents the element  $t_0 + t_1 \alpha + \dots + t_m \alpha^m \in \mathbf{GBT}_n$ . Since  $\omega^{n+1} = 1$ , multiplication of  $t_i$  by  $\omega$  is equivalent to a circular shift one place to the right in the bit string corresponding to  $t_i$ . Therefore, multiplication by  $\alpha = 2 - \omega$  amounts to multiplication of the bit string (as a column vector) by the matrix  $B$ . If  $t_0 + t_1 \alpha + \dots + t_m \alpha^m = a_0 + a_1 \omega + \dots + a_n \omega^n$ , where  $\mathbf{a} = (a_0, a_1, \dots, a_n)^T = \mathbf{t}_0 + B\mathbf{t}_1 + B^2\mathbf{t}_2 + \dots + B^m\mathbf{t}_m$ , then the inverse image under the addressing map  $\Phi$  is  $\mathbf{x} = a_0 \mathbf{v}_0 + a_1 \mathbf{v}_1 + \dots + a_n \mathbf{v}_n = V\mathbf{a}$ .

Concerning statement (2), the procedure for finding the standard sequence  $t_0, t_1, \dots$  of an element  $\gamma \in \mathbf{GBT}_n$  in the inverse limit is to let  $\gamma_0 = \gamma$  and repeat the two steps: (1) find  $t_i \in T$  such that  $\gamma_i \equiv t_i \pmod{\alpha}$  and (2) let  $\gamma_{i+1} = (\gamma_i - t_i)/\alpha$ . Under the addressing map  $\Phi$ , the cell  $\mathbf{x}$  is mapped in  $\mathbf{GBT}_n$  to  $\sum_{i=0}^n a_i \omega^i = \sum_{i=0}^n a_i 2^i \pmod{q}$  because  $\omega = 2 \pmod{\alpha}$ . By Lemma 4.2, congruence of an integer  $\pmod{\alpha}$  is the same as congruence  $\pmod{q}$ . Therefore, step (1) is exactly the mapping  $\Theta$ , the first step in the algorithm of statement (2). Multiplication by  $\alpha$  is equivalent to multiplication by  $B$ . Therefore, division by  $\alpha$  is multiplication by  $B^{-1}$  and step (2) is the second step in the algorithm of statement (2).  $\square$

**Example 5.6.** Consider the cell  $\mathbf{x} = a_0 \mathbf{v}_0 + a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2$  in  $\mathbb{R}^2$  with canonical address 362. The standard address is

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

To find the location of this cell, use Proposition 5.5 (1), implementing multiplication by  $B$  completely in terms of binary bit string operations:

$$\begin{aligned} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} &= \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + B \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + B^2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + B \left[ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 00 \\ 01 \\ 00 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right] \\ &= \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + B \begin{pmatrix} 00 \\ 11 \\ 00 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 000 \\ 011 \\ 000 \end{pmatrix} - \begin{pmatrix} 00 \\ 00 \\ 11 \end{pmatrix} = \begin{pmatrix} 1 \\ 111 \\ -11 \end{pmatrix}. \end{aligned}$$

Therefore, the vector  $\mathbf{x} = \mathbf{v}_0 + 7\mathbf{v}_1 - 3\mathbf{v}_2$ .

Conversely, to find the canonical address  $s_0s_1s_2$  of the cell  $\mathbf{x} = \mathbf{v}_0 + 7\mathbf{v}_1 - 3\mathbf{v}_2$ , use Proposition 5.5 (2).

$$\begin{aligned} s_0 &= 1 + 7 \cdot 2 - 3 \cdot 4 = 3, & \mathbf{a}_1 &= B^{-1}[(1, 7, -3) - (1, 1, 0)]^T = (0, 3, 0)^T, \\ s_1 &= 3 \cdot 2 = 6, & \mathbf{a}_2 &= B^{-1}[(0, 3, 0) - (0, 1, 1)]^T = (0, 1, 0)^T, \\ s_2 &= 1 \cdot 2 = 2, & \mathbf{a}_3 &= B^{-1}(0, 0, 0)^T = (0, 0, 0)^T. \end{aligned}$$

Therefore, the address is 362.

## 6. Aggregates

The addressing on the permutohedral cells induces tilings of  $\mathbb{R}^n$  by aggregates as follows. Recall that a compact set  $C$  is said to tile  $\mathbb{R}^n$  if  $\mathbb{R}^n$  can be written as the union countably many copies of  $C$ , where the interiors of distinct copies are pairwise disjoint. Let  $t_0(\mathbf{x})t_1(\mathbf{x})\dots$  denote the standard address or canonical address of a cell  $\mathbf{x}$ . Let

$$A_k = \{\mathbf{x} \in A_n^* : t_i(\mathbf{x}) = 0 \text{ for all } i \geq k\},$$

and call  $A_k$  the  $k$ th-aggregate. Examples of the first, second- and third-level aggregates in 2-dimensions are given in Figs. 1, 2, and 3, respectively. Theorem 5.2 implies that every cell is contained in some aggregate. From the definitions it follows that there is

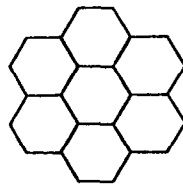


Fig. 1. The first level aggregate.

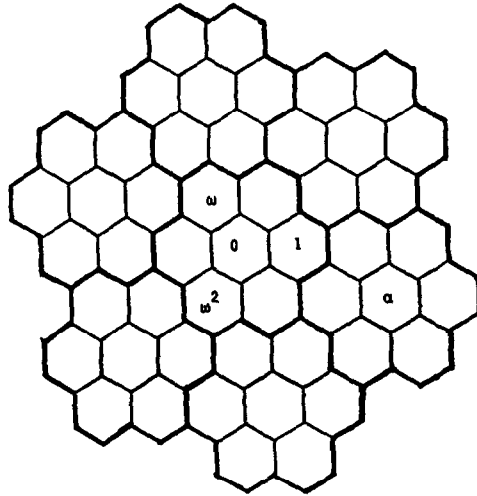


Fig. 2. The second level aggregate and vector  $\alpha$ .

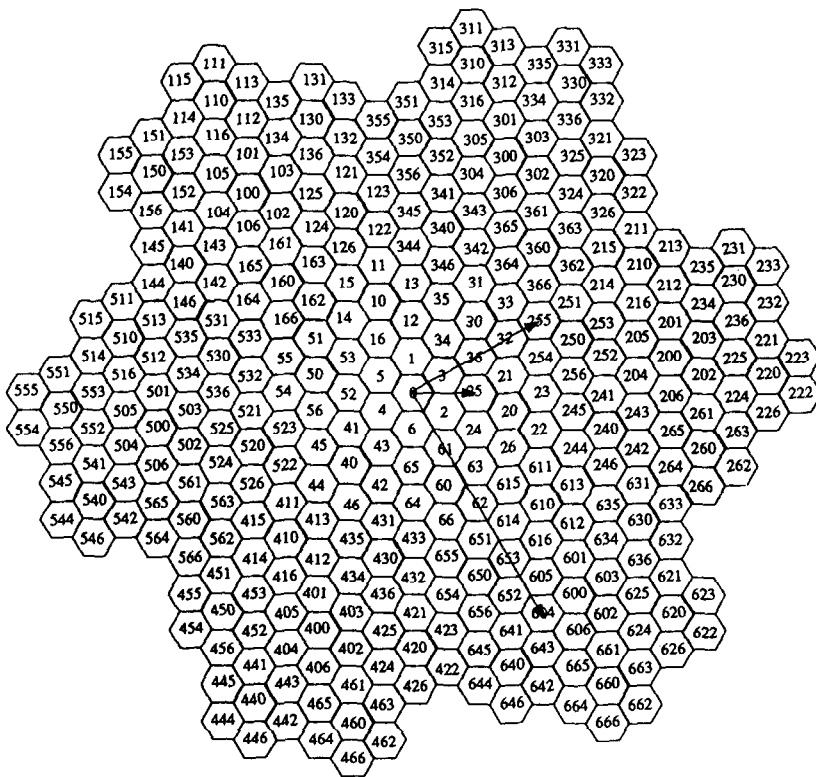


Fig. 3. The third level aggregate with the GBT product of 255 and 25 displayed.

a one-to-one correspondence between the cells of the  $k$ th-aggregate and the elements of  $\mathbf{GBT}_n/\alpha^k\mathbf{GBT}_n$ . Therefore,  $A_k$  inherits a ring structure from  $\mathbf{GBT}_n/\alpha^k\mathbf{GBT}_n$ . If  $q = 2^{n+1} - 1$ , then by the results in Section 4 this ring is isomorphic to  $\mathbb{Z}/q^k\mathbb{Z}$  when  $n + 1$  and  $q$  are relatively prime.

It is equivalent to define the  $k$ th-aggregate by  $A_0 = 0$  and for  $k \geq 1$ ,

$$A_k = \Phi^{-1}(T \oplus \alpha T \oplus \alpha^2 T \oplus \cdots \oplus \alpha^k T).$$

Lemma 4.1 implies  $\mathbf{GBT}_n = T \oplus \alpha T \oplus \alpha^2 T \oplus \cdots \oplus \alpha^k T \oplus \alpha^{k+1} \mathbf{GBT}_n$ . Looking at the inverse image under  $\Phi$ ,

$$A_n^* = A_k \oplus \Phi^{-1}(\alpha^{k+1} \mathbf{GBT}),$$

$$A_{k+1} = A_k \oplus \Phi^{-1}(\alpha^{k+1} T).$$

From these formulas it follows that there is a nested family of tessellations of  $\mathbb{R}^n$  by aggregates.

**Proposition 6.1.** *For each  $k = 0, 1, \dots$ ,*

- (1) *the space  $\mathbb{R}^n$  is tiled by copies of the  $k$ th aggregate,*
- (2) *the  $(k + 1)$ st aggregate is tiled by  $2^{n+1} - 1$  copies of the  $k$ th aggregate, and*
- (3) *every cell lies in some aggregate.*

## 7. Concluding remarks

An obvious question is what form should Theorem 4.4 take if  $\gcd(n + 1, 2^{n+1} - 1) > 1$ ? Also while we have not mentioned any topology in this paper, the two rings in Theorem 4.4 could be described as completions of finite sequences. These completions will be topologically equivalent to the standard Cantor set.

## Acknowledgments

Finally, we would like to express our gratitude to Dean Lucas, who has not only stimulated our interest in the area but has also supplied us with a variety of reprints and preprints and has taken the time to discuss in phone conversations a variety of technical details concerning the cells and their addresses. Many of the propositions and techniques expressed as examples, as well as the matrix  $B$  and its inverse, were known to him. He also suggested the terminology Generalized Balanced Ternary and Extended Generalized Balanced Ternary.

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