Stochastic Processes and their Applications 23 (1986) 259-267
North-Holland

THE PRESERVATION OF LIKELIHOOD RATIO ORDERING UNDER CONVOLUTION

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Received 1 October 1985
Revised 7 April 1986

Unlike stochastic ordering ($\geq^{st}$), which is preserved under convolution (i.e., summation of independent random variables), so far it is only known that likelihood ratio ordering ($\geq^{lr}$) is preserved under convolution of log-concave (PF_2) random variables. In this paper we define a stronger version of likelihood ratio ordering, termed *shifted likelihood ratio ordering* ($\geq^{lr}_{\uparrow}$) and show that it is preserved under convolution. An application of this closure property to closed queueing network is given. Other properties of shifted likelihood ratio ordering are also discussed.

AMS Subject Classifications: Primary 60G99; Secondary 15A45.

shifted likelihood ratio ordering * log-concavity * queueing networks * total positivity * conditional stochastic order

1. Introduction

Likelihood ratio ordering (Karlin, 1957; Lehmann, 1959) prevalent in stochastic processes is very useful in developing bounds and approximation for performance measures of stochastic systems. Two random variables X and Y are ordered in the sense of likelihood ratio ordering (i.e. $X \geq^{lr} Y$) when the ratio $f_X(x)/f_Y(x)$ of their probability density functions is non-decreasing in x. Several useful properties of likelihood ratio ordering, its equivalence to uniform stochastic ordering and its extension to the multivariate setting can be found in Karlin and Rinott (1980), Keilson and Sumita (1982), and Whitt (1980, 1982). In our attempt to develop bounds for the throughput of closed queueing networks, we needed to consider the

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sums of such ordered pairs of independent random variables, say $(X_i \geq^{\mathrm{lr}} Y_i)$, $i = 1, 2, \ldots, m$. Therefore it is of interest to know whether or not the following is true:

$$\sum_{i=1}^{m} X_i \geq^{\mathrm{lr}} \sum_{i=1}^{m} Y_i. \tag{1.1}$$

Without additional condition on (X_i, Y_i), $i = 1, 2, \ldots, m$, the above inequality need not be true. From Keilson and Sumita (1982, Theorem 2.1(d)) it is known that if Z has a log-concave density function and $X \geq^{\mathrm{lr}} Y$, then $X + Z \geq^{\mathrm{lr}} Y + Z$. If $X_i \geq^{\mathrm{lr}} Y_i$, $i = 1, 2$ and both Y_1 and X_2 have log-concave density functions one easily sees that $X_1 + X_2 \geq^{\mathrm{lr}} Y_1 + X_2 \geq^{\mathrm{lr}} Y_1 + Y_2$. Using this and the preservation of the log-concavity under convolution (Karlin and Proschan, 1960), one has

Lemma 1.1. *Suppose* (X_i, Y_i), $i = 1, 2, \ldots, m$, *are independent pairs of random variables such that*:

(a) $X_i \geq^{\mathrm{lr}} Y_i$, $i = 1, 2, \ldots, m$, *and*

(b) X_i, Y_i, $i = 1, 2, \ldots, m$, *all have log-concave density function, except for possibly one* X_l *and one* Y_k $(l \neq k)$. *Then* (1.1) *is true.*

In this paper we define a stronger version of likelihood ratio ordering called *shifted likelihood ratio ordering* $(\geq^{\mathrm{lr}}_{\uparrow})$ and show that it is closed under convolution. The condition needed for this closure property is, however, weaker than (b).

In Section 2 we give some preliminaries needed in our analysis. The main results of this paper are presented in Section 3. Section 4 contains an application of our main result to closed queueing networks.

2. Preliminaries

The following notations and conventions will be used throughout this paper. $N(N_+, N_{++})$ denotes the set of (non-negative, positive) integers; $R(R_+, R_{++})$ denotes the set of (non-negative, positive) real numbers. Two functions $f, g : R \to R$ satisfying $f(x) \geq g(x)$ for all x in the definition domain will be denoted by $f \geq g$. $f\uparrow$ $(f\downarrow)$ means that f is an increasing (decreasing) function. The terms increasing and decreasing are not used in the strict sense. Let $S(X)$ denote the support of a random variable X, i.e. $S(X) = \{x : f_X(x) > 0, x \in R\}$ where f_X is the probability density function of X. Except in Remark 3.5, for any random variable X under consideration, $S(X) = \{0, 1, 2, \ldots, N\}$ where N is a finite or an infinite integer.

Definition 2.1. The equilibrium rate of a random variable X, $r_X(n)$, is a function: $S(X) \to R_+$, defined as

$$r_X(0) = 0, \qquad r_X(n) = P[X = n-1]/P[X = n], \quad n \in S(X) - \{0\}.$$

Definition 2.2. A random variable X (or its distribution) is said to be log-concave or PF_2 (Polya frequency of order 2), denoted as $X \in \mathrm{PF}_2$, if $r_X\uparrow$.

Definition 2.3. Two random variables X and Y have a common support. X dominates Y in the sense of likelihood ratio, denoted as $X \geqslant^{\mathrm{lr}} Y$ if $r_X \leqslant r_Y$.

Remark 2.4. It is known that $X \geqslant^{\mathrm{lr}} Y \to X \geqslant^{\mathrm{st}} Y$ while the reverse need not be true (Ross, 1983, Chapter 8; $\geqslant^{\mathrm{st}}$ denotes stochastic ordering).

The next definition and lemma will be needed to define and characterize the shifted likelihood ratio ordering.

Definition 2.5. $f, g: R \to R$ are two functions. Define $f \geqslant_{\uparrow} g$ if for any x, y in their definition domain $x \geqslant y \to f(x) \geqslant g(y)$.

Lemma 2.6. (i) $f \geqslant_{\uparrow} g \to f \geqslant g$.

(ii) *If either* $f\uparrow$ *or* $g\uparrow$ *(or both) then* $f \geqslant_{\uparrow} g \leftrightarrow f \geqslant g$.

(iii) $f \geqslant_{\uparrow} g \leftrightarrow f(x+y) \geqslant g(y)$, $x \geqslant 0$.

(iv) $f \geqslant_{\uparrow} g \leftrightarrow$ *there exists an* $h\uparrow$ *such that* $f \geqslant h \geqslant g$.

Proof. (i), (ii) and (iii) are obvious. For (iv) $\to$, let $h(x) = \max_{y \leqslant x} g(y)$, and for (iv) $\leftarrow$ observe that for any $x \geqslant y$, $f(x) \geqslant h(x) \geqslant h(y) \geqslant g(y)$.

Combining Definitions 2.1, 2.3 and 2.5, we define a stronger version of likelihood ratio ordering.

Definition 2.7. X and Y are two random variable with the same support. X is larger than Y in the shifted likelihood ratio ordering (denoted $X \geqslant^{\mathrm{lr}}_{\uparrow} Y$) if $r_Y \geqslant_{\uparrow} r_X$.

Taking into account Definitions 2.2, 2.3 and 2.5, we have, parallel to Lemma 2.6.

Theorem 2.8. (i) $X \geqslant^{\mathrm{lr}}_{\uparrow} Y \to X \geqslant^{\mathrm{lr}} Y$.

(ii) *If either* $X \in \mathrm{PF}_2$ or $Y \in \mathrm{PF}_2$ *(or both) then* $X \geqslant^{\mathrm{lr}}_{\uparrow} Y \leftrightarrow X \geqslant^{\mathrm{lr}} Y$.

(iii) $X \geqslant^{\mathrm{lr}}_{\uparrow} Y \leftrightarrow X \geqslant^{\mathrm{lr}} \{Y - y \mid Y \geqslant y\}$, $y \in N_+$, $N = +\infty$.

(iv) $X \geqslant^{\mathrm{lr}}_{\uparrow} Y \leftrightarrow$ *there exists a* $Z \in PF_2$ *such that* $X \geqslant^{\mathrm{lr}} Z \geq^{\mathrm{lr}} Y$.

3. The main results

We will first establish a stochastic monotonicity property of birth–death processes. We will use this result to establish the closure property of shifted likelihood ratio ordering under convolution. Furthermore, this monotonicity property complements the results of Keilson and Kester (1977, 1978) and Keilson and Sumita (1982) for birth–death processes and may be of independent interest on its own.

Theorem 3.1. *$\{N^i(t)\}$ is a birth–death process on $\{0, 1, \ldots, n\}$ governed by downward transition rates $\mu^i(k) \equiv r_1^i(k)$ $(0 \leqslant k \leqslant n)$ $[r_1^i(0) = 0]$ and upward transition rates $\lambda^i(k) \equiv r_2^i(n-k)$ $(0 \leqslant k \leqslant n)$ $[r_2^i(0) = 0]$ for $i = 1, 2$. Let $D_j^i(t)$ be the number of downward $(j = 1)$ and upward $(j = 2)$ transitions of $\{N^i(t)\}$ in the interval $(0, t]$. Then $\{(D_1^2(t), D_2^2(t))\} \geqslant^{\mathrm{st}} \{(D_1^1(t), D_2^1(t))\}$, if*

(i) $r_j^2 \geqslant_\uparrow r_j^1, j = 1, 2$, *and*
(ii) $N^1(0) =^{\mathrm{st}} N^2(0)$.

Proof. For notational convenience let $X_1^i(t) = N^i(t), X_2^i(t) = n - X_1^i(t)$, $\boldsymbol{X}^i(t) = (X_1^i(t), X_2^i(t))$ and $\boldsymbol{D}^i(t) = (D_1^i(t),\ D_2^i(t))$, $t \geqslant 0$ $(i = 1, 2)$. We shall construct the processes $\{(\hat{\boldsymbol{X}}(t), \hat{\boldsymbol{D}}^i(t))\}$ $(i = 1, 2)$ on the same probability space (Ω, F, P), such that they have the same probability distributions as $\{(\boldsymbol{X}^i(t), \boldsymbol{D}^i(t))\}$ $(i = 1, 2)$. The conclusion of Theorem 3.1 will then be proved by proving the pathwise probability-one dominance $\hat{\boldsymbol{D}}^2(t) \geqslant \hat{\boldsymbol{D}}^1(t)$.

Let $\boldsymbol{T} = (T_k)_{k=0}^\infty$ $(T_0 = 0)$ be the sequence of arrival epochs of a Poisson process with rate $\lambda = 2 \max_{1 \leqslant k \leqslant n} \{r_j^i(k) \mid i = 1, 2; j = 1, 2\}$, and $\boldsymbol{U}_l = (U_{lk})_{k=1}^\infty$ $(l = 1, 2)$ be two sequences of i.i.d. random variables uniformly distributed on $(0, 1)$. For $\omega \in \Omega$, let $t_k = T_k(\omega)$ and $u_{lk} U_{lk}(\omega)$ $(l = 1, 2;\ k = 1, 2, \ldots)$ be the sample points of $\boldsymbol{T}$ and $\boldsymbol{U}_l$. For $t_0 = 0$, set

$$\hat{\boldsymbol{X}}^1(0) = \hat{\boldsymbol{X}}^2(0) \quad \text{and} \quad \hat{\boldsymbol{D}}^1(0) = \hat{\boldsymbol{D}}^2(0) = (0, 0).$$

Consider $w \in \Omega$. For the processes to be constructed, possible changes of states can only occur at time points t_k $(k = 1, 2, \ldots)$. Given the states of $\hat{\boldsymbol{X}}^i$ and $\hat{\boldsymbol{D}}^i$ $(i = 1, 2)$ at t_{k-1}, we specify in the following the states at t_k. At t_{k-1}, the downward and upward transition rates are $r_j^i(\hat{X}_j^i(t_{k-1})), j = 1, 2$, for $i = 1, 2$; denote these simply as r_j^i. Let $R_j = \max\{r_j^1, r_j^2\}$ $(j = 1, 2)$. The specifications of $\hat{X}_j^i$ and $\hat{D}_j^i$ (at t_k) are summarized in Table 1 (for the time being, ignore the last four columns in the table).

Table 1
State transitions (from t_{k-1} to t_k).

Situation (at t_{k-1})		Changes (at t_k)											
		$\hat{X}_1^1$	$\hat{D}_1^1$	$\hat{X}_1^2$	$\hat{D}_1^2$	$\hat{X}_2^1$	$\hat{D}_2^1$	$\hat{X}_2^2$	$\hat{D}_2^2$	δ_1	Δ_1	δ_2	Δ_2
(I)	$(R_1+R_2)/\lambda < u_{1k}$	0	0	0	0	0	0	0	0	0	0	0	0
(II)	$R_1/\lambda < u_{1k} \leqslant (R_1+R_2)/\lambda$												
(i)	$R_2 = r_2^2 \geqslant r_2^1, r_2^1/r_2^2 \geqslant u_{2k}$	+1	0	+1	0	−1	+1	−1	+1	0	0	0	0
(ii)	$R_2 = r_2^2 \geqslant r_2^1, r_2^1/r_2^2 < u_{2k}$	0	0	+1	0	0	0	−1	+1	−1	0	+1	+1
(iii)	$R_2 = r_2^1 > r_2^2, r_2^2/r_2^1 \geqslant u_{2k}$	+1	0	+1	0	−1	+1	−1	+1	0	0	0	0
(iv)	$R_2 = r_2^1 > r_2^2, r_2^2/r_2^1 < u_{2k}$	+1	0	0	0	−1	+1	0	0	+1	0	−1	−1
(III)	$R_1/\lambda \geqslant u_{1k}$												
(i)	$R_1 = r_1^2 \geqslant r_1^1, r_1^1/r_1^2 \geqslant u_{2k}$	−1	+1	−1	+1	+1	0	+1	0	0	0	0	0
(ii)	$R_1 = r_1^2 \geqslant r_1^1, r_1^1/r_1^2 < u_{2k}$	0	0	−1	+1	0	0	+1	0	+1	+1	−1	0
(iii)	$R_1 = r_1^1 > r_1^2, r_1^2/r_1^1 \geqslant u_{2k}$	−1	+1	−1	+1	+1	0	+1	0	0	0	0	0
(iv)	$R_1 = r_1^1 > r_1^2, r_1^2/r_1^1 < u_{2k}$	−1	+1	0	0	+1	0	0	0	−1	−1	+1	0

For instance, situation II(i) corresponds to the case that in both birth–death processes an upward transition occurs. ("0" indicates no change.)

It can be readily verified that the processes so constructed, $\{(\hat{\boldsymbol{X}}^i(t), \hat{\boldsymbol{D}}^i(t))\}$ $(i = 1, 2)$, have the same transition rates and hence the same probability laws as $(\{\boldsymbol{X}^i(t)\}, \{\boldsymbol{D}^i(t)\})$ $(i = 1, 2)$, respectively.

Let $\delta_j(t_k) = \hat{X}_j^1(t_k) - \hat{X}_j^2(t_k)$ and $\Delta_j(t_k) = \hat{D}_j^2(t_k) - \hat{D}_j^1(t_k)$, for all $k = 1, 2, \ldots$, and $j = 1, 2$. To prove the proposition, it suffices to prove $\Delta_j(t_k) \geqslant 0$ for $j = 1, 2$ and all k. We prove, instead, the following stronger result:

$$\Delta_j(t_k) \geqslant \max\{\delta_j(t_k), 0\}. \tag{3.1}$$

We use induction. Changes of δ_j and Δ_j $(j = 1, 2)$ at t_k (from t_{k-1}) are summarized in the last four columns of Table 1. The validity of (3.1) at t_1 is readily checked based on the initial conditions and Table 1 (in this case, situations II(iv) and III(iv) cannot happen, due to assumption (i)). Suppose (3.1) is valid at t_{k-1}, then it is obviously valid at t_k under all situations in Table 1, except (perhaps not so obviously) II(iv) and III(iv). Consider II(iv): because of (i) at t_{k-1} we must have $\hat{X}_2^1 > \hat{X}_2^2$ or equivalently $\hat{X}_1^1 < \hat{X}_1^2$, i.e., $\delta_1(t_{k-1}) < 0$ and $\delta_2(t_{k-1}) > 0$. Therefore $\delta_1(t_k) = \delta_1(t_{k-1}) + 1 \leqslant 0$ and $\delta_2(t_k) = \delta_2(t_{k-1}) - 1 \geqslant 0$, and hence $\Delta_1(t_k) = \Delta_1(t_{k-1}) \geqslant 0 = \max\{\delta_1(t_k), 0\}$, and $\Delta_2(t_k) = \Delta_2(t_{k-1}) - 1 = \max\{\delta_2(t_{k-1}), 0\} - 1 = \max\{\delta_2(t_k), 0\}$. That is, (3.1) is valid at t_k. Situation III(iv) can be similarly argued.

This completes the induction and concludes the proof of the proposition. □

Before proceeding to present our main results we will see how Theorem 3.1 can be applied to closed queueing networks. Consider a cyclic queueing network with two stations $(j = 1, 2)$ and n jobs which cyclically go through the two stations for services. Service times at both stations are exponentially distributed and the queue discipline is first-come-first-served (FCFS). For station j $(j = 1, 2)$, the service rate is a function of the queue length n_j (i.e., the number of jobs in queue and in service at station j), $\mu_j(n_j)$. Assume $0 < \mu_j(n_j) < \infty$ for all $n_j \geqslant 1$ and $j = 1, 2$. Denote the network by $C(n, \mu_1, \mu_2)$. This is a special case of the closed queueing network of Gordon and Newell (1967). Observe that the number of jobs at station 1 follows a birth–death process with downward transition rates $\mu_1(k)$ $(0 \leqslant k \leqslant n)$ and upward transition rates $\mu_2(n-k)$ $(0 \leqslant k \leqslant n)$ when there are k jobs. Since the number of downward transitions $D_1(t)$ is the number of jobs serviced by station 1 during $(0, t]$, and the throughput of this network is $\mathrm{TH}[C(n, \mu_1, \mu_2)] = \lim_{t\to\infty} D_1(t)/t$, one has, from Theorem 3.1,

Corollary 3.2. *For the cyclic queueing network as described above $\mu_j^2 \geqslant_\uparrow \mu_j^1$, $j = 1, 2$, implies*

$$\mathrm{TH}[C(n, \mu_1^2, \mu_2^2)] \geqslant \mathrm{TH}[C(n, \mu_1^1, \mu_2^1)], \quad n \geqslant 1.$$

We will next present a lemma that appears to be weaker than the closure property we are after. However, we will soon see that this lemma is indeed equivalent to our main theorem to follow.

Lemma 3.3. *Y_1^i and Y_2^i $(i=1,2)$ are two pairs of independent random variables.*

$$Y_j^1 \geq_{\uparrow}^{\mathrm{lr}} Y_j^2 \ (j=1,2) \rightarrow Y_1^1 + Y_2^1 \geq^{\mathrm{lr}} Y_1^2 + Y_2^2.$$

Proof. To prove Lemma 3.3, we use the construction of a cyclic queueing network. Consider two independent random variables Y_1 and Y_2 with common support $S=\{0, 1, \ldots, N\}$ $(N \leq \infty)$. First notice the following relation:

$$r_{Y_1+Y_2}(n) = \frac{\sum_{k=0}^{n-1} P[Y_1=k]P[Y_2=n-1-k]}{\sum_{k=0}^{n} P[Y_1=k]P[Y_2=n-k]}$$

$$= \frac{\sum_{k=0}^{n-1} \bar{r}_1(k)\bar{r}_2(n-1-k)}{\sum_{k=0}^{n} \bar{r}_1(k)\bar{r}_2(n-k)} \quad (n \geq 1) \tag{3.2}$$

where for $j=1,2$, $\bar{r}_j(k) = [r_{Y_j}(1) \cdots r_{Y_j}(k)]^{-1}$ $(k \geq 1)$ and $\bar{r}_j(0)=1$.

Let $X=(X_1, X_2)$ denote the equilibrium joint queue lengths of the two station cyclic queueing network described earlier. It is known that

$$P[(X_1, X_2) = (n_1, n_2)] = f_1(n_1) f_1(n_2)/G(n) \quad (n_1+n_2=n), \tag{3.3}$$

where, for $j=1,2$,

$$f_j(n_j) = \left[\prod_{k=1}^{n_j} \mu_j(k)\right]^{-1} \quad (n_j \geq 1) \quad \text{and} \quad f_j(0)=1,$$

and

$$G(n) = \sum_{k=0}^{n} f_1(k) f_2(n-k)$$

is a normalizing constant.

The throughput of the network, $\mathrm{TH}[C(n, \mu_1, \mu_2)]$ can be derived as follows:

$$\mathrm{TH}[C(n, \mu_1, \mu_2)] = \sum_{k=0}^{n} \mu_1(k) P[X_1=k] = \sum_{k=0}^{n} \mu_2(k) P[X_2=k]$$

$$= \frac{\sum_{k=0}^{n-1} f_1(k) f_2(n-1-k)}{\sum_{k=0}^{n} f_1(k) f_2(n-k)} \quad (n \geq 1). \tag{3.4}$$

Comparing (3.4) and (3.2), we have

$$r_{Y_1+Y_2}(n) = \mathrm{TH}[C(n, r_{Y_1}, r_{Y_2})] \quad (n \geq 1). \tag{3.5}$$

That is, the equilibrium rate of the convolution Y_1+Y_2 is equal to the throughput of a two-station cyclic queueing network with r_{Y_1} and r_{Y_2} being the service rate functions of the two stations. This combined with Corollary 3.2 and Definition 2.3 completes the proof. □

Now we can present the closure property of shifted likelihood ratio ordering under convolution.

Theorem 3.4. $\{Y_j^i; j=1,\dots,m\}$ $(i=1,2)$ *are two families of independent random variables. Then*

$$Y_j^1 \geq_{\uparrow}^{\mathrm{lr}} Y_j^2 \,(j=1,2,\dots,m) \to \sum_{j=1}^{m} Y_j^1 \geq_{\uparrow}^{\mathrm{lr}} \sum_{j=1}^{m} Y_j^2.$$

Proof. Consider Theorem 3.4 for $m=2$. From Theorem 2.8 (iv), we know that, for $j=1,2$,

$$Y_j^1 \geq_{\uparrow}^{\mathrm{lr}} Y_j^2 \to Y_j^1 \geq^{\mathrm{lr}} Z_j \geq^{\mathrm{lr}} Y_j^2,$$

where $Z_1 \in \mathrm{PF}_2$ and $Z_2 \in \mathrm{PF}_2$ are independent. From Theorem 2.8(ii), this then leads to $Y_j^1 \geq_{\uparrow}^{\mathrm{lr}} Z_j \geq_{\uparrow}^{\mathrm{lr}} Y_j^2$. Therefore,

$$Y_1^1 + Y_2^1 \geq^{\mathrm{lr}} Z_1 + Z_2 \geq^{\mathrm{lr}} Y_1^2 + Y_2^2,$$

according to Lemma 3.3, and hence, from Theorem 2.8(iv),

$$Y_2^1 + Y_2^1 \geq_{\uparrow}^{\mathrm{lr}} Y_1^2 + Y_2^2,$$

since $Z_1 + Z_2 \in \mathrm{PF}_2$. Repeated use of this result (for $m=2$) establishes the closure property for general m. $\square$

Remark 3.5. Let $\boldsymbol{p}$ and $\boldsymbol{q}$ be the probability mass vectors (row vectors) of X and Y respectively. Then, from Definition 2.7, we have

$$X \geq_{\uparrow}^{\mathrm{lr}} Y \leftrightarrow \begin{vmatrix} q_{m+k} & q_{n+k} \\ p_m & p_n \end{vmatrix} \geq 0 \quad \text{for all } m \leq n,\, k \geq 0. \tag{3.6}$$

The continuous version of (3.6) is

$$X \geq_{\uparrow}^{\mathrm{lr}} Y \leftrightarrow \begin{vmatrix} g(x+a) & g(y+a) \\ f(x) & g(y) \end{vmatrix} \geq 0 \quad \text{for all } x \leq y \text{ and } a \geq 0.$$

where $f(\cdot)$ and $g(\cdot)$ are densities of X and Y, respectively. Therefore, Theorem 3.4 also holds in the continuous case through discretization and taking limit.

4. An application

One application of the results developed here is to bound the throughput of product-form closed queueing networks (CQN). Consider such a CQN with N jobs and M stations. Let ν_i and $\mu_i(n_i)$ denote the visit ratio and the service rate (which is a function of the queue length) of station i $(i=1,\dots,M)$. Here, we allow $\mu_i(\cdot)$ to be *any* function of the queue length at station i $(i=1,\dots,M)$: increasing, decreasing, non-monotonic, and etc. We only require $0<\mu_i(n_i)<\infty$ for all $n_i \geq 1$ and for all i. Notice that to date all bounds on the throughput of CQN's have been restricted to networks with fixed service rates, i.e., the μ_i's are fixed constants. Refer to Eager and Sevcik (1983), for instance. For all i, define

$$\mu_i^U = \max_{1\le n\le N} \mu_i(n) \quad \text{and} \quad \mu_i^L = \min_{1\le n\le N} \mu_i(n),$$

and let $\mu_i(n) = \mu_i^U$ for all $n > N$. Suppose the visit ratios are adequately normalized such that $\nu_i < \mu_i^L$ for all i. Let $\{Y_i\ (i=1,\dots,M)\}$ be a set of independent random variables with the common support $S = \{0, 1, \dots, N, \dots\}$, and equilibrium rates $r_{Y_i}(n) = \mu_i(n)/\nu_i\ (i=1,\dots,M)$. It is known (Shanthikumar and Yao, 1986) that the throughput of the CQN described above, TH(N), is equal to the equilibrium rate of $\sum_{i=1}^M Y_i$. That is, for any $N \ge 1$,

$$\mathrm{TH}(N) = P\left[\sum_{i=1}^{M} Y_i = N-1\right] \Big/ P\left[\sum_{i=1}^{M} Y_i = N\right]. \tag{4.1}$$

Now, if we write TH(N) as TH$(\{\mu_i(n)\})$, where $\{\mu_i(n)\}$ denotes the set of queue-length dependent service rates of the CQN, and define accordingly TH$\{\mu_i^{\mathrm{U}}\}$ and TH$\{\mu_i^{\mathrm{L}}\}$ (the throughputs of two CQN's with fixed service rates, $\{\mu_i^{\mathrm{U}}\}$ and $\{\mu_i^{\mathrm{L}}\}$, respectively), then based on the results developed earlier in this paper, we have

$$\mathrm{TH}\{\mu_i^{\mathrm{L}}\} \le \mathrm{TH}\{\mu_i(n)\} \le \mathrm{TH}\{\mu_i^{\mathrm{U}}\}. \tag{4.2}$$

The existing bounds for CQN's with fixed service rates can then be used to bound TH$\{\mu_i^{\mathrm{L}}\}$ and TH$\{\mu_i^{\mathrm{U}}\}$, and hence TH$\{\mu_i(n)\}$.

In some special cases, e.g., when the stations have multiple parallel servers, bounds tighter than those in (4.2) can be developed. For details, refer to Shanthikumar and Yao (1985).

Acknowledgment

The authors would like to thank the referees for their suggestions which helped to improve the presentation of this paper.

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