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Some new solitonary solutions of the modified Benjamin–Bona–Mahony equation

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ABSTRACT

In this paper, we use the exp-function method to construct some new soliton solutions of the Benjamin–Bona–Mahony and modified Benjamin–Bona–Mahony equations. These equations have important and fundamental applications in mathematical physics and engineering sciences. The exp-function method is used to find the soliton solution of a wide class of nonlinear evolution equations with symbolic computation. This method provides the concise and straightforward solution in a very easier way. The results obtained in this paper can be viewed as a refinement and improvement of the previously known results.

1. Introduction

Many phenomena in engineering and applied sciences are modeled by nonlinear evolution equations. Solitonary solutions of nonlinear evolution equations provide better understanding of the physical mechanism of phenomena. Nonlinear evolution equations also characterize the wave phenomena in fluid dynamics, hydro magnetic waves in cold plasma, acoustic waves in crystals, elastic media, optical fiber and some other branches of engineering and applied sciences; see [1–20] and the references therein. A substantial amount of work has been invested for solving such models. Several analytical techniques for solving nonlinear evolution equations have been presented, such as the inverse scattering method, the perturbation method, the sine–cosine method, the homotopy perturbation method, Backlund transformation, Hirota's method, Darboux transformation, Painleve expansions, extended tanh-function, the F-expansion method, the extended F-expansion method and so on.

In this paper, we consider a well-known nonlinear evolution equation which is also known as the generalized Benjamin–Bona–Mahony equation [4]:

$$u_t + u_x + au^n u_x + u_{xxt} = 0, \quad n \geq 1, \tag{1}$$

where a is a constant and n is the order of nonlinearity involved in the equation. We consider its two special cases which are widely used in nonlinear phenomena. The case $n = 1$, was proposed by Benjamin et al. [4] in 1972. It describes the unidirectional propagation of long waves in certain nonlinear dispersive media. The second case $n = 2$, is the modified Benjamin–Bona–Mahony equation. Due to the importance of the generalized Benjamin–Bona–Mahony equation, a great deal of research work has been carried out to find solitonary, periodic and exact traveling wave solutions of this equation. Several effective techniques including the homogeneous balance method [1] by Abdel Rady et al., an algebraic method [16] by Tang et al., the variable-coefficient balancing-act method [21] by Chen et al., the factorization technique [6] by Estevez

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et al. and the Jacobi elliptic function expansion method [3] by An and Zhang have been investigated for the solitonary periodic and exact traveling wave solutions of this equation. Mammeri [9] found some long time bounds for the periodic Benjamin–Bona–Mahony equation by using a transformation. Gomez et al. [7] have applied tanh–coth method for finding some new periodic and soliton solutions for the generalized BBM and Burgers-BBM equation. Recently, the variational iteration method and the exp-function method have been coupled together by Gomez and Salas [8] to construct traveling wave solutions of this equation.

He and Wu [22] developed the exp-function method and has been used extensively to seek the solitonary, periodic and compacton like solutions of nonlinear differential equations; see [2,5,22,10–15,23–25,17,18] and the references therein. The expression of the exp-function method is more general than the tanh-function method [25]; the solution procedure using Maple, Matlab or Mathematica, is of utter simplicity and the exp-function method is more convenient and effective than other analytic techniques. Noor et al. [11–15,23] have successfully applied the exp-function method for finding soliton, periodic and exact traveling wave solutions of several known partial differential equations like Boussinesq equation, good Boussinesq equation, master partial differential equation, Calogero–Degasperis–Fokas equation, Lax equation and nonlinear evolution equations.

In this paper, we use the exp-function method to construct some new solitonary solutions of the known nonlinear evolution equation such as the Benjamin–Bona–Mahony equation and its variant forms. We present the graphical representation of the soliton solution of Benjamin–Bona–Mahony equations. We hope that this technique can be applied for finding the soliton solutions of other nonlinear evolution equations. It is worth mentioning that the exp-function method has been modified using some novel ideas and techniques; see, for example, [18–20].

2. Exp-function method

We consider the general nonlinear partial differential equation of the type:

$$P\left(u, u_t, u_x, u_t, u_{xx}, u_{xt}, u_{xxt}, \ldots\right) = 0. \tag{2}$$

Using a transformation

$$\eta = kx + \omega t, \tag{3}$$

where k and ω are real constants. We can rewrite Eq. (2) in the following form of nonlinear ordinary differential equation:

$$Q\left(u, u', u'', u''', \ldots\right) = 0, \tag{4}$$

where the prime denotes derivative with respect to η. According to the exp-function method, which was developed by He and Wu [22], we assume that the wave solution can be expressed in the following form:

$$u(\eta) = \frac{\sum_{n=-c}^{d} a_n \exp[n\eta]}{\sum_{m=-p}^{q} b_m \exp[m\eta]} \tag{5}$$

where p, q, c and d are unknown parameters which can be further determined. a_n and b_m are unknown constants. We can rewrite Eq. (5) in the following equivalent form:

$$u(\eta) = \frac{a_c \exp[c\eta] + \cdots + a_{-d} \exp[-d\eta]}{b_p \exp[p\eta] + \cdots + b_{-q} \exp[-q\eta]}. \tag{6}$$

This equivalent formulation plays an important and fundamental role for finding the analytic solutions of problems. To determine the value of c and p, we balance the linear term of highest order of Eq. (4) with the highest order nonlinear term. Similarly, to determine the value of d and q, we balance the linear term of lowest order of Eq. (4) with lowest order nonlinear term.

3. Numerical applications

In this section, we apply the exp-function method to construct some new soliton solutions of the Benjamin–Bona–Mahony and modified Benjamin–Bona–Mahony equations.

Example 3.1 (*[1,3,4,6–9,16]*). Consider the generalized Benjamin–Bona–Mahony equation (1) with $n = 1$, known as Benjamin–Bona–Mahony equation:

$$u_t + u_x + a u u_x + u_{xxt} = 0. \tag{7}$$

Introducing a transformation as $\eta = kx + \omega t$, we can covert Eq. (7) into the following ordinary differential equation:

$$(\omega + k)\, u' + a\, k\, u u' + \omega\, k^2 u''' = 0. \tag{8}$$

The trial solution of Eq. (8) can be expressed in the form of (6)

$$u(\eta) = \frac{a_c \exp[\eta] + \cdots + a_{-d}\exp[-d\eta]}{b_p \exp[p\eta] + \cdots + b_{-q}\exp[-q\eta]}.$$

To determine the value of c and p, we balance the linear term of highest order of Eq. (8) with the highest order nonlinear term

$$u''' = \frac{c_1 \exp[(7p + c)\eta] + \cdots}{c_2 \exp[8p\eta] + \cdots}, \tag{9}$$

and

$$uu' = \frac{c_3 \exp[(4p + 2c)\eta] + \cdots}{c_4 \exp[6p\eta] + \cdots} = \frac{c_3 \exp[(6p + 2c)\eta] + \cdots}{c_4 \exp + [8p\eta]\cdots}, \tag{10}$$

where c_i are constants depending on a_1, k, ω etc. Balancing the highest order of exp-function in (9) and (10), we have

$$p = c. \tag{11}$$

To determine the value of d and q, we balance the linear term of lowest order of Eq. (8) with the lowest order nonlinear term

$$u''' = \frac{\cdots + d_1 \exp[(-d - 7q)\eta]}{\cdots + d_2 \exp[-8q\eta]}, \tag{12}$$

and

$$u'u'' = \frac{\cdots + d_3 \exp[(-4q - 2d)\eta]}{\cdots + d_4 \exp[-6q\eta]} = \frac{\cdots + d_3 \exp[(-2d - 6q)\eta]}{\cdots + d_4 \exp[-8q\eta]}, \tag{13}$$

where d_i are determined coefficients. Now, balancing the lowest order of exp-function in (12) and (13), we have

$$q = d. \tag{14}$$

We consider the following case.

Case 3.1.1. $(p = c = q = d = 1)$.

Taking $p = c = 1$ and $q = d = 1$ in (6), we obtain the trial solution in following form:

$$u(\eta) = \frac{a_1 \exp[\eta] + a_0 + a_{-1}\exp[-\eta]}{b_1 \exp[\eta] + a_0 + b_{-1}\exp[-\eta]}. \tag{15}$$

By substituting the trial solution (15) into Eq. (8), we have

$$\frac{1}{A}[C_3 \exp(3\eta) + C_2 \exp(2\eta) + C_1 \exp(\eta) + C_0 + C_{-1}\exp(-\eta) + C_{-2}\exp(-2\eta) + C_{-3}\exp(-3\eta)] = 0 \tag{16}$$

where $A = (b_1 \exp(\eta) + b_0 + b_{-1}\exp(-\eta))^4$, C_i $(i = -3, -2, \ldots, 2, 3)$ are coefficients that are obtained by using Maple 7.

Equating the coefficients of $\exp(n\eta)$ to zero, we obtain a system of equations

$$\{C_{-3} = 0, C_{-2} = 0, C_{-1} = 0, C_0 = 0, C_1 = 0, C_2 = 0, C_3 = 0\}. \tag{17}$$

By solving (17), we have the following values

$$\Big\{ b_1 = \frac{1}{4}\frac{b_0^2}{b_{-1}}, b_{-1} = b_{-1}, b_0 = b_0, a_{-1} = -\frac{b_{-1}(\omega + \omega k^2 + k)}{ak},$$
$$a_1 = -\frac{1}{4}\frac{b_0^2(\omega + \omega k^2 + k)}{akb_{-1}}, a_0 = \frac{b_0(-\omega + 5\omega k^2 - k)}{ak} \Big\}. \tag{18}$$

By substituting the values from (18) into trail solution (15), we have the following soliton solution $u(x, t)$ of Eq. (7):

$$u(x,t) = \frac{-\frac{1}{4}\frac{b_0^2(\omega + \omega k^2 + k)e^{(kx+\omega t)}}{akb_{-1}} + \frac{b_0(-\omega + 5\omega k^2 - k)}{ak} - \frac{b_{-1}(\omega + \omega k^2 + k)e^{(-kx-\omega t)}}{ak}}{\frac{1}{4}\frac{b_0^2 e^{(kx+\omega t)}}{b_{-1}} + b_0 + b_{-1}e^{(-kx-\omega t)}}, \tag{19}$$

where b_{-1}, b_0, a, ω and k are real numbers (Fig. 3.1).

Example 3.2 ([3,4,6–8,16,17]). Consider the generalized Benjamin–Bona–Mahony equation (1) with $n = 2$, which is also known as the modified Benjamin–Bona–Mahony equation

$$u_t + u_x + au^2u_x + u_{xxt} = 0. \tag{20}$$

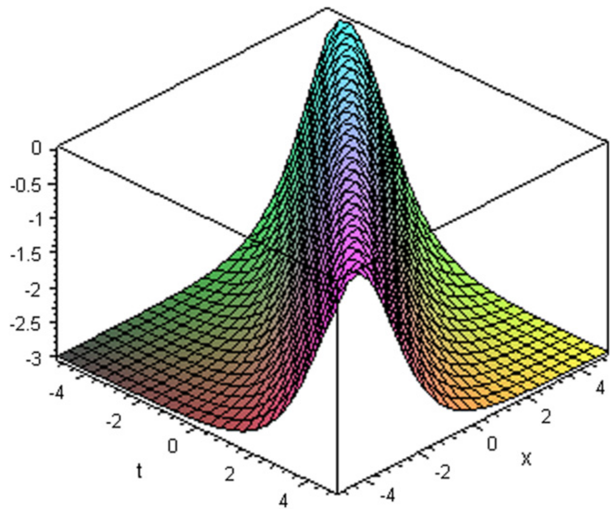

Fig. 3.1. Soliton solution of Eq. (7) with $b_{-1} = b_0 = a = k = \omega = 1$.

Introducing the same transformation as in the previous example, i.e $\eta = kx + \omega t$, we have the following ordinary differential equation from Eq. (20):

$$(\omega + k)\, u' + a\, k\, u^2 u' + \omega\, k^2 u''' = 0. \tag{21}$$

The trail solution of Eq. (21) can be expressed in the following form:

$$u(\eta) = \frac{a_c \exp[c\eta] + \cdots + a_{-d} \exp[-d\eta]}{b_p \exp[p\eta] + \cdots + b_{-q} \exp[-q\eta]}.$$

Proceeding as before, we have the following results, $p = c$ and $q = d$.

Case 3.2.1. ($p = c = q = d = 1$).

By setting $p = c = 1$ and $q = d = 1$, we have the following trial solution as obtained before in (15):

$$u\,(\eta) = \frac{a_1 \exp\,[\eta] + a_0 + a_{-1} \exp\,[-\eta]}{b_1 \exp\,[\eta] + a_0 + b_{-1} \exp\,[-\eta]}.$$

By substituting the trial solution into Eq. (21), we have

$$\frac{1}{A}\,[C_4 \exp\,(4\eta) + C_3 \exp\,(3\eta) + C_2 \exp\,(2\eta) + C_1 \exp\,(\eta) + C_0 + C_{-1} \exp\,(-\eta) + C_{-2} \exp\,(-2\eta) + C_{-3} \exp\,(-3\eta) + C_{-4} \exp\,(-4\eta)] = 0 \tag{22}$$

where $A = (b_1 \exp\,(\eta) + b_0 + b_{-1} \exp\,(-\eta))^4$, C_i $(i = -4, -3, \ldots, 3, 4)$ are coefficients that are obtained by using Maple 7.

Equating the coefficients of $\exp\,(n\eta)$ equal to zero, we have the following system of equations

$$\{C_{-4} = 0, C_{-3} = 0, C_{-2} = 0, C_{-1} = 0, C_0 = 0, C_1 = 0, C_2 = 0, C_3 = 0, C_4 = 0\}. \tag{23}$$

From (23), we have the following two sets of unknown values:

$$\left\{b_1 = -\frac{1}{24}\frac{a a_0^2 (k^2+1)}{b_{-1} k^2}, b_{-1} = b_{-1}, b_0 = 0, a_{-1} = 0, a_1 = 0, \omega = -\frac{k}{1+k^2}, a_0 = a_0\right\}, \tag{24}$$

and

$$\left\{b_1 = -\frac{1}{8}\frac{b_0^2(3k^2 b_{-1}^2 + a_{-1}^2 a k^2 - 2 a a_{-1}^2)}{a a_{-1}^2 b_{-1}(k^2+1)}, a_0 = -\frac{b_0(3k^2 b_{-1}^2 + 2a_{-1}^2 a k^2 - a a_{-1}^2)}{a a_{-1} b_{-1}(k^2+1)}, b_{-1} = b_{-1},\right.$$
$$\left. a_1 = -\frac{1}{8}\frac{b_0^2(3k^2 b_{-1}^2 + a_{-1}^2 a k^2 - 2 a a_{-1}^2)}{a a_{-1} b_{-1}^2 (k^2+1)}, \omega = -\frac{k(b_{-1}^2 + a_{-1}^2 a)}{(1+k^2) b_{-1}^2}, b_0 = b_0, a_{-1} = a_{-1}\right\}. \tag{25}$$

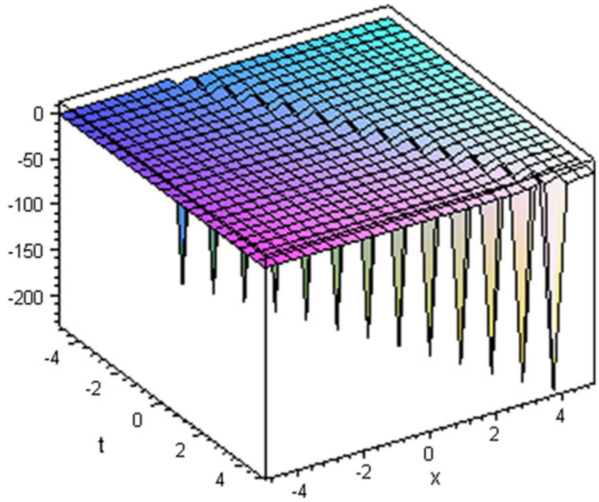

Fig. 3.2.1. Soliton solution of the modified BBM equation (20) with $b_{-1} = a_0 = a = k = 1$.

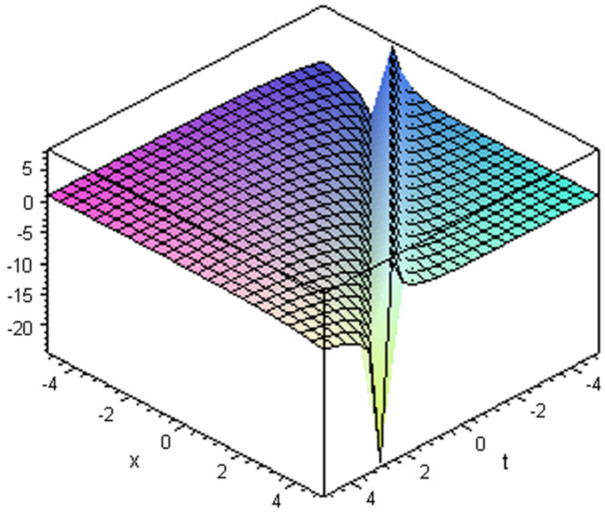

Fig. 3.2.2. Soliton solution of the modified BBM equation (20) with $b_{-1} = b_0 = a = k = a_{-1} = 1$.

By substituting the values from (24) into the trial solution, we have the following soliton solution of the modified Benjamin–Bona–Mahony equation (20):

$$u(x,t) = \frac{a_0}{-\frac{1}{24}\frac{a_0^2 a\left(k^2+1\right)e^{\left(kx-\frac{k}{k^2+1}t\right)}}{b_{-1}k^2} + b_{-1}e^{\left(-kx+\frac{k}{k^2+1}t\right)}}. \tag{26}$$

Now by using the values from (25), we have another soliton solution of the modified Benjamin–Bona–Mahony equation (20):

$$u(\eta) = \frac{-\frac{1}{8}\frac{b_0^2\left(3k^2b_{-1}^2+a_{-1}^2ak^2-2aa_{-1}^2\right)e^{\eta}}{aa_{-1}^2b_{-1}\left(k^2+1\right)} - \frac{b_0\left(3k^2b_{-1}^2+2a_{-1}^2ak^2-aa_{-1}^2\right)e^{\eta}}{aa_{-1}b_{-1}\left(k^2+1\right)} + a_{-1}e^{-\eta}}{-\frac{1}{8}\frac{b_0^2\left(3k^2b_{-1}^2+a_{-1}^2ak^2-2aa_{-1}^2\right)e^{\eta}}{aa_{-1}^2b_{-1}\left(k^2+1\right)} + b_0 + b_{-1}e^{-\eta}}, \tag{27}$$

where $\eta = kx - \frac{k\left(b_{-1}^2+a_{-1}^2a\right)}{(1+k^2)b_{-1}^2}t$, and b_{-1}, b_0, a, a_{-1} and k are real numbers (Figs. 3.2.1 and 3.2.2).

4. Conclusion

In this paper, we have applied the exp-function method with computerized symbolic computation to obtain some new soliton solutions of the Benjamin–Bona–Mahony equation and the modified Benjamin–Bona–Mahony equation. The obtained solutions satisfy both the Benjamin–Bona–Mahony equation and the modified Benjamin–Bona–Mahony equation. Graphical representation of the results is also given. It is noticed that Zhang [18,19] has applied the exp-function method for finding soliton solutions of some evolution and higher-dimensional evolution equations. We would also like to mention that Zhang et al. [20] have used the exp-function method in a new dimension for solving fractional Riccati differential equation. It is also noticed that the modified exp-function method of Zhang et al. [19,20] can easily be extended for solving a wide class of nonlinear evolution equations. We remark that the exp-function method is a very powerful mathematical tool for solving nonlinear evolution equations in particular and many other nonlinear partial differential equations.

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