Available online at www.sciencedirect.com
ScienceDirect
Computers and Mathematics with Applications 56 (2008) 1176–1187
An International Journal computers & mathematics with applications
www.elsevier.com/locate/camwa

The optimum output quantity of a duopoly market under a fuzzy decision environment

Gin-Shuh Liang[a,*], Ling-Yuan Lin[b], Chin-Feng Liu[c]

[a] *Department of Shipping and Transportation Management, National Taiwan Ocean University, 2 Pei Ning Road, Keelung, Taiwan, ROC*
[b] *Department of Business Administration, Ching Yun University, 229 Chien Hsin Road, Jung-Li, Taiwan, ROC*
[c] *Department of Finance and Taxation, Dahan Institute of Technology, 1 Shjen Street, Sincheng, Hualien, Taiwan, ROC*

Received 18 April 2005; received in revised form 9 July 2007; accepted 6 February 2008

Abstract

The main purpose of this paper is to develop a new optimum output quantity decision analysis of a duopoly market under a fuzzy decision environment. To efficiently handle the fuzziness of the decision variables, the linguistic values, subjectively represented by the trapezoidal fuzzy numbers, are used to act as the evaluation tool of decision variables such as fixed cost and unit variable cost. This paper will apply fuzzy set theory to construct an optimum output quantity decision model based on aiming for the maximum profit of a duopoly market. By using this decision model, the decision-makers' fuzzy assessments with various variables can be considered in the decision process to assure more convincing and accurate decision-making.

Keywords: Trapezoidal fuzzy number; Cournot model; Stackelberg model; Optimum output quantity; Duopoly market

1. Introduction

To maximize profit and minimize overall cost, the optimum output quantity decision has been an important issue for industrial organization. In a duopoly market, two models can be used to optimize the output quantity. One of them is the Cournot model [1–3], and the other is the Stackelberg model [4]. The Cournot and Stackelberg models are similar because, in both, competition occurs in terms of quantity. The Cournot model forms a situation in which each firm chooses its output independently. The Stackelberg model is a two-stage leadership in which the leader chooses its output quantity before the follower does. The follower then notes the leader's choice of output quantity and chooses its own output quantity [5]. Both Cournot and Stackelberg model play a vital role in such fields as economics, behavioral sciences, management, and politics [6–8].

Alepuz and Urbano [9] analyzed how learning behavior can change the outcome of competition in a duopoly industry facing demand uncertainty. They found that each firm will increase its first period quantity for the myopic choice to make price a more informative signal. Wang [10] studied and compared fee and royalty licensing in a differentiated Cournot duopoly. The study results showed licensing by a royalty may be better than a fixed-fee from the

* Corresponding author. Tel.: +886 2 2462 2192; fax: +886 2 2463 1903.
E-mail address: gsliang@mail.ntou.edu.tw (G.-S. Liang).

0898-1221/$ - see front matter
doi:10.1016/j.camwa.2008.02.018

viewpoint of the patent-holding firm. However, the consumer preferred the fixed-fee licensing. Barr and Saraceno [11] examined the effects of both environmental and organizational factors on the outcome of repeated Cournot games.

Eiselt [12] examined a facility planner's advantage resulting from knowledge of his competitor's opinion in the location Stackelberg games. Some valuable findings are obtained. For example, given perfect information, the leader always has an advantage over the follower. Lavigne et al. [13] presented a general method to study the electricity market of a country or region, under various pricing mechanisms. The study results showed monopolistic pricing chosen by the producer to minimize its costs while knowing the optimal consumers' reaction to the proposed price of electricity leads to a Stackelberg-type equilibrium. Nie [6] explored discrete time dynamic Stackelberg games with an open loop complete state. The study pointed out that both feedback and closed loop dynamic Stackelberg games with complete information are valuable in explaining some social and economic phenomena. Yang and Zhou [8] analyzed the effects of the duopolistic retailers' different competitive behaviors - Cournot Collusion and Stackelberg - on the optimal decisions of the manufacturer and the duopolistic retailers themselves. The results showed that the total profit of the duopolistic retailers who act as the followers will exceed the more powerful manufacturer's profit as long as the dissimilarity between the duopolistic retailers' market demands is sufficient.

In a duopoly market, four patterns to market structure can be formed. They are: (1) both companies A and B are followers; (2) company A is a leader and company B is a follower; (3) company B is a leader and company A is a follower; (4) both companies A and B are leaders. In pattern (2) and (3), the leader makes the optimum output quantity decision by considering the response function of the follower, then, the follower decides his optimum output quantity based on the leader's decision. In pattern (1), the optimum output quantities of companies A and B can be solved by the simultaneous-equation models constructed by their response function respectively. In pattern (4), the optimum output quantities of companies A and B can be achieved when they are recognized as leaders.

In conventional precision-based models of duopoly, decision variables are expressed in crisp values [14,15]. However, because of the insufficiency and uncertainty of information in the decision environment, it is difficult to find the exact economic assessment data, such as the prices of products, volume of activity, per unit variable costs, and total fixed costs. Therefore, the precision-based decision may be ineffective. In fact, when decision-makers decide, they assess based on their professional knowledge, experience, and subjective judgment. Linguistic values, such as "about 2000 dollars", "about 40%", are usually used to suggest their estimations. Fuzzy set theory can play a significant role in this decision-making environment.

Fuzzy set theory was introduced by Zadeh [16] to solve problems in which a source of vagueness existed. Linguistic values can be expressed fittingly by the approximate reasoning of fuzzy set theory [17]. To deal with the ambiguities involved in the process of linguistic estimations effectively, the trapezoidal fuzzy numbers are used to characterize fuzzy measure of linguistic values [18]. At the same time, combining the Cournot and Stackelberg models, a fuzzy duopoly model which considers the factors of market demand, business cost and market position is developed to answer the optimum output quantity of duopoly market under a fuzzy decision environment.

This paper is organized as follows. Section 2 introduces fuzzy set theory. The fuzzy model of duopoly is developed in Section 3. A numerical example to explain the computational process of a fuzzy model of duopoly will be presented in Section 4. Finally, conclusions are given in Section 5.

2. Fuzzy set theory

In this section, some concepts of fuzzy set theory used in this paper are briefly introduced.

2.1. Trapezoidal fuzzy numbers

A fuzzy number A [19] is described as a special fuzzy subset of real numbers whose membership function f_A satisfies five conditions. (1) f_A is a continuous mapping from $\Re$ (real line) to a closed interval [0, 1], (2) $f_A(x) = 0$, for all $x \in (-\infty, c] \cup [d, \infty)$, (3) $f_A(x)$ is strictly increasing in the interval $[c, a]$, (4) $f_A(x) = 1$, for all $x \in [a, b]$, (5) $f_A(x)$ is strictly decreasing in the interval $[b, d]$. Where c, a, b, d are real numbers, and $-\infty < c \le a \le b \le d < \infty$. For convenience, f_A^L is named as the left membership function of fuzzy number A, defining $f_A^L(x) = f_A(x)$, for all $x \in [c, a]$; and f_A^R is named as the right membership function of fuzzy number A, defining $f_A^R(x) = f_A(x)$, for $x \in [b, d]$. The fuzzy number A in $\Re$ is a trapezoidal fuzzy number if its membership function $f_A : \Re \to [0, 1]$ is

$$f_A(x) = \begin{cases} (x-c)/(a-c), & c \le x \le a \\ 1, & a \le x \le b \\ (x-d)/(b-d), & b \le x \le d \\ 0, & \text{otherwise} \end{cases} \quad (1)$$

where $-\infty < c \le a \le b \le d < \infty$.

The trapezoidal fuzzy number, as given by Eq. (1), can be represented by (c, a, b, d). The interval $[a, b]$ of trapezoidal fuzzy number A gives the maximal grade of $f_A(x)$, that is, $f_A(x) = 1$, $x \in [a, b]$; it is the most probable value of the evaluation data. In addition, c and d are the lower and upper bounds of the available area of the evaluation data. They are used to reflect the fuzziness of the evaluation data. The narrower the interval $[c, d]$ is, the lower the fuzziness of the evaluation data.

The trapezoidal fuzzy numbers are easily used and interpreted. For example, 'approximately between 800 and 810' can be represented by (795, 800, 810, 815); and the more blurry representation can be represented as (790, 800, 810, 820). In addition, an exact number, 'a' can be represented by (a, a, a, a). For example, '40' can be represented by (40, 40, 40, 40).

By the extension principle [16], the extended algebraic operations of two trapezoidal fuzzy numbers, $A_1 = (c_1, a_1, b_1, d_1)$ and $A_2 = (c_2, a_2, b_2, d_2)$, can be expressed as:

$$A_1 \oplus A_2 = (c_1 + c_2, a_1 + a_2, b_1 + b_2, d_1 + d_2), \qquad k \otimes A = (kc, ka, kb, kd), k \in \mathfrak{R}, \quad k \ge 0.$$

2.2. The ranking of fuzzy numbers

In the fuzzy model of duopoly, ranking the fuzzy profits and fuzzy total profits being considered is important and essential. Many methods of ranking fuzzy numbers have been proposed [20–27]. The graded mean integration representation method [28] not only improves on some drawbacks of existing ranking methods, but also has the advantage of easy implementation and power in problem solving. So, it will be used to characterize the presentation value of the trapezoidal fuzzy number and rank the fuzzy profits and fuzzy total profits.

Let $A_i = (c_i, a_i, b_i, d_i)$, $i = 1, 2, \ldots, n$, be n trapezoidal fuzzy numbers. The graded mean integration representation $P(A_i)$ of A_i is

$$P(A_i) = \frac{c_i + 2a_i + 2b_i + d_i}{6}. \quad (2)$$

Let $P(A_i)$ and $P(A_j)$ be the graded mean integration representations of A_i and A_j, respectively. Define that

$$\begin{aligned} A_i > A_j &\quad \text{if and only if } P(A_i) > P(A_j), \\ A_i < A_j &\quad \text{if and only if } P(A_i) < P(A_j), \\ A_i = A_j &\quad \text{if and only if } P(A_i) = P(A_j). \end{aligned}$$

3. Fuzzy model of duopoly

Notation

- q_i: The output quantity of company i.
- Q: The total output quantity of duopoly market.
- $P(Q)$: The demand price under Q.
- FTC_i: The fuzzy cost function of company i.
- f_i: The fuzzy fixed cost of company i.
- c_i: The fuzzy unit variable cost of company i.
- $FT\pi_i$: The fuzzy profit function of company i.
- $q_{F_{ik}}$: The optimum output quantity of company (follower) i in pattern k.
- $q_{L_{ik}}$: The optimum output quantity of company (leader) i in pattern k.
- P_k: The market equilibrium price in pattern k.

$FT\pi_{F_{ik}}$: The fuzzy maximum profit of company (follower) i in pattern k.
$FT\pi_{L_{ik}}$: The fuzzy maximum profit of company (leader) i in pattern k.

Let q_A and q_B represent the output quantities of companies A and B, respectively. Let Q be the total output quantity of duopoly market. That is, $Q = q_A + q_B$. Suppose the demand price $P(Q)$ is

$$P(Q) = \alpha + \beta Q = \alpha + \beta (q_A + q_B). \tag{3}$$

In addition, we define the fuzzy cost functions of companies A and B, represented by FTC_A and FTC_B, as follows.

$$FTC_A = f_A \oplus c_A \otimes q_A \tag{4}$$
$$FTC_B = f_B \oplus c_B \otimes q_B \tag{5}$$

where, f_A and f_B stand for the fuzzy fixed costs, c_A and c_B represent the fuzzy unit variable costs, and

$$f_A = (f_A^1, f_A^2, f_A^3, f_A^4), \qquad c_A = (c_A^1, c_A^2, c_A^3, c_A^4),$$
$$f_B = (f_B^1, f_B^2, f_B^3, f_B^4), \qquad c_B = (c_B^1, c_B^2, c_B^3, c_B^4).$$

Then the fuzzy profit functions of companies A and B, represented by $FT\pi_A$ and $FT\pi_B$, can be calculated by Eqs. (6) and (7), respectively.

$$\begin{aligned} FT\pi_A &= TR_A \Theta FTC_A \\ &= p(Q) q_A \Theta FTC_A \\ &= (\alpha q_A + \beta q_A^2 + \beta q_A q_B) \Theta (f_A^1 + c_A^1 q_A, f_A^2 + c_A^2 q_A, f_A^3 + c_A^3 q_A, f_A^4 + c_A^4 q_A) \\ &= ((\alpha - c_A^4) q_A + \beta q_A q_B + \beta q_A^2 - f_A^4, (\alpha - c_A^3) q_A + \beta q_A q_B + \beta q_A^2 - f_A^3, \\ &\quad (\alpha - c_A^2) q_A + \beta q_A q_B + \beta q_A^2 - f_A^2, (\alpha - c_A^1) q_A + \beta q_A q_B + \beta q_A^2 - f_A^1) \end{aligned} \tag{6}$$

and

$$\begin{aligned} FT\pi_A &= P(Q) q_B \Theta FTC_B \\ &= (\alpha + \beta (q_A + q_B)) q_B \Theta (f_B^1 + c_B^1 q_B, f_B^2 + c_B^2 q_B, f_B^3 + c_B^3 q_B, f_B^4 + c_B^4 q_B) \\ &= ((\alpha - c_B^4) q_B + \beta q_A q_B + \beta q_B^2 - f_B^4, (\alpha - c_B^3) q_B + \beta q_A q_B + \beta q_B^2 - f_B^3, \\ &\quad (\alpha - c_B^2) q_B + \beta q_A q_B + \beta q_B^2 - f_B^2, (\alpha - c_B^1) q_B + \beta q_A q_B + \beta q_B^2 - f_B^1). \end{aligned} \tag{7}$$

By Eq. (2), the graded mean integration representation $P(FT\pi_A)$ of $FT\pi_A$ is

$$\begin{aligned} P(FT\pi_A) &= \frac{\left[\left(\alpha - c_A^4\right) q_A + \beta q_A q_B + \beta q_A^2 - f_A^4\right] + 2\left[\left(\alpha - c_A^3\right) q_A + \beta q_A q_B + \beta q_A^2 - f_A^3\right]}{6} \\ &\quad + \frac{2\left[\left(\alpha - c_A^2\right) q_A + \beta q_A q_B + \beta q_A^2 - f_A^2\right] + \left[\left(\alpha - c_A^1\right) q_A + \beta q_A q_B + \beta q_A^2 - f_A^1\right]}{6} \\ &= \left[\alpha - \frac{c_A^1 + 2c_A^2 + 2c_A^3 + c_A^4}{6}\right] q_A + \beta q_A q_B + \beta q_A^2 - \frac{f_A^1 + 2f_A^2 + 2f_A^3 + f_A^4}{6}. \end{aligned}$$

Similarly, the graded mean integration representation of $FT\pi_B$ is

$$P(FT\pi_B) = \left[\alpha - \frac{c_B^1 + 2c_B^2 + 2c_B^3 + c_B^4}{6}\right] q_B + \beta q_A q_B + \beta q_B^2 - \frac{f_B^1 + 2f_B^2 + 2f_B^3 + f_B^4}{6}.$$

Pattern 1: *Both companies A and B are followers.*

Step 1: Find the response functions of companies A and B.

By considering the maximum profit of company A, we can calculate the first-order partial of $P(FT\pi_A)$ versus q_A and suppose the derivative result equals zero. Then the response function of company A can be found. That is, let

$$\frac{\partial P(FT\pi_A)}{\partial q_A} = \alpha + 2\beta q_A + \beta q_B - \frac{c_A^1 + 2c_A^2 + 2c_A^3 + c_A^4}{6} = 0. \tag{8}$$

Then, by Eq. (8), we can find the response function of Company A.

$$q_A = \frac{c_A^1 + 2c_A^2 + 2c_A^3 + c_A^4 - 6\alpha - 6\beta q_B}{12\beta}. \tag{9}$$

Similarly, we can find the response function of company B.

$$q_B = \frac{c_B^1 + 2c_B^2 + 2c_B^3 + c_B^4 - 6\alpha - 6\beta q_A}{12\beta}. \tag{10}$$

Step 2: By Eqs. (9) and (10), the optimum output quantities of companies A and B, represented by $q_{F_{A1}}$ and $q_{F_{B1}}$, can be found. That is

$$q_{F_{A1}} = \frac{2c_A^1 + 4c_A^2 + 4c_A^3 + 2c_A^4 - c_B^1 - 2c_B^2 - 2c_B^3 - c_B^4 - 6\alpha}{18\beta} \tag{11}$$

$$q_{F_{B1}} = \frac{2c_B^1 + 4c_B^2 + 4c_B^3 + 2c_B^4 - c_A^1 - 2c_A^2 - 2c_A^3 - c_A^4 - 6\alpha}{18\beta}. \tag{12}$$

Step 3: By taking Eqs. (11) and (12) into Eq. (3), we can find the market equilibrium price. That is,

$$P_1 = \frac{c_A^1 + 2c_A^2 + 2c_A^3 + c_A^4 + c_B^1 + 2c_B^2 + 2c_B^3 + c_B^4 + 6\alpha}{18\beta}. \tag{13}$$

Step 4: By taking $q_{F_{A1}}$ and $q_{F_{B1}}$ into Eqs. (6) and (7), the fuzzy maximum profits $FT\pi_{F_{A1}}$ and $FT\pi_{F_{B1}}$ of companies A and B can be found.

$$\begin{aligned} FT\pi_{F_{A1}} = {} & ((\alpha - c_A^4)q_{F_{A1}} + \beta q_{F_{A1}} q_{F_{B1}} + \beta q_{F_{A1}}^2 - f_A^4, (\alpha - c_A^3)q_{F_{A1}} + \beta q_{F_{A1}} q_{F_{B1}} + \beta q_{F_{A1}}^2 - f_A^3, \\ & (\alpha - c_A^2)q_{F_{A1}} + \beta q_{F_{A1}} q_{F_{B1}} + \beta q_{F_{A1}}^2 - f_A^2, (\alpha - c_A^1)q_{F_{A1}} + \beta q_{F_{A1}} q_{F_{B1}} + \beta q_{F_{A1}}^2 - f_A^1) \end{aligned} \tag{14}$$

$$\begin{aligned} FT\pi_{F_{B1}} = {} & ((\alpha - c_B^4)q_{F_{B1}} + \beta q_{F_{A1}} q_{F_{B1}} + \beta q_{F_{B1}}^2 - f_B^4, (\alpha - c_B^3)q_{F_{B1}} + \beta q_{F_{A1}} q_{F_{B1}} + \beta q_{F_{B1}}^2 - f_B^3, \\ & (\alpha - c_B^2)q_{F_{B1}} + \beta q_{F_{A1}} q_{F_{B1}} + \beta q_{F_{B1}}^2 - f_B^2, (\alpha - c_B^1)q_{F_{B1}} + \beta q_{F_{A1}} q_{F_{B1}} + \beta q_{F_{B1}}^2 - f_B^1). \end{aligned} \tag{15}$$

Step 5: By Eqs. (14) and (15), the fuzzy total profit $FT\pi_1$ in pattern 1 is as below.

$$FT\pi_1 = FT\pi_{F_{A1}} \oplus FT\pi_{F_{B1}}. \tag{16}$$

Pattern 2: *Company A is a leader and company B is a follower.*

Step 1: By taking the response function of company B, represented by Eq. (10), into Eq. (6), the fuzzy profits function $FT\pi_A$ is as below:

$$\begin{aligned} FT\pi_A = {} & \left(\left(\frac{6\alpha - 12c_A^4 + c_B^1 + 2c_B^2 + 2c_B^3 + c_B^4}{12} \right) q_A + \frac{\beta}{2} q_A^2 - f_A^4, \right. \\ & \left(\frac{6\alpha - 12c_A^3 + c_B^1 + 2c_B^2 + 2c_B^3 + c_B^4}{12} \right) q_A + \frac{\beta}{2} q_A^2 - f_A^3, \\ & \left(\frac{6\alpha - 12c_A^2 + c_B^1 + 2c_B^2 + 2c_B^3 + c_B^4}{12} \right) q_A + \frac{\beta}{2} q_A^2 - f_A^2, \\ & \left. \left(\frac{6\alpha - 12c_A^1 + c_B^1 + 2c_B^2 + 2c_B^3 + c_B^4}{12} \right) q_A + \frac{\beta}{2} q_A^2 - f_A^1 \right). \end{aligned} \tag{17}$$

Then the graded mean integration representation $P\,(FT\pi_A)$ of $FT\pi_A$ is

$$P\,(FT\pi_A) = \frac{\left(6\alpha + c_B^1 + 2c_B^2 + 2c_B^3 + c_B^4 - 2c_A^1 - 4c_A^2 - 4c_A^3 - 2c_A^4\right) q_A}{12}$$

$$+\frac{6\beta q_A^2 - 2f_A^1 - 4f_A^2 - 4f_A^3 - 2f_A^4}{12}.$$

Step 2: Let $q_{L_{A2}}$ be the optimum quantity of company A. It can be found by taking first-order partial of $P(FT\pi_A)$ versus q_A and assuming the derivative result equals zero. That is, let

$$\frac{\partial P(FT\pi_A)}{\partial q_A} = \frac{\left(6\alpha + c_B^1 + 2c_B^2 + 2c_B^3 + c_B^4 - 2c_A^1 - 4c_A^2 - 4c_A^3 - 2c_A^4\right) + 12\beta q_A}{12} = 0. \tag{18}$$

By solving the equation above, we can find $q_{L_{A2}}$ as below:

$$q_{L_{A2}} = \frac{2c_A^1 + 4c_A^2 + 4c_A^3 + 2c_A^4 - c_B^1 - 2c_B^2 - 2c_B^3 - c_B^4 - 6\alpha}{12\beta}. \tag{19}$$

Step 3: By taking Eq. (19) into Eq. (17), we can find the fuzzy maximum profit $FT\pi_{L_{A2}}$ of company A.

$$\begin{aligned} FT\pi_{L_{A2}} = \Bigg(&\left(\frac{6\alpha - 12c_A^4 + c_B^1 + 2c_B^2 + 2c_B^3 + c_B^4}{12}\right) q_{L_{A2}} + \frac{\beta}{2} q_{L_{A2}}^2 - f_A^4,\\ &\left(\frac{6\alpha - 12c_A^3 + c_B^1 + 2c_B^2 + 2c_B^3 + c_B^4}{12}\right) q_{L_{A2}} + \frac{\beta}{2} q_{L_{A2}}^2 - f_A^3,\\ &\left(\frac{6\alpha - 12c_A^2 + c_B^1 + 2c_B^2 + 2c_B^3 + c_B^4}{12}\right) q_{L_{A2}} + \frac{\beta}{2} q_{L_{A2}}^2 - f_A^2,\\ &\left(\frac{6\alpha - 12c_A^1 + c_B^1 + 2c_B^2 + 2c_B^3 + c_B^4}{12}\right) q_{L_{A2}} + \frac{\beta}{2} q_{L_{A2}}^2 - f_A^1\Bigg). \end{aligned} \tag{20}$$

Step 4: By taking Eq. (19) into Eq. (10), we can find the optimum output quantity $q_{F_{B2}}$ of company B.

$$q_{F_{B2}} = \frac{3c_B^1 + 6c_B^2 + 6c_B^3 + 3c_B^4 - 2c_A^1 - 4c_A^2 - 4c_A^3 - c_A^4 - 6\alpha}{24\beta}. \tag{21}$$

Step 5: By taking Eqs. (19) and (21) into Eq. (3), the market equilibrium price P_2 in pattern 2 can be found.

$$P_2 = \frac{2c_A^1 + 4c_A^2 + 4c_A^3 + 2c_A^4 + c_B^1 + 2c_B^2 + 2c_B^3 + c_B^4 + 6\alpha}{24}. \tag{22}$$

Step 6: By taking $q_{L_{A2}}$ and $q_{F_{B2}}$ into Eq. (7), the fuzzy maximum profit $FT\pi_{L_{B2}}$ of company B in pattern 2 is as below:

$$\begin{aligned} FT\pi_{F_{B2}} = (&(\alpha - c_B^4)q_{F_{B2}} + \beta q_{L_{A2}} q_{F_{B2}} + \beta q_{F_{B2}}^2 - f_B^4, (\alpha - c_B^3)q_{F_{B2}} + \beta q_{L_{A2}} q_{F_{B2}} + \beta q_{F_{B2}}^2 - f_B^3,\\ &(\alpha - c_B^2)q_{F_{B2}} + \beta q_{L_{A2}} q_{F_{B2}} + \beta q_{F_{B2}}^2 - f_B^2, (\alpha - c_B^1)q_{F_{B2}} + \beta q_{L_{A2}} q_{F_{B2}} + \beta q_{F_{B2}}^2 - f_B^1). \end{aligned} \tag{23}$$

Step 7: Then the fuzzy total profit $FT\pi_2$ in pattern 2 is

$$FT\pi_2 = FT\pi_{L_{A2}} \oplus FT\pi_{F_{B2}}. \tag{24}$$

Pattern 3: *Company B is a leader and company A is a follower.*

By copying the analysis of pattern 2, the following results can be found:

The optimum output quantity $q_{L_{B3}}$ of company B is

$$q_{L_{B3}} = \frac{2c_B^1 + 4c_B^2 + 4c_B^3 + 2c_B^4 - c_A^1 - 2c_A^2 - 2c_A^3 - c_A^4 - 6\alpha}{12\beta}. \tag{25}$$

The fuzzy maximum profit $FT\pi_{L_{B3}}$ of company B is

$$FT\pi_{L_{B3}} = \left(\left(\frac{6\alpha - 12c_B^4 + c_A^1 + 2c_A^2 + 2c_A^3 + c_A^4}{12}\right)q_{L_{B3}} + \frac{\beta}{2}q_{L_{B3}}^2 - f_B^4,\right.$$
$$\left(\frac{6\alpha - 12c_B^3 + c_A^1 + 2c_A^2 + 2c_A^3 + c_A^4}{12}\right)q_{L_{B3}} + \frac{\beta}{2}q_{L_{B3}}^2 - f_B^3,$$
$$\left(\frac{6\alpha - 12c_B^2 + c_A^1 + 2c_A^2 + 2c_A^3 + c_A^4}{12}\right)q_{L_{B3}} + \frac{\beta}{2}q_{L_{B3}}^2 - f_B^2,$$
$$\left.\left(\frac{6\alpha - 12c_B^1 + c_A^1 + 2c_A^2 + 2c_A^3 + c_A^4}{12}\right)q_{L_{B3}} + \frac{\beta}{2}q_{L_{B3}}^2 - f_B^1\right). \quad (26)$$

Step 1: By taking Eq. (25) into Eq. (9), we can find the optimum output quantity $q_{F_{A3}}$ of company A, it can be expressed as

$$q_{F_{A3}} = \frac{3c_A^1 + 6c_A^2 + 6c_A^3 + 3c_A^4 - 2c_B^1 - 4c_B^2 - 4c_B^3 - 2c_B^4 - 6\alpha}{24\beta}. \quad (27)$$

Step 2: By taking Eqs. (25) and (27) into Eq. (3), we can find the market equilibrium price P_3 in pattern 3, that is

$$P_3 = \frac{2c_B^1 + 4c_B^2 + 4c_B^3 + 2c_B^4 + c_A^1 + 2c_A^2 + 2c_A^3 + c_A^4 + 6\alpha}{24}.$$

Step 3: By taking $q_{L_{B3}}$ and $q_{F_{A3}}$ into Eq. (6), the fuzzy maximum profit $FT\pi_{F_{A3}}$ is

$$FT\pi_{F_{A3}} = ((\alpha - c_A^4)\cdot q_{F_{A3}} + \beta q_{F_{A3}}q_{L_{B3}} + \beta q_{F_{A3}}^2 - f_A^4, (\alpha - c_A^3)\cdot q_{F_{A3}} + \beta q_{F_{A3}}q_{L_{B3}} + \beta q_{F_{A3}}^2 - f_A^3,$$
$$(\alpha - c_A^2)\cdot q_{F_{A3}} + \beta q_{F_{A3}}q_{L_{B3}} + \beta q_{F_{A3}}^2 - f_A^2, (\alpha - c_A^1)\cdot q_{F_{A3}} + \beta q_{F_{A3}}q_{L_{B3}} + \beta q_{F_{A3}}^2 - f_A^1). \quad (28)$$

Step 4: Then the fuzzy total profit $FT\pi_3$ of companies A and B is

$$FT\pi_3 = FT\pi_{F_{A3}} \oplus FT\pi_{L_{B3}}. \quad (29)$$

Pattern 4: *Both companies A and B are leaders.*

Step 1: When company A is a leader and company B is a follower, the optimum output quantities of companies A and B are as below, respectively.

$$q_{L_{A2}} = \frac{2c_A^1 + 4c_A^2 + 4c_A^3 + 2c_A^4 - c_B^1 - 2c_B^2 - 2c_B^3 - c_B^4 - 6\alpha}{12\beta}$$
$$q_{F_{B2}} = \frac{3c_B^1 + 6c_B^2 + 6c_B^3 + 3c_B^4 - 2c_A^1 - 4c_A^2 - 4c_A^3 - c_A^4 - 6\alpha}{24\beta}.$$

And the fuzzy maximum profits $FT\pi_{L_{A2}}$ and $FT\pi_{F_{B2}}$ are expressed as Eqs. (20) and (23).

Step 2: When company B is a leader and company A is a follower, the optimum output quantities of the two companies are as below:

$$q_{L_{B3}} = \frac{2c_B^1 + 4c_B^2 + 4c_B^3 + 2c_B^4 - c_A^1 - 2c_A^2 - 2c_A^3 - c_A^4 - 6\alpha}{12\beta}$$
$$q_{F_{A3}} = \frac{3c_A^1 + 6c_A^2 + 6c_A^3 + 3c_A^4 - 2c_B^1 - 4c_B^2 - 4c_B^3 - 2c_B^4 - 6\alpha}{24\beta}.$$

And the fuzzy maximum profits $FT\pi_{L_{A2}}$ and $FT\pi_{F_{B2}}$ are expressed as Eqs. (26) and (28).

Step 3: When both companies A and B are leaders, the optimum output quantities of the two companies are as below:

Table 1
The optimum output quantities, fuzzy maximum profits, fuzzy total profits and equilibrium market prices of the four different market structures

		Company B			
		Leader		Follower	
Company A	Leader	$FT\pi_{L_{A4}}$ $FT\pi_{L_{B4}}$ $FT\pi_4$	$q_{L_{A4}}$ $q_{L_{B4}}$ P_4	$FT\pi_{L_{A2}}$ $FT\pi_{F_{B2}}$ $FT\pi_2$	$q_{L_{A2}}$ $q_{L_{B2}}$ P_2
	Follower	$FT\pi_{F_{A3}}$ $FT\pi_{L_{B3}}$ $FT\pi_3$	$q_{L_{A3}}$ $q_{L_{B3}}$ P_3	$FT\pi_{F_{A1}}$ $FT\pi_{F_{B1}}$ $FT\pi_1$	$q_{F_{A1}}$ $q_{F_{B1}}$ P_1

The optimum output quantity of company A is

$$q_{L_{A4}} = q_{L_{A2}} = \frac{2c_A^1 + 4c_A^2 + 4c_A^3 + 2c_A^4 - c_B^1 - 2c_B^2 - 2c_B^3 - c_B^4 - 6\alpha}{12\beta}. \tag{30}$$

The optimum output quantity of company B is

$$q_{L_{B4}} = q_{L_{B3}} = \frac{2c_B^1 + 4c_B^2 + 4c_B^3 + 2c_B^4 - c_A^1 - 2c_A^2 - 2c_A^3 - c_A^4 - 6\alpha}{12\beta}. \tag{31}$$

By taking $q_{L_{A4}}$ and $q_{L_{B4}}$ into Eqs. (6) and (7), the fuzzy maximum profits and fuzzy total profit, $FT\pi_{L_{A4}}$, $FT\pi_{L_{B4}}$ and $FT\pi_4$, are as below:

$$\begin{aligned}
FT\pi_{L_{A4}} &= ((\alpha - c_A^4)q_{L_{A4}} + \beta q_{L_{A4}}q_{L_{B4}} + \beta q_{L_{A4}}^2 - f_A^4, (\alpha - c_A^3)q_{L_{A4}} + \beta q_{L_{A4}}q_{L_{B4}} + \beta q_{L_{A4}}^2 - f_A^3,\\
&\quad (\alpha - c_A^2)q_{L_{A4}} + \beta q_{L_{A4}}q_{L_{B4}} + \beta q_{L_{A4}}^2 - f_A^2, (\alpha - c_A^1)q_{L_{A4}} + \beta q_{L_{A4}}q_{L_{B4}} + \beta q_{L_{A4}}^2 - f_A^1)\\
FT\pi_{L_{B4}} &= ((\alpha - c_B^4)q_{L_{B4}} + \beta q_{L_{A4}}q_{L_{B4}} + \beta q_{L_{B4}}^2 - f_B^4, (\alpha - c_B^3)q_{L_{B4}} + \beta q_{L_{A4}}q_{L_{B4}} + \beta q_{L_{B4}}^2 - f_B^3,\\
&\quad (\alpha - c_B^2)q_{L_{B4}} + \beta q_{L_{A4}}q_{L_{B4}} + \beta q_{L_{B4}}^2 - f_B^2, (\alpha - c_B^1)q_{L_{B4}} + \beta q_{L_{A4}}q_{L_{B4}} + \beta q_{L_{B4}}^2 - f_B^1) \qquad (32)\\
FT\pi_4 &= FT\pi_{L_{A4}} \oplus FT\pi_{L_{B4}}.
\end{aligned}$$

And by taking Eqs. (30) and (31) into Eq. (3), we can find the market equilibrium price P_4 in pattern 4, that is,

$$P_4 = \frac{c_A^1 + 2c_A^2 + 2c_A^3 + c_A^4 + c_B^1 + 2c_B^2 + 2c_B^3 + c_B^4}{12}.$$

By understanding the analysis of four cases described above, we can obtain the optimum output quantities, fuzzy maximum profits, fuzzy total profits and market equilibrium prices of the four different market structures. These results are shown as Table 1.

4. Numerical example

In this section, a hypothetical optimum output quantity decision problem of duopoly market is designed to explain the computational process of the fuzzy Stackelberg model proposed here.

Assume that companies A and B need to make decision of the optimum output quantities in a duopoly market. The estimated fuzzy fixed costs, fuzzy variable costs are given as follows:

For company A, the fuzzy fixed cost is about 65,000 dollars, and the fuzzy variable cost a unit is about 200 dollars. That is, $f_A = (60000, 65000, 65000, 68000)$ and $c_A = (150, 200, 200, 230)$.

For company B, the fuzzy fixed cost is about 60,000 dollars, and the fuzzy variable cost a unit is about 250 dollars. That is, $f_B = (55000, 60000, 60000, 63000)$, and $c_A = (200, 250, 250, 280)$.

Then, the fuzzy total costs of companies A and B are

$$FTC_A = (60000 + 150q_A, 65000 + 200q_A, 65000 + 200q_A, 68000 + 230q_A)$$
$$FTC_B = (55000 + 200q_B, 60000 + 250q_B, 60000 + 250q_B, 63000 + 280q_B).$$

In addition, let demand price P be $P = 2100 - 2\,(q_A + q_B)$.

Pattern 1: *Both companies A and B are followers.*

In this pattern, the optimum output quantities, fuzzy maximum profits, fuzzy total profit and market equilibrium price of the two companies are as below:

$$q_{F_{A1}} = 304, \qquad q_{F_{B1}} = 308, \qquad P_1 = 848$$
$$FT\pi_{F_{A1}} = (128188, 140313, 140313, 160521);$$
$$FT\pi_{F_{B1}} = (120458, 132708, 132708, 153125);$$
$$FT\pi_1 = (248646, 273021, 273021, 313646).$$

Pattern 2: *Company A is a leader and company B is a follower.*

In this pattern, the optimum output quantities, fuzzy maximum profits, fuzzy total profit and market equilibrium price of the two companies are as below:

$$q_{L_{A2}} = 488, \qquad q_{F_{B2}} = 219, \qquad P_2 = 685$$
$$FT\pi_{L_{A2}} = (154192, 171842, 171842, 201258);$$
$$FT\pi_{F_{B2}} = (25763, 35338, 35338, 51296);$$
$$FT\pi_2 = (179954, 207179, 207179, 252554).$$

Pattern 3: *Company B is a leader and company A is a follower.*

In this pattern, the optimum output quantities, fuzzy maximum profits, fuzzy total profit and market equilibrium price of the two companies are as below:

$$q_{L_{B3}} = 451, \qquad q_{F_{A3}} = 250, \qquad P_3 = 698$$
$$FT\pi_{L_{B3}} = (125223, 141748, 141748, 169290);$$
$$FT\pi_{F_{A3}} = (49070, 59582, 59582, 77103);$$
$$FT\pi_3 = (174293, 201330, 201330, 246393).$$

Pattern 4: *Both companies A and B are leaders.*

When both companies A and B are leaders, they decide the optimum output quantity is based on profit itself.

1. Company A is a leader and company B is a follower:

$$q_{L_{A2}} = 488, \qquad q_{F_{B2}} = 219$$
$$FT\pi_{L_{A2}} = (154192, 171842, 171842, 201258);$$
$$FT\pi_{F_{B2}} = (25763, 35338, 35338, 51296);$$
$$FT\pi_2 = (179954, 207179, 207179, 252554).$$

2. Company B is a leader and company A is a follower:

$$q_{L_{B3}} = 451, \qquad q_{F_{A3}} = 250$$
$$FT\pi_{L_{B3}} = (125223, 141748, 141748, 169290);$$
$$FT\pi_{F_{A3}} = (49070, 59582, 59582, 77103);$$
$$FT\pi = (174293, 201330, 201330, 246393).$$

3. When both companies A and B are leaders, they want their fuzzy profit to be maximal.
   To pursue the maximum individual fuzzy profit, they want to be leaders.
   The company A will produce the output quantity $q_{L_{A4}} = 488$.
   And company B will produce the output quantity $q_{L_{B4}} = 451$.

$$FT\pi_{L_{A4}} = (-71904, -54264, -54264, -24864);$$
$$FT\pi_{L_{B4}} = (-91304, -73664, -73664, -44264);$$
$$FT\pi_4 = (-163208, -127928, -127928, -69128);$$
$$P_4 = 222.$$

Table 2
The optimum output quantities, fuzzy maximum profits, fuzzy total profits and equilibrium market prices of four different market structures

		Company B	
		Leader	Follower
Company A	Leader	$FT\pi_{L_{A4}} = (-71904, -54264, -54264, -24864)$ $FT\pi_{L_{B4}} = (-91304, -73664, -73664, -44264)$ $FT\pi_4 = (-163208, -127928, -127928, -69128)$ $q_{L_{A4}} = 488;\ q_{L_{B4}} = 451;\ P_4 = 222$	$FT\pi_{L_{A2}} = (154192, 171842, 171842, 201258)$ $FT\pi_{F_{B2}} = (25763, 35338, 35338, 51296)$ $FT\pi_2 = (179954, 207179, 207179, 252554)$ $q_{L_{A2}} = 488;\ q_{F_{B2}} = 219;\ P_2 = 685$
	Follower	$FT\pi_{L_{B3}} = (125223, 141748, 141748, 169290)$ $FT\pi_{F_{A3}} = (49070, 59582, 59582, 77103)$ $FT\pi_3 = (174293, 201330, 201330, 246393)$ $q_{L_{B3}} = 451;\ q_{F_{A3}} = 250;\ P_3 = 698$	$FT\pi_{F_{A1}} = (128188, 140313, 140313, 160521)$ $FT\pi_{F_{B1}} = (120458, 132708, 132708, 153125)$ $FT\pi_1 = (248646, 273021, 273021, 313646)$ $q_{F_{A1}} = 304;\ q_{F_{B1}} = 308;\ P_1 = 848$

By understanding the analysis results of four cases described above, we can find the optimum output quantities, fuzzy maximum profits, fuzzy total profits and market equilibrium prices of four different market structures. They are expressed as Table 2.

For company A, the fuzzy profits of four different market structures are as below:

$$FT\pi_{F_{A1}} = (128188, 140131, 140131, 160521);$$
$$FT\pi_{L_{A2}} = (154192, 171842, 171842, 201258);$$
$$FT\pi_{F_{A3}} = (49070, 59582, 59582, 77103);$$
$$FT\pi_{L_{A4}} = (-71094, -54264, -54264, -24864).$$

Based on the graded mean integration representation method,

$$P\left(FT\pi_{F_{A1}}\right) = \frac{128{,}188 + 2 \times 140{,}313 + 2 \times 140{,}313 + 160,521}{6} = 141{,}660$$
$$P\left(FT\pi_{L_{A2}}\right) = \frac{154{,}192 + 2 \times 171{,}842 + 2 \times 171{,}842 + 210,258}{6} = 173{,}803$$
$$P\left(FT\pi_{F_{A3}}\right) = \frac{49{,}070 + 2 \times 59{,}582 + 2 \times 59{,}582 + 77,103}{6} = 60{,}750$$
$$P\left(FT\pi_{L_{A4}}\right) = \frac{(-71{,}904) + 2 \times (-54{,}264) + 2 \times (-54{,}264) + (-24{,}864)}{6} = -52{,}304.$$

Since $P\left(FT\pi_{L_{A2}}\right) > P\left(FT\pi_{F_{A1}}\right) > P\left(FT\pi_{F_{A3}}\right) > P\left(FT\pi_{L_{A4}}\right)$,

So $FT\pi_{L_{A2}} > FT\pi_{F_{A1}} > FT\pi_{F_{A3}} > FT\pi_{L_{A4}}$.

For company B, the fuzzy profits of four different market structures are as below:

$$FT\pi_{F_{B1}} = (120458, 132708, 132708, 153125);$$
$$FT\pi_{F_{B2}} = (25763, 35338, 35338, 51296);$$
$$FT\pi_{L_{B3}} = (125223, 141748, 141748, 169290);$$
$$FT\pi_{L_{B4}} = (-91304, -73664, -73664, -44264).$$

Based on the graded mean integration representation method,

$$P\left(FT\pi_{F_{B1}}\right) = \frac{120{,}458 + 2 \times 132{,}708 + 2 \times 132{,}708 + 153,125}{6} = 134{,}069$$
$$P\left(FT\pi_{F_{B2}}\right) = \frac{25{,}763 + 2 \times 35{,}338 + 2 \times 35{,}338 + 51,296}{6} = 36{,}402$$
$$P\left(FT\pi_{L_{B3}}\right) = \frac{125{,}223 + 2 \times 141{,}748 + 2 \times 141{,}748 + 169,290}{6} = 143{,}584$$
$$P\left(FT\pi_{L_{B4}}\right) = \frac{(-91{,}304) + 2 \times (-73{,}664) + 2 \times (-73{,}664) + (-44{,}264)}{6} = -71{,}704.$$

Since $P(FT\pi_{L_{B3}}) > P(FT\pi_{F_{B1}}) > P(FT\pi_{F_{B2}}) > P(FT\pi_{L_{B4}})$,

So $FT\pi_{L_{B3}} > FT\pi_{F_{B1}} > FT\pi_{F_{B2}} > FT\pi_{L_{B4}}$.

The fuzzy total profits of companies A and B are as below:

$$FT\pi_1 = (248646, 273021, 273021, 313646);$$
$$FT\pi_2 = (179954, 207179, 207179, 252554);$$
$$FT\pi_3 = (174293, 201330, 201330, 246393);$$
$$FT\pi_4 = (-163208, -127928, -127928, -69128).$$

Based on the graded mean integration representation method,

$$P\,(FT\pi_1) = \frac{248{,}646 + 2 \times 273{,}021 + 2 \times 273{,}021 + 313,646}{6} = 275{,}729$$
$$P\,(FT\pi_2) = \frac{179{,}954 + 2 \times 207{,}179 + 2 \times 207{,}179 + 252,554}{6} = 210{,}204$$
$$P\,(FT\pi_3) = \frac{174{,}293 + 2 \times 201{,}330 + 2 \times 201{,}330 + 246,393}{6} = 204{,}334$$
$$P\,(FT\pi_4) = \frac{(-163{,}208) + 2 \times (-127{,}928) + 2 \times (-127{,}928) + (-69{,}128)}{6} = -124{,}008.$$

Since $P(FT\pi_1) > P(FT\pi_2) > P(FT\pi_3) > P(FT\pi_4)$.

So $FT\pi_1 > FT\pi_2 > FT\pi_3 > FT\pi_4$.

Based on the analysis results stated above, we find:

1. When both companies A and B are followers. The equilibrium market price and the total fuzzy profit are maximal. That is,

$$848 > 698 > 685 > 222$$
$$FT\pi_1 > FT\pi_2 > FT\pi_3 > FT\pi_4.$$

And, the company individual fuzzy profit is almost maximal.

Since, for company A: $FT\pi_{L_{A2}} > FT\pi_{F_{A1}} > FT\pi_{F_{A3}} > FT\pi_{L_{A4}}$.

And for company B: $FT\pi_{L_{B3}} > FT\pi_{L_{B1}} > FT\pi_{F_{B2}} > FT\pi_{L_{B4}}$.

2. When both companies A and B are leaders and both companies produce many products, the equilibrium market price and total fuzzy profit are minimal. That is,

$$222 < 685 < 698 < 848$$
$$FT\pi_1 < FT\pi_2 < FT\pi_3 < FT\pi_4.$$

And, the company individual fuzzy profit will be the lowest.

Since, for company A: $FT\pi_{L_{A4}} > FT\pi_{F_{A3}} > FT\pi_{F_{A1}} > FT\pi_{L_{A2}}$.

And, for company B: $FT\pi_{L_{B4}} > FT\pi_{L_{B2}} > FT\pi_{F_{B1}} > FT\pi_{L_{B3}}$.

Therefore, the best decision is 'both companies A and B are followers'.

5. Conclusions

In business decision analysis, obviously, the difficulty in finding precise assessment data such as fixed cost, unit variable cost of product caused the vagueness and uncertainty of the relevant information in deciding. So, the conventional precision-based decision analysis method is less effective in suggesting available information in such an imprecise and fuzzy decision environment.

This paper improves the estimating methods in which we can send the estimation of decision variables by the linguistic values characterized by trapezoidal fuzzy numbers. Further, in this paper we examined an analytical method of effectively carrying out the "optimum output quantity" decision of duopoly market under a fuzzy environment. The proposed fuzzy model of duopoly not only considered the factors of market demand, business cost and market position but also could be used to analyze the interaction tactic behavior between enterprises. Besides, the proposed model can be computerized to enable the decision-makers to achieve the best decision automatically either by conducting fuzzy or non-fuzzy assessment.

References

[1] D. John, C. Schultz, Renegotiation in a repeated Cournot duopoly, Economics Letters 47 (1995) 143–148.
[2] C. James, M. Walker, Learning to play Cournot duopoly strategies, Journal of Economic Behavior and Organization 36 (1998) 141–161.
[3] Morton, A cake for Cournot, Economics Letters 70 (2001) 349–355.
[4] H.V. Stackelberg, Marktform und Gleichgewicht, Juliu Springer, Vienna, 1934.
[5] M.R. Caputo, A dual vista of the Stackelberg duopoly reveals it fundamental qualitative structure, International Journal of Industrial organization 16 (1998) 333–352.
[6] P.Y. Nie, Dynamic Stackelberg games under open-loop complete information, Journal of Franklin Institute 342 (2005) 737–748.
[7] J.F. Tsai, C.P. Chu, Economic analysis of collecting parking fees by a private firm, Transportation Research Part A 40 (2006) 690–697.
[8] S.N. Yang, Y.W. Zhou, Two-echelon supply chain models: Considering duopolistic retailer's different competitive behaviors, International Journal of Production Economics 103 (2006) 104–116.
[9] M.D. Alepuz, A. Urbano, Duopoly experimentation: Cournot competition, Mathenatical Social Sciences 37 (1999) 165–188.
[10] X.H. Wang, Fee versus royalty licensing in a differentiated Cournot duopoly, Journal of Economics and Business 54 (2002) 253–266.
[11] J. Barr, F. Saraceno, Cournot competition, organization and learning, Journal of Economic & Control 29 (2005) 277–295.
[12] H.A. Eiselt, Perception and information in a competitive location model, European Journal of Operational Research 108 (1998) 94–105.
[13] D. Lavigne, R. Loulou, G. Savard, Pure competition, regulated and Stackelberg equilibria: Application to the energy system of Quebec, European Journal of Operational Research 125 (2000) 1–17.
[14] T. Au, T.P. Au, Engineering Economics for Capital Investment Analysis, Allyn Bacon Inc., Boston, 1983.
[15] F.E. Brigmann, L.C. Gapensiki, Financial Management: Theory and Practice, 7th ed., The Dryden Press, 1994.
[16] L.A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338–353.
[17] L.A. Zadeh, The concept of a linguistic variable and its application to approximate reasoning, part 1, 2 and 3, Information Sciences 8 (1975) 199–249; 9, (1976) 43–58.
[18] G.S. Liang, M.J.J. Wang, A fuzzy multi-criteria decision-making method for facility site selection, International Journal of Production of Production Research 29 (11) (1991) 2313–2330.
[19] D. Dubois, H. Prade, Operations on fuzzy numbers, The International Journal of Systems Sciences 9 (1978) 613–626.
[20] R.R. Yager, A procedure for ordering fuzzy subsets of the unit interval, Information Sciences 24 (1981) 101–143.
[21] G. Bortolan, R. Degani, A review of some methods for ranking fuzzy subsets, Fuzzy Sets and Systems 15 (1985) 1–19.
[22] S.H. Chen, Ranking fuzzy numbers with maximizing and minimizing set, Fuzzy Sets and Systems 17 (1985) 13–129.
[23] J.J. Buckley, A fuzzy ranking of fuzzy numbers, Fuzzy Sets and Systems 33 (1989) 119–121.
[24] F. Choobineh, H. Li, A new index for ranking fuzzy numbers, in: Proceeding of International Symposium on Uncertainty Modeling and Analysis, IEEE Computer Press, 1990, pp. 387–391.
[25] K. Kim, K.S. Park, Ranking fuzzy numbers with index of optimism, Fuzzy Sets and Systems 35 (1990) 143–150.
[26] T.S. Liou, M.J.J. Wang, Ranking fuzzy numbers with integral value, Fuzzy Sets and Systems 50 (1992) 247–255.
[27] O.P. Dias, Ranking alternatives using fuzzy numbers—a computational approach, Fuzzy Sets and Systems 56 (1993) 247–252.
[28] S.H. Chen, C.H. Hsieh, Representation, ranking, distance, and similarity of L-R type fuzzy number and Application, Australian Journal of Intelligent Processing System 6 (4) (2000) 217–229.