JOURNAL OF COMBINATORIAL THEORY Series A 40, 459–462 (1985)

Note

Blocking Sets in Block Designs

DAVID A. DRAKE*

Department of Mathematics, University of Florida, Gainesville, Florida 32611

Communicated by A. Barlotti

Received January 27, 1984

A lower bound is obtained for the cardinality of a blocking set in a non-symmetric block design. The known lower bound for blocking sets in symmetric block designs is proved to hold (if and) only if the blocking set is a Baer subdesign.

A *blocking set* in an incidence structure Σ is a subset B of the point set of Σ such that every line of Σ intersects both B and the complement of B. In 1970 and 1971, Bruen proved the following theorem (see Theorems 1 and 2 in [2] and Theorem 3.9 in [3]).

THEOREM 1 (Bruen). *Let B be a blocking set of a projective plane Π of order n. Then $|B| \geqslant n + n^{1/2} + 1$, and $|B| = n + n^{1/2} + 1$ if and only if B is the point set of a Baer subplane of Π.*

In 1982, de Resmini [4] generalized part of the Bruen theorem to symmetric block designs. She proved

THEOREM 2 (de Resmini). *Let B be a blocking set in a symmetric block design with parameters v, k, λ. Then $|B| \geqslant (k + n^{1/2})/\lambda$, where n denotes $k - \lambda$.*

de Resmini observed that "Baer subdesigns" are blocking sets whose cardinalities satisfy the lower bound in Theorem 2. We prove (Theorem 6 below) that the Baer subdesigns are the only blocking sets which satisfy the lower bound. This result completes the generalization of the Bruen theorem and yields a new characterization of Baer subdesigns. The main result of the paper (Theorem 3) generalizes Theorems 2 and 6 to designs Σ that are

* Research supported by the National Science Foundation.

0097-3165/85 $3.00

not necessarily symmetric and to point sets B that may not cover all the lines of Σ.

A finite incidence structure Σ is called a *group divisible design* if the line set can be partitioned into a subset $\mathscr{G}$ whose members are called *groups* and a subset $\mathscr{B}$ whose members are called *blocks* and if there exist integers $g \geqslant 1$, $l \geqslant 2$, λ_1, λ_2, such that the following conditions hold: (1) $\mathscr{G}$ is empty if $g=1$, and the lines of $\mathscr{G}$ partition the points of Σ if $g>1$; (2) $|G|=g$ for each G in $\mathscr{G}$; (3) $|L|=l$ for each L in $\mathscr{B}$; (4) each pair of points lies in $\lambda_1 - 1$ common blocks if the points are contained in a common group and in λ_2 common blocks if the points are not contained in a common group.

A *block design* $B(v, k, \lambda)$ is a group divisible design on $v \geqslant k+2$ points with $g=1$, $k=l$, and $\lambda=\lambda_2$. It is well known that every point of a $B(v, k, \lambda)$ lies on exactly $r=(v-1)\lambda/(k-1)$ lines and that the total number of lines is $b=vr/k$.

If S is a subset of the point set of an incidence structure Σ, one says that S *covers* a line L if and only if S contains a point of L. The *substructure induced* by Σ on S is the set S together with all lines of Σ that contain at least two points of S together with the induced incidence relation.

THEOREM 3. *Let Σ be a $B(v, k, \lambda)$. Let S be a subset of the point set of Σ, $|S|=w$, $m \geqslant \max|S \cap L|$ as L varies over all lines of Σ. Let x, y be the integers that satisfy $(w-1)\lambda=(m-1)x+y$, $0 \leqslant y \leqslant m-2$. Then S covers at most d lines of Σ, where*

$$d = w\left(r - 1 + \frac{1}{y+1} - \frac{(w-1)\lambda - y}{m}\right).$$

Furthermore, S covers exactly d lines if and only if Σ induces on S a group divisible design with block size m, group size $y+1$, and $\lambda_1=\lambda_2=\lambda$.

Proof. For each point p of S, define the weight of p, denoted by $\mathrm{wt}(p)$, to be the sum of the reciprocals of the integers $|S \cap L|$ as L varies over all lines incident with p. If c denotes the number of lines of Σ covered by S, then

$$c = \sum_L 1 = \sum_L \sum_{p \in S \cap L} (1/|S \cap L|).$$

Reversing the order of summation yields $c = \sum_{p \in S} \mathrm{wt}(p)$.

The largest possible weight of a point p occurs if x lines through p intersect S in m points and some line through p intersects S in $y+1$ points. Thus each point p in S satisfies $\mathrm{wt}(p) \leqslant e$, where

$$e = \frac{x}{m} + \frac{1}{y+1} + (r - x - 1).$$

Further, $\mathrm{wt}(p) < e$ unless p does lie on x induced lines of size m and (unless $y = 0$) on one induced line of size $y + 1$. Then $c = \Sigma\ \mathrm{wt}(p) \leqslant we = d$. It is also clear that $c = d$ if and only if every point of S lies on x induced lines of size m and (if $y \neq 0$) on one line of size $y + 1$. If $y \neq 0$, the induced lines of size $y + 1$ partition the points of S, and, hence, may be taken to be the groups of a group divisible design.

LEMMA 4. *Let d' denote $w(r - (w - 1)\lambda/m)$. Under the assumptions of Theorem 3, $d = d'$ if $y = 0$ and $d < d'$ if $y \neq 0$. In particular, if S covers d' lines, then $y = 0$, so Σ induces a block design on S.*

LEMMA 5. *Let B be a blocking set in a $B(v, k, \lambda)$. If $|B| = w$, then $|B \cap L| \leqslant w\lambda - r + 1$ for every line L.*

Proof. Let L be a line of the block design, p be a point of $L \backslash B$; let N denote the number of flags (x, X) with x in B and p in X. One obtains $w\lambda = N \geqslant |B \cap L| + (r - 1)$.

A block design $B(v, k, \lambda)$ is said to be *symmetric* if $r = k$; equivalently, if $b = v$. A substructure Π of a symmetric $B(v, k, \lambda)$ is said to be a *Baer subdesign* (see [1]) if Π is a symmetric $B(v^*, k^*, \lambda)$ with $k^* = (k - \lambda)^{1/2} + 1$.

THEOREM 6. *Let Σ be a symmetric $B(v, k, \lambda)$ with a blocking set B. If $|B| = (k + n^{1/2})/\lambda$, where n denotes $k - \lambda$, then Σ induces a Baer subdesign on B.*

Proof of Theorems 2 *and* 6. Let B be a blocking set in Σ, a symmetric $B(v, k, \lambda)$. Apply Theorem 3 with Lemmas 4 and 5 to conclude that $v \leqslant d'$, where d' is evaluated by putting w equal to $|B|$ and m equal to $w\lambda - k + 1$. Using the fact that $(v - 1)\lambda = k(k - 1)$, one sees that $v \leqslant d'$ is equivalent to

$$0 \leqslant w^2\lambda^2 - 2wk\lambda + (k^2 - k + \lambda).$$

Both or neither of the inequalities is strict. The roots are $w^{\pm} = (k \pm n^{1/2})/\lambda$. Counting flags (x, X) with x in B yields $wk \geqslant v$, hence $w > (k - 1)/\lambda > w^-$. Thus $w \geqslant w^+$, and the proof of Theorem 2 is complete.

If $w = w^+$, then $v = d'$; thus Lemma 4 yields the conclusion that Σ induces a block design on B. By Theorem 3 the block size in the subdesign is $m = (w^+)\lambda - k + 1 = n^{1/2} + 1$, so the induced design is a Baer subdesign. The proof of Theorem 6 is complete.

THEOREM 7. *Let Σ be a $B(v, k, \lambda)$ with a blocking set B of cardinality w. Then $k \neq 3$, and $w \geqslant w_0$, where*

$$w_0 = \frac{v}{2} - \frac{1}{2k}(v^2k^2 - 4v^2k + 4vk)^{1/2}.$$

Proof. Apply Theorem 3 and Lemma 4 with $S=B$, $m=k-1$. One obtains

$$w^2\lambda + w(-rk+r-\lambda)+(bk-b)\leqslant 0.$$

Using $r=(v-1)\lambda/(k-1)$ and $b=vr/k$ and dividing by λ, one obtains

$$g(w) := w^2 - wv + \frac{v(v-1)}{k} \leqslant 0.$$

Then $|B|$ must lie between the roots of $g(w)$. Since w_0 is the smaller root, the asserted inequality holds. If $k=3$, the discriminant D is a positive multiple of $-v^2+4v$. Since $v \geqslant k+2=5$, D is negative. Then the roots of g are not real, so $g(w)>0$ for all w.

Remark 8. For $k=4$, the inequality of Theorem 7 simplifies to $|B| \geqslant (v-v^{1/2})/2$.

Acknowledgments

We are grateful to Professors S. S. Shrikhande and N. Singhi for pointing out that Theorem 3 can be used to prove de Resmini's Theorem 2 as well as to obtain the characterization result (Theorem 6).

References

1. R. C. Bose and S. S. Shrikhande, Baer subdesigns of symmetric balanced incomplete block designs, *in* "Essays in Probability and Statistics" (S. Ikeda et al., Eds.), pp. 1–16, Shinko Tsusho, Tokyo, 1976.
2. A. Bruen, Baer subplanes and blocking sets, *Bull. Amer. Math. Soc.* **76** (1970), 342–344.
3. A. Bruen, Blocking sets in finite projective planes, *SIAM J. Appl. Math.* **21** (1971), 380–392.
4. M. J. de Resmini, On blocking sets in symmetric BIBD's with $\lambda \geqslant 2$, *J. Geometry* **18** (1982), 194–198.