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Laplacian spectral characterization of some graph products

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ABSTRACT

This paper studies the Laplacian spectral characterization of some graph products. We consider a class of connected graphs: $\mathscr{G} = \{G : |EG| \leqslant |VG|\}$, and characterize all graphs $G \in \mathscr{G}$ such that the products $G \times K_m$ are L-DS graphs. The main result of this paper states that, if $G \in \mathscr{G}$, except for C_6, is a L-DS graph, so is the product $G \times K_m$. In addition, the L-cospectral graphs with $C_6 \times K_m$ have been found.

1. Introduction

We start with some basic conceptions of graphs followed from [1]. Let $G = (VG, EG)$ be a graph with vertex set VG and edge set EG, where EG is a collection of 2-subsets of VG. All graphs considered here are simple and undirected. The *adjacency matrix* $A(G) = (a_{u,v})$ $(u, v \in VG)$ of G is a matrix whose rows and columns are labeled by VG, with $a_{u,v} = 1$ if $\{u, v\} \in EG$ and $a_{u,v} = 0$ otherwise. The matrix $L(G) = D(G) - A(G)$ is called the *Laplacian matrix* of G, where $D(G)$ is a diagonal matrix whose diagonal entry is the degree of the corresponding vertex. Since the matrix $L(G)$ is real and symmetric, its eigenvalues are real numbers and called the *Laplacian eigenvalues* of G. It can be shown that $L(G)$ is positive semidefinite. Assuming that $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n (= 0)$ are these eigenvalues, the multiset $\mathrm{Spec}(G) = \{\lambda_1, \ldots, \lambda_n\}$ is called the *Laplacian spectrum* of G. For simplicity, we write $[\lambda_i]^{m_i} \in \mathrm{Spec}(G)$ to denote that the multiplicity of λ_i is m_i. Two graphs are said to be *L-cospectral* if they share the same Laplacian spectrum. Two graphs G and H are said to be *isomorphic* if there is a bijection between VG and VH which induces a bijection between EG and EH. Throughout this paper,

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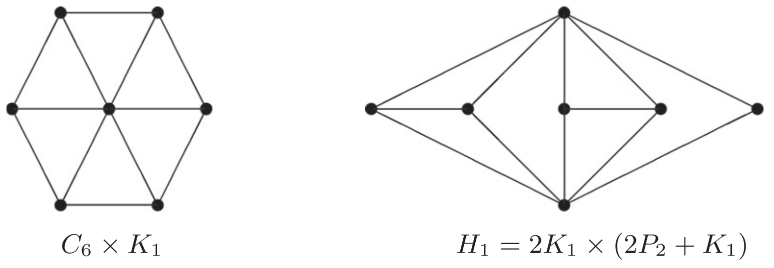

Fig. 1. The L-cospectral graphs $C_6 \times K_1$ and H_1.

we write $G = H$ whenever G and H are isomorphic. A graph G is called to be *determined by its Laplacian spectrum*, or *L-DS graph* for short, if all graphs L-cospectral with G are isomorphic to G.

Given two graphs G_1 and G_2 with disjoint vertex sets VG_1 and VG_2 and edge sets EG_1 and EG_2, the *disjoint union*, or *addition* for convenience, of G_1 and G_2 is defined to be the graph $G = (VG_1 \cup VG_2, EG_1 \cup EG_2)$, denoted by $G_1 + G_2$. Especially, $\underbrace{G + \cdots + G}_{m}$ is denoted by mG. The *product* of graphs G_1 and G_2 is the graph $G_1 + G_2$ together with all the edges joining VG_1 and VG_2, denoted by $G_1 \times G_2$. Let K_m be the *complete graph* of m vertices, P_m the *path* of m vertices, and C_m the *cycle* of m vertices, respectively. Clearly, the complete graph K_m can be written as the product of m isolated vertices. Let K_1 be an isolated vertex, then $K_m = \underbrace{K_1 \times \cdots \times K_1}_{m}$. Similarly, $mK_1 = \underbrace{K_1 + \cdots + K_1}_{m}$ denotes the disjoint union of m isolated vertices. A connected graph is called a *tree* if it contains no cycle, and *unicyclic* if exactly one cycle. Let G be a connected graph. A subgraph S of G is called a *spanning tree* of G if S is a tree and $VS = VG$. Denote by $s(G)$ the number of spanning trees of G. Obviously, $s(G) = 0$ if G is disconnected. These notations will be fixed throughout this paper.

This paper is to characterize which graph products are determined by their Laplacian spectra. It is motivated by [7,14] that we propose the following problem.

Problem 1. Characterize all graphs G such that $G \times K_m$ are L-DS graphs.

In [14], the wheel graph $C_n \times K_1$ for $n \neq 6$ is proved to be L-DS graph. In the conclusion, the authors posed an interesting question. The question is that which graphs satisfy the following relation:

Relation 1. If G is a L-DS graph, then $G \times K_1$ is also a L-DS graph.

Clearly, Relation 1 is just a special case of Problem 1. It is known that if G is disconnected, i.e., G has at least two components, then G always satisfies Relation 1 (see Proposition 4 in [3]). If G is connected, we know that cycle C_n with $n \neq 6$ and path P_n satisfy Relation 1 [7,14].

In this paper, we consider a class of connected graphs: $\mathscr{G} = \{G : |EG| \leqslant |VG|\}$, and characterize all graphs G among $\mathscr{G}$ such that $G \times K_m$ are L-DS graphs. Indeed, $\mathscr{G}$ consists of all connected trees and connected unicyclic graphs. In Section 3, we investigate all connected trees. It is shown that if a connected tree T is L-DS, so is $T \times K_m$. The characterization for unicyclic graphs are investigated in Section 4. We prove that if a connected unicyclic graph $U \neq C_6$ is L-DS, then $U \times K_m$ is also L-DS. Moreover, L-cospectral graphs $C_6 \times K_m$ and $H_1 \times K_{m-1}$ are found. See Fig. 1 for the case $m = 1$, which has been posed in [14]. Indeed, L-cospectral graphs shown in Fig. 1 can also be figured out by our proof in Section 4.

2. Preliminaries

In this section, we mention some results, which will be used later.

Lemma 2.1 [1]. *Let $\{\lambda_1, \ldots, \lambda_{n-1}, 0\}$ be the Laplacian spectrum of the graph G. Then*

$$s(G) = \frac{\lambda_1\lambda_2\cdots\lambda_{n-1}}{n}.$$

Lemma 2.2 [6,2]. *Let G be a graph. The following can be determined by its Laplacian spectrum:*

(1) *The number of vertices of G.*
(2) *The number of edges of G.*
(3) *The number of components of G.*
(4) *The number of spanning trees of G.*
(5) *The sum of the squares of degrees of vertices.*

Lemma 2.3 [11]. *Let G and H be two graphs with $|VG| = n$ and $|VH| = m$. Suppose $Spec(G) = \{\mu_1, \mu_2, \ldots, \mu_{n-1}, 0\}$ and $Spec(H) = \{\nu_1, \nu_2, \ldots, \nu_{m-1}, 0\}$. Then the Laplacian spectrum of the product $G \times H$ is*

$$Spec(G \times H) = \{n + m, m + \mu_1, \ldots, m + \mu_{n-1}, n + \nu_1, \ldots, n + \nu_{m-1}, 0\}.$$

Lemma 2.4. *Suppose G is a L-DS graph. If there is a graph H and a positive integer m such that $Spec(G \times K_m) = Spec(H \times K_m)$, then we have $G = H$.*

Proof. Since $\text{Spec}(G \times K_m) = \text{Spec}(H \times K_m)$, Lemma 2.3 implies that $\text{Spec}(G) = \text{Spec}(H)$. Therefore, $G = H$ since G is a L-DS graph. $\square$

Lemma 2.5 [4]. *Let G be a connected graph with n vertices. Then n is the Laplacian eigenvalue with multiplicity k if and only if G is the product of exactly $k + 1$ graphs.*

Lemma 2.6 [10]. *Let G be a graph and $\lambda(G)$ the largest Laplacian eigenvalue of G. Denote by $d(v)$ the vertex degree of $v \in VG$. Then*

$$\lambda(G) \leqslant \max\{d(v) + m(v) | v \in VG\},$$

where $m(v) = \frac{1}{d(v)} \sum_{\{u,v\} \in EG} d(u)$ is the average of degrees for all neighbors of v.

3. Laplacian spectral characterization of the products of trees and complete graphs

In this section, the main result states that the products of L-DS trees and complete graphs are L-DS graphs. To prove this result, we first need one number theoretic result.

Lemma 3.1. *Let s and t be two positive integers. If $x_0, x_1, \ldots, x_k$ is a sequence of integers with $\sum_{i=0}^{k} x_i = t$ and $x_i \geqslant s$ for all i, then we have*

$$\sum_{i=0}^{k} x_i^2 \leqslant (t - ks)^2 + ks^2, \tag{3.1}$$

where the equality of (3.1) holds if and only if all x_i are identically s but one equals to $t - ks$.

Proof. Denote by $y_i = x_i - s$ for all $i = 0, 1, \ldots, k$. Then we have $\sum_{i=0}^{k} y_i = t - (k+1)s$ and $y_i \geq 0$. Note that

$$\sum_{i=0}^{k} y_i^2 \leq \left(\sum_{i=0}^{k} y_i\right)^2 = \big(t - (k+1)s\big)^2,$$

where the equality holds if and only if all y_i are 0 but one is $t-(k+1)s$. Substituting $y_i = x_i - s$ to above inequality and applying $\sum_{i=0}^{k} x_i = t$, we can easily obtain (3.1). This completes the proof. $\square$

Lemma 3.2. *If a tree T is L-DS, so is the product $T \times K_1$.*

Proof. To prove $T \times K_1$ is L-DS, assume that G is a graph L-cospectral to $T \times K_1$. We need to prove that G is isomorphic to $T \times K_1$. If $|VT| = n$, by Lemma 2.2, G is a connected graph with $|VG| = n+1$. By Lemmas 2.3 and 2.5, G can be written as the product of two graphs, then say $G = G_1 \times G_2$. Fix the following notations:

$$v_1 = |VG_1|, \quad e_1 = |EG_1|, \quad e_2 = |EG_2|.$$

Without loss of generality, we assume $|VG| \geq 2|VG_1|$, i.e., $n+1 \geqslant 2v_1$. Counting the edges of both G and $T \times K_1$ and applying Lemma 2.2, we obtain $e_1 + e_2 + v_1(n+1-v_1) = 2n-1$. It follows that

$$e_1 + e_2 = (2 - v_1)n + v_1^2 - v_1 - 1. \tag{3.2}$$

From Lemma 2.4, we only need to show that $v_1 = 1$, viz. $G = K_1 \times G_2$. Now suppose $v_1 \geqslant 2$. Applying $n+1 \geqslant 2v_1$ and $v_1 \geq 2$ to (3.2), we have

$$e_1 + e_2 \leqslant (2 - v_1)(2v_1 - 1) + v_1^2 - v_1 - 1 = -(v_1 - 1)(v_1 - 3). \tag{3.3}$$

Note that $e_1 + e_2 \geq 0$. It forces $v_1 = 2$ or 3. Then our proof will be complete with the following cases.

Case 1. $v_1 = 2$. Eq. (3.2) implies $e_1 + e_2 = 1$. Then we have $e_1 = 1$ or $e_1 = 0$.

Case 1.1. $e_1 = 1$. Since $v_1 = 2$, it is easily seen that $G_1 = K_2 = K_1 \times K_1$. It follows that $G = G_1 \times G_2 = K_1 \times (K_1 \times G_2)$. Since G is L-cospectral to $T \times K_1$, applying Lemma 2.4, we have $G = T \times K_1$.

Case 1.2. $e_1 = 0$. Applying $v_1 = 2$ and $e_1 + e_2 = 1$, we can easily obtain that $G_1 = 2K_1$ and $G_2 = (n-3)K_1 + P_2$. Since $G = G_1 \times G_2$, by routine calculations, we have $\mathrm{Spec}(G_2) = \{2, [0]^{n-2}\}$. Applying Lemma 2.3, we have

$$\mathrm{Spec}(G) = \{n+1, n-1, 4, [2]^{n-3}, 0\}.$$

Since $\mathrm{Spec}(T \times K_1) = \mathrm{Spec}(G)$, by Lemma 2.3, the Laplacian spectrum of T is

$$\mathrm{Spec}(T) = \{n-2, 3, [1]^{n-3}, 0\}.$$

By Lemma 2.1, the number of spanning trees of T is given by $s(T) = \frac{3(n-2)}{n}$. But obviously $s(T) = 1$. It follows that $n = 3$. Hence, $G_2 = P_2$, and then $G = 2K_1 \times P_2 = K_1 \times P_3$. Now we can complete this case easily by applying Lemma 2.4.

Case 2. $v_1 = 3$. Eq. (3.3) implies $e_1 = e_2 = 0$. Applying $v_1 = 3$ and $e_1 = e_2 = 0$ to (3.2), we can obtain $n = 5$. It follows that $G_1 = 3K_1$ and $G_2 = 3K_1$, and then $G = 3K_1 \times 3K_1$. Its Laplacian spectrum is $\{6, [3]^4, 0\}$. Since $\mathrm{Spec}(T \times K_1) = \mathrm{Spec}(G)$, by Lemma 2.3, the Laplacian spectrum of T is $\{[2]^4, 0\}$. Apply Lemma 2.1, we have $s(T) = \frac{16}{5}$, which is a contradiction. This completes the proof. $\square$

Theorem 3.3. *If a tree T is L-DS, so is the product $T \times K_m$ for all positive integers m.*

Proof. Suppose the graph G is L-cospectral to $T \times K_m$. We shall use induction on m to show that $G = T \times K_m$. The case $m = 1$ is stated in Lemma 3.2. Now we assume $m \geq 2$. Note that

$$T \times K_m = T \times \underbrace{K_1 \times \cdots \times K_1}_{m}.$$

Since $\mathrm{Spec}(G) = \mathrm{Spec}(T \times K_m)$, by Lemma 2.5, G is the product of $m+1$ graphs, denoted

$$G = G_0 \times G_1 \times \cdots \times G_m.$$

Fix notations as follows:

$$n = |VT|, \quad e_i = |EG_i|, \quad v_i = |VG_i| \quad \text{for } i = 0, 1, \ldots, m. \tag{3.4}$$

Without loss of generality, assume $v_0 \geqslant v_1 \geqslant \cdots \geqslant v_m$. It is obvious that $\sum_{i=0}^{m} v_i = n+m$ by Lemma 2.2. In the following, we are going to prove $v_m = 1$ by contradiction. Now suppose $v_m \geqslant 2$. It follows that $v_i \geq 2$ for all $i = 0, \ldots, m$. Then we have $m + n = \sum_{i=0}^{m} v_i \geq 2(m+1)$, so $n \geq m+2$. For convenience, we list those conclusions as follows:

$$m \geq 2, \quad v_0 \geq \cdots \geq v_m \geq 2, \quad m + n = \sum_{i=0}^{m} v_i, \quad n \geq m+2. \tag{3.5}$$

Combining $v_0 \geq \cdots \geq v_m \geq 2$ with $\sum_{i=0}^{m} v_i = n + m$, by Lemma 3.1, we have

$$\sum_{i=0}^{m} v_i^2 \leq (n-m)^2 + 4m. \tag{3.6}$$

Since $\mathrm{Spec}(G) = \mathrm{Spec}(T \times K_m)$, Lemma 2.2 implies that G and $T \times K_m$ have the same number of edges. Counting the edges of both G and $T \times K_m$, we have

$$\sum_{i=0}^{m} e_i + \sum_{0 \leqslant i < j \leqslant m} v_i v_j = n - 1 + mn + \frac{m(m-1)}{2}. \tag{3.7}$$

Since $\sum_{i=0}^{m} v_i = n + m$, we have

$$\sum_{0 \leqslant i < j \leqslant m} v_i v_j = \frac{1}{2}\left(\left(\sum_{i=0}^{m} v_i\right)^2 - \sum_{i=0}^{m} v_i^2\right) = \frac{1}{2}\left((n+m)^2 - \sum_{i=0}^{m} v_i^2\right), \tag{3.8}$$

Applying (3.8) to (3.7), we obtain

$$\sum_{i=0}^{m} e_i = \frac{1}{2}\left(\sum_{i=0}^{m} v_i^2 - n^2 - m\right) + n - 1. \tag{3.9}$$

Applying (3.6) to (3.9), we have

$$\sum_{i=0}^{m} e_i \leqslant (1-m)n + \frac{1}{2}(m^2 + 3m) - 1. \tag{3.10}$$

Applying $m \geqslant 2$ and $n \geq m+2$ of (3.5) to (3.10), we have

$$\sum_{i=0}^{m} e_i \leqslant -\frac{1}{2}(m^2 - m - 2). \tag{3.11}$$

Notice that $-\frac{1}{2}(m^2 - m - 2) \leq 0$ for $m \geq 2$, but $\sum_{i=0}^{m} e_i \geq 0$. It follows that

$$m = 2, \quad e_i = 0 \quad \text{for } i = 0, 1, 2. \tag{3.12}$$

Combining (3.12), (3.10), and $n \geq m+2$ of (3.5), we obtain $n = 4$. So far, we have obtained that $G = G_0 \times G_1 \times G_2$ satisfies

$$|VG_0| \geq |VG_1| \geq |VG_2| \geq 2, \quad |EG_0| = |EG_1| = |EG_2| = 0, \quad \text{and } |VG| = m + n = 6.$$

It follows that

$$G = 2K_1 \times 2K_1 \times 2K_1.$$

Then we have $\mathrm{Spec}(G) = \{[6]^2, [4]^3, 0\}$. Since $\mathrm{Spec}(G) = \mathrm{Spec}(T \times K_m)$, applying Lemma 2.3, we have $\mathrm{Spec}(T) = \{[2]^3, 0\}$. By Lemma 2.1, the number of spanning trees of T is $s(T) = 2$. Note the fact that T is a tree. It is a contradiction. Now we have shown that $v_m = 1$, and then $G_m = K_1$. From $\mathrm{Spec}(T \times K_m) = \mathrm{Spec}(G)$, we have

$$\mathrm{Spec}(K_1 \times (T \times K_{m-1})) = \mathrm{Spec}(K_1 \times (G_0 \times \cdots \times G_{m-1})).$$

By Lemma 2.3, we have

$$\mathrm{Spec}(T \times K_{m-1}) = \mathrm{Spec}(G_0 \times \cdots \times G_{m-1}).$$

By the induction hypothesis of $m - 1$,

$$T \times K_{m-1} = G_0 \times \cdots \times G_{m-1}.$$

Thus $T \times K_m = G_0 \times \cdots \times G_m = G$. The proof is complete. $\square$

Remark 3.4. Up until now, there are so many trees are proved to be L-DS graphs, for examples, path P_n [2], graphs Z_n, T_n, and W_n [13], starlike tree S [12], etc. Therefore, Theorem 3.3 implies that $P_n \times K_m$, $Z_n \times K_m$, $T_n \times K_m$, $W_n \times K_m$ and $S \times K_m$ are also L-DS graphs.

4. Laplacian spectral characterization of the products of unicyclic graphs and complete graphs

This section is devoted to the Laplacian spectral characterization of the products of unicyclic graphs and complete graphs. Recall that a unicyclic graph is a connected graph containing exactly one cycle. In other words, a connected graph $G = (VG, EG)$ is unicyclic iff $|VG| = |EG|$. In notations, we write the unicyclic graph as U. For $k \leqslant n$, denote by $\mathcal{U}(n, k)$ the collection of all unicyclic graphs U with $|VU| = n$ and containing the cycle C_k as a subgraph. Recall in Lemma 2.6 that given a vertex v of the graph G, $d(v)$ denotes the degree of v, and $m(v)$ is defined to be $m(v) = \dfrac{1}{d(v)} \sum_{\{u,v\}\in EG} d(u)$.

Lemma 4.1. *With above notations, for all $U \in \mathcal{U}(n, k)$, we have*

$$\max\{d(v) + m(v) \mid v \in VU\} \leqslant n - k + 3 + \frac{2}{n - k + 2}. \tag{4.1}$$

The equality of (4.1) holds if and only if U is the graph obtained by appending $n - k$ vertices to a vertex of the cycle C_k.

Proof. Since $U \in \mathcal{U}(n, k)$ contains the cycle C_k as a subgraph, then $|VU \backslash VC_k| = n - k$. It is easily seen that the maximum vertex degree of U is $n - k + 2$, viz.

$$d(v) \leq n - k + 2 \quad \text{for all } v \in VU. \tag{4.2}$$

Given $v_0 \in VU$, we shall prove (4.1) by studying the following cases of $d(v_0)$.

Case 1. $d(v_0) = 1$. Clearly, $v_0 \notin VC_k$ and there is a unique vertex adjacent to v_0, denoted $v \in VU$. By (4.2), we have $d(v) \leqslant n - k + 2$. Thus

$$d(v_0) + m(v_0) = d(v_0) + d(v) \leqslant n - k + 3.$$

Case 2. $n - k + 2 \geq d(v_0) \geq 2$. To prove (4.1), viz. to find the maximum value $m(v_0)$ for v_0 with fixed $d(v_0)$, it is enough to find the maximum value of the sum

$$\sum_{\{v,v_0\}\in EG} d(v). \tag{4.3}$$

Note that U is unicyclic with the cycle C_k. Consider the following vertex set

$$V_0 = \{u \in VU \backslash VC_k \mid u \text{ is not adjacent to } v_0\}.$$

Since U is unicyclic, v_0 has at most two neighbors in VC_k. And v_0 has two neighbors in VC_k occurs only when $v_0 \in VC_k$. It implies that

$$|V_0| \le n - k - d(v_0) + 2. \tag{4.4}$$

In order to make the sum (4.3) as large as possible, assume that all vertices of V_0 are adjacent to neighbors of v_0. Now, the sum (4.3) equals

$$n - k - d(v_0) + 2 + d(v_0) + 2 = n - k + 4.$$

Clearly, this is the maximum value for the sum (4.3). Thus, in general, we have

$$\sum_{\{v,v_0\}\in EU} d(v) \leqslant n - k + 4.$$

It follows that

$$d(v_0) + m(v_0) \leqslant d(v_0) + \frac{n-k+4}{d(v_0)}.$$

Now we are going to find an upper bound of $d(v_0) + \frac{n-k+4}{d(v_0)}$ with $2 \le d(v_0) \le n-k+2$. Note that the maximum value of $d(v_0) + \frac{n-k+4}{d(v_0)}$ occurs only when $d(v_0) = 2$ or $n-k+2$. On the other hand, to compare these two values, we have

$$\left(n-k+2+\frac{n-k+4}{n-k+2}\right) - \left(2+\frac{n-k+4}{2}\right) = \frac{n-k+2}{2} + \frac{2}{n-k+2} - 2 \geqslant 0.$$

It is easily seen that $d(v_0) + \frac{n-k+4}{d(v_0)}$ is maximum iff $d(v_0) = n-k+2$. Hence,

$$d(v_0) + m(v_0) \leqslant n - k + 3 + \frac{2}{n-k+2},$$

where the equality holds iff $d(v_0) = n-k+2$. Note that $d(v_0) = n-k+2$ implies that $V_0 = \emptyset$ by (4.4). This completes the proof. □

Lemma 4.2. *If a unicyclic graph U is L-DS and $U \neq C_6$, then $U \times K_1$ is L-DS. Moreover, the unique L-cospectral graph of $C_6 \times K_1$ is $2K_1 \times (2P_2 + K_1)$, see Fig. 1.*

Proof. The idea of the proof is almost the same as Lemma 3.2. Similarly, assume that G is a graph L-cospectral to $U \times K_1$. We shall determine the condition, under which G is isomorphic to $U \times K_1$. Let $|VU| = n$, by Lemma 2.2, then G is a connected graph with $|VG| = n + 1$. By Lemmas 2.3 and 2.5, G can be written as the product of two graphs G_1 and G_2, i.e., $G = G_1 \times G_2$. Fix the following notations:

$$v_1 = |VG_1|, \quad e_1 = |EG_1|, \quad e_2 = |EG_2|.$$

Without loss of generality, we assume $|VG| \ge 2|VG_1|$, i.e., $n + 1 \geqslant 2v_1$. Counting the edges of both G and $U \times K_1$ and applying Lemma 2.2, we obtain $e_1 + e_2 + v_1(n + 1 - v_1) = 2n$. It follows that

$$e_1 + e_2 = (2 - v_1)n + v_1^2 - v_1. \tag{4.5}$$

From Lemma 2.4, it would be enough if we obtain $v_1 = 1$, viz. $G = K_1 \times G_2$. Now suppose $v_1 \geqslant 2$. Applying $n + 1 \geqslant 2v_1$ and $v_1 \ge 2$ to (4.5), we have

$$e_1 + e_2 \leqslant (2 - v_1)(2v_1 - 1) + v_1^2 - v_1 = -(v_1 - 2)^2 + 2. \tag{4.6}$$

Notice the fact $e_1 + e_2 \geq 0$. It forces $v_1 = 2$ or 3. Then our proof will be complete with the following cases.

Case 1. $v_1 = 2$. Applying $v_1 = 2$ to (4.5), we have $e_1 + e_2 = 2$. Notice that $v_1 = |VG_1| = 2$ implies $e_1 = |EG_1| \leq 1$. Then $e_1 = 1$ or 0.

Case 1.1. $e_1 = 1$. Since $v_1 = 2$, it is clear that $G_1 = K_2$. Then we have

$$G = K_2 \times G_2 = K_1 \times (K_1 \times G_2).$$

Since $\mathrm{Spec}(U \times K_1) = \mathrm{Spec}(G)$ and U is L-DS, by Lemma 2.4, we obtain that $K_1 \times G_2$ and U are isomorphic. Clearly, $G = U \times K_1$.

Case 1.2. $e_1 = 0$. It is clear that $G_1 = 2K_1$. Since $e_1 + e_2 = 2$, then we have $e_2 = |EG_2| = 2$. Depending on two edges of G_2 either adjacent or not, G_2 may be isomorphic to $P_3 + (n-4)K_1$ or $2P_2 + (n-5)K_1$. Thus, we have

$$G = 2K_1 \times (P_3 + (n-4)K_1)\,, \text{ or } G = 2K_1 \times (2P_2 + (n-5)K_1)\,.$$

Case 1.2.1. $G = 2K_1 \times (P_3 + (n-4)K_1)$. Since $\mathrm{Spec}(P_3) = \{3, 1, 0\}$, applying Lemma 2.3, we have

$$\mathrm{Spec}(G) = \{n+1, n-1, 5, 3, [2]^{n-4}, 0\}.$$

Since $\mathrm{Spec}(G) = \mathrm{Spec}(U \times K_1)$, applying Lemma 2.3 again, we obtain

$$\mathrm{Spec}(U) = \{n-2, 4, 2, [1]^{n-4}, 0\}.$$

By Lemma 2.1, the number of the spanning trees of U is $s(U) = \frac{8(n-2)}{n}$. It forces $n = 4$, 8 or 16.

Case 1.2.1.1. $n = 4$. Clearly, we have $G = 2K_1 \times P_3$. Notice that the unicyclic graph U has $s(U) = \frac{8(n-2)}{n} = 4$ spanning trees. It follows that the cycle of U is C_4. But $|VU| = n = 4$, then $U = C_4$. On the other hand, it is easily seen that $2K_1 \times P_3 = K_1 \times C_4$, viz. $G = K_1 \times U$ in this case.

Case 1.2.1.2. $n = 8$. Since $s(U) = \frac{8(n-2)}{n} = 6$ and U is unicyclic, then the cycle C_6 is a subgraph of U, i.e., $U \in \mathcal{U}(8, 6)$. By Lemma 4.1, we have $d(v) + m(v) \leqslant 5.5$ for all $v \in VU$. Notice that the maximum Laplacian eigenvalue of U is $\lambda(U) = n - 2 = 6$. It is a contradiction by Lemma 2.6.

Case 1.2.1.3. $n = 16$. Similar as *Case 1.2.1.2*, we have $s(U) = 7$ and $U \in \mathcal{U}(16, 7)$. By Lemma 4.1, $d(v) + m(v) \leqslant 12 + \frac{2}{11}$ for all $v \in VU$. So it is a contradiction by Lemma 2.6 since the maximum Laplacian of U is $\lambda(U) = n - 2 = 14$ for the current case.

Case 1.2.2. $G = 2K_1 \times (2P_2 + (n-5)K_1)$. Similar as *Case 1.2.1*, applying Lemma 2.3, we obtain that

$$\mathrm{Spec}(G) = \{n+1, n-1, [4]^2, [2]^{n-4}, 0\}, \text{ and } \mathrm{Spec}(U) = \{n-2, [3]^2, [1]^{n-4}, 0\}.$$

It follows that the number of spanning trees of U is $s(U) = \frac{9(n-2)}{n}$, so $n = 6$, 9, or 18.

Case 1.2.2.1. $n = 6$. Similar as *Case 1.2.1.2*, we have $U \in \mathcal{U}(6, 6)$, which implies U is exactly the cycle C_6. By routine calculations, we can check that

$$\mathrm{Spec}(K_1 \times C_6) = \mathrm{Spec}(2K_1 \times (2P_2 + K_1)).$$

But it is easily seen that $K_1 \times C_6$ and $2K_1 \times (2P_2 + K_1)$ are not isomorphic, see Fig. 1.

Case 1.2.2.2. $n = 9$. Similar as *Case 1.2.1.2*, we have $U \in \mathcal{U}(9, 7)$, and then $d(v) + m(v) \leqslant 5.5$ for all $v \in VU$. But $\lambda(U) = 7$ in this case, so it is a contradiction by Lemma 2.6.

Case 1.2.2.3. $n = 18$. The arguments in this case is also similar as *Case 1.2.1.2*. It is a contradiction by Lemma 2.6 since $U \in \mathcal{U}(18, 8)$ and $m(v) + d(v) \leqslant 13 + \frac{1}{6}$ for all $v \in VU$, but $\lambda(U) = 16$.

Case 2. $v_1 = 3$. Applying $v_1 = 3$ to (4.5), we have $e_1 + e_2 = 6 - n$. Using the fact $e_1 + e_2 \geq 0$, then $n \leqslant 6$. Notice that $|VG| = n + 1 \geqslant 2|VG_1| = 6$ implies $n \geqslant 5$. Thus, $n = 5$ or 6.

Case 2.1. $n = 6$. Applying $v_1 = 3$ and $n = 6$ to (4.5), we can obtain $e_1 + e_2 = 0$, viz. $e_1 = e_2 = 0$. Then we have $G = 3K_1 \times 4K_1$, whose Laplacian spectrum is $\mathrm{Spec}(G) = \{7, [4]^2, [3]^3, 0\}$. Since $\mathrm{Spec}(G) = \mathrm{Spec}(U \times K_1)$, applying Lemma 2.3, we have $\mathrm{Spec}(U) = \{[3]^2, [2]^3, 0\}$. Using Lemma 2.1, we have $s(U) = 12$. However, $|VU| = n = 6$, it follows that $s(U) \leq 6$, a contradiction.

Case 2.2. $n = 5$. Our arguments are similar as *Case 2.1*. When $n = 5$, we have $e_1 + e_2 = 1$, and then $G = (P_2 + K_1) \times 3K_1$. Similar as *Case 2.1*, we can easily obtain $s(U) = \frac{32}{5}$. Note the fact that $s(U)$ is an integer, a contradiction. This completes the proof. □

The following result is obvious from Lemmas 4.2 and 2.3. Indeed, L-cospectral graphs shown in Fig. 1 have also been given in [14].

Corollary 4.3. *Graphs $C_6 \times K_m$ and $H_1 \times K_{m-1}$ are L-cospectral graphs for all integers $m \geqslant 2$, where $H_1 = 2K_1 \times (2P_2 + K_1)$ as shown in Fig. 1.*

Theorem 4.4. *If a unicyclic graph U is L-DS and $U \neq C_6$, then the product $U \times K_m$ is L-DS for all positive integers m.*

Proof. The idea to prove this theorem is similar as the proof of Theorem 3.3. In the following, we borrow all of arguments and notations ahead of (3.7) in the proof of Theorem 3.3, except that the tree T is replaced by the unicyclic graph U. In the following, we prove the theorem by induction on m. Note that the case $m = 1$ is Lemma 4.2. Now assume $m \geq 2$. We are going to prove $v_m = 1$ by contradiction. Suppose $v_m \geqslant 2$. Since $|EU| = |VU| = n$, instead of (3.7), we have

$$\sum_{i=0}^{m} e_i + \sum_{0 \leqslant i < j \leqslant m} v_i v_j = n + mn + \frac{m(m-1)}{2}. \tag{4.7}$$

Then by the same arguments as Theorem 3.3, instead of (3.9), (3.10), and (3.11), we have

$$\sum_{i=0}^{m} e_i = \frac{1}{2}\left(\sum_{i=0}^{m} v_i^2 - n^2 - m\right) + n; \tag{4.8}$$

$$\sum_{i=0}^{m} e_i \leq (1-m)n + \frac{1}{2}(m^2 + 3m); \tag{4.9}$$

$$\sum_{i=0}^{m} e_i \leq -\frac{1}{2}(m^2 - m - 4). \tag{4.10}$$

Note that $\sum_{i=0}^{m} e_i \geq 0$. Applying the assumption $m \geq 2$ to (4.10), we have $m = 2$. Then (4.9) becomes

$$e_0 + e_1 + e_2 \leq -n + 5.$$

It follows that $n \leq 5$. On the other hand, following the arguments of Theorem 3.3, we also have, similar as (3.5), $n \geq m + 2 = 4$. Combining them together, we have $n = 4$ or 5.

Case 1. $n = 4$. It is easily obtained that $v_0 + v_1 + v_2 = m + n = 6$. Recall the assumption $v_0 \geqslant v_1 \geqslant v_2 \geqslant 2$. It follows that $v_0 = v_1 = v_2 = 2$. Now applying $m = 2$, $n = 4$, and $v_0 = v_1 = v_2 = 2$ to (4.8), we have $e_0 + e_1 + e_2 = 1$. It is easily seen that

$$G = K_2 \times 2K_1 \times 2K_1 = C_4 \times K_2.$$

Since $\mathrm{Spec}(G) = \mathrm{Spec}(U \times K_2)$, by Lemma 2.4, we have $U = C_4$, and then $G = U \times K_2$.

Case 2. $n = 5$. Clearly, $v_0 + v_1 + v_2 = 7$. Since $v_0 \geqslant v_1 \geqslant v_2 \geqslant 2$, then we have

$$v_0 = 3, \quad v_1 = 2, \quad v_2 = 2.$$

Applying these to (4.8), we obtain $e_0 = e_1 = e_2 = 0$. It means that

$$G = 3K_1 \times 2K_1 \times 2K_1.$$

From Lemma 2.3, by routine calculations, we have

$$\mathrm{Spec}(G) = \{[7]^2, [5]^2, [4]^2, 0\}.$$

Since $\mathrm{Spec}(G) = \mathrm{Spec}(U \times K_2)$, applying Lemma 2.3 again, we have

$$\mathrm{Spec}(U) = \{[3]^2, [2]^2, 0\}.$$

It is a contradiction since the number of spanning tree of U is $s(U) = \frac{36}{5}$ by Lemma 2.1. So far, what we have obtained is $v_m = 1$, i.e., $G_m = K_1$. Since $\mathrm{Spec}(G) = \mathrm{Spec}(U \times K_m)$, namely,

$$\mathrm{Spec}((G_0 \times \cdots \times G_{m-1}) \times K_1) = \mathrm{Spec}((U \times K_{m-1}) \times K_1),$$

by Lemma 2.3, it is easy to obtain that

$$\mathrm{Spec}(G_0 \times \cdots \times G_{m-1}) = \mathrm{Spec}(U \times K_{m-1}).$$

From the induction hypothesis of $m - 1$, we have

$$G_0 \times \cdots \times G_{m-1} = U \times K_{m-1}.$$

Obviously, we have $G = U \times K_m$. This completes the proof. □

Up until now, there are only few unicyclic graphs have been proved to be L-DS graphs. For example, *lollipop graph*, which is a graph obtained by attaching a pendant vertex of a path to a cycle, and graph $H(n; q, n_1, n_2)$ with order n, which contains a cycle C_q and two hanging paths P_{n_1} and P_{n_2} attached at the same vertex of the cycle, are proved to be L-DS graphs, respectively [5,8]. Thus we can trivially get the following results.

Corollary 4.5. *Let G be the lollipop graph. Then $G \times K_m$ is L-DS for all positive integers m.*

Corollary 4.6. *Let $G = H(n; q, n_1, n_2)$. Then $G \times K_m$ is L-DS for all positive integers m.*

5. Conclusion

In this paper, we mainly consider a class of connected graphs: $\mathscr{G} = \{G : |EG| \leqslant |VG|\}$, and characterize all graphs $G \in \mathscr{G}$ such that the products $G \times K_m$ are L-DS graphs. Indeed, if we enlarge $\mathscr{G}$ to be $\mathscr{G} = \{G : |EG| \leqslant |VG| + k\}$ with integral $k \geqslant 1$, our method is also valid. For example, further work in [9], the case of connected bicyclic graphs, that is $|EG| = |VG| + 1$, are considered. It is proved that if a bicyclic graph B is L-DS, so is the product of $B \times K_m$, except for $B = \Theta_{3,2,5}$, where $\Theta_{3,2,5}$ denotes the graph consisting of two cycles C_3 and C_5 who share a common path $P_2 = C_3 \cap C_5$. However, the implementation becomes more complicated.

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