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The Deficiency Index of a Certain Class of Ordinary Self-Adjoint Differential Operators*

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1. PROLOGUE

One method of attacking the problem of the deficiency index of an ordinary self-adjoint differential operator has been through the use of asymptotic methods. For example, one may consult the book of Naimark [5] where various theorems utilizing this approach have been given. It is important to be able to compute the deficiency index for a self-adjoint differential operator, with real coefficients, since this gives the number of linearly independent self-adjoint boundary conditions that one needs in order to get a self-adjoint extension of the minimal operator.

In [9] we gave a general approach to the asymptotic method in the deficiency index problem using a theorem of Levinson [1, p. 92], [4]. The general theorem we gave yielded, as rather special cases, many of the known theorems which had appeared in the literature. Unfortunately, the proof we gave contained a gap. Although the general idea of the paper is correct we had failed to take into account the fact that it is possible to have several solutions of a linear differential equation which are not square integrable and yet some nontrivial linear combination of these solutions may be square integrable. It turns out that by adding an extra mild hypothesis to the statement of our original "theorem" it is possible to close this gap and still obtain all of the special theorems which we had mentioned in our original paper. However, since it would be almost impossible to present, out of context, a proof of the corrected theorem as an errata, we have felt it would be valuable to present this exposition.

In [2], M. V. Fedorjuk has given a deficiency index theorem also based on an asymptotic theorem. The theorem we shall present in this

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exposition does not contain Fedorjuk's theorem, and as far as we can tell his theorem does not contain our theorem or any of the special theorems which are implied by our theorem. Very recently we have been able to obtain a generalization of Levinson's asymptotic theorem which we think will allow us to prove a deficiency index theorem which contains the theorem we give here as well as Fedorjuk's deficiency index theorem. However, since all of the details of this are not yet clear we shall postpone an exposition of these matters for another occasion.

2. Introduction

We shall use the concept of an ordinary differential operator as expounded in Naimark's book [5] and shall assume the reader has some familiarity with it.

We are interested in computing the deficiency index of the formal self-adjoint differential operator

$$l(y) = (-1)^n (q_0 y^{(n)})^{(n)} + (-1)^{n-1} (q_1 y^{(n-1)})^{(n-1)} + \cdots + q_n y, \qquad (2.1)$$

defined on the interval $(0, \infty)$ with the coefficients q_k real and measurable. The formal operator (2.1) is called regular if $1/q_0$, $q_1, \ldots, q_n$ are integrable on every finite interval $(0, t)$, *and we assume this in what follows.*

Since the coefficients of (2.1) are only measurable a sense must be given to the formal expression (2.1). For this purpose Naimark introduces the notion of quasiderivatives. These are defined as follows:

$$\begin{aligned} y^{[0]} &= y, \\ y^{[k]} &= d^k y/dt^k, \qquad 1 \leqslant k \leqslant n-1, \\ y^{[n]} &= q_0(d^n y/dt^n), \\ y^{[n+k]} &= q_k \frac{d^{n-k}y}{dt^{n-k}} - \frac{d}{dt}(y^{[n+k-1]}), \qquad 1 \leqslant k \leqslant n. \end{aligned}$$

The formal operator $l(y)$ is then defined for those functions y for which all of the quasiderivatives up to the $(2n-1)$-th order, inclusive, exist and are absolutely continuous in every finite subinterval $[t_1, t_2]$ of the interval $(0, \infty)$. The formal operator l is then defined as

$$l(y) = y^{[2n]}.$$

As is well known [5], corresponding to the operator l there is a minimal operator L_0, and a maximal operator L so that

$$L_0^* = L. \qquad (2.2)$$

Since L_0 permutes with the ordinary conjugation operator, it has self-adjoint extensions, and every self-adjoint extension lies between L_0 and L. The deficiency index (m, m) counts the number m of linearly independent, square-integrable y such that

$$L_0^*(y) = L(y) = zy, \qquad \operatorname{Im} z \neq 0. \tag{2.3}$$

If the formal operator l has one regular endpoint it is well known [5] that $n \leqslant m \leqslant 2n$.

In Section 5 we will show how a number of special cases fit into the general framework of the main theorem of this paper. As a by-product of our result we can obtain the result, originally proved by Glasmann [3], that for operators which are regular at the origin the number m can take on every value between n and $2n$. Indeed our considerations show how to construct explicitly and easily any number of differential operators with a given deficiency index (m, m), $n \leqslant m \leqslant 2n$.

3. Preliminaries

The equation (2.3) is equivalent to the vector equation

$$\frac{du(t)}{dt} = B(t, z)\, u(t), \tag{3.1}$$

where

$$B = \left[\begin{array}{cccc|cccc}
0 & 1 & & & & & & 0 \\
 & 0 & 1 & & & & \cdot & \\
 & & \ddots & \ddots & & \cdot & & \\
 & & & 0 & 1/q_0 & & & \\
\hline
 & & & q_1 & 0 & -1 & & \\
 & & \cdot & & & \ddots & \ddots & \\
 & q_{n-1} & & & & & 0 & -1 \\
q_n - z & & & & & & & 0
\end{array}\right],$$

all functions being evaluated at t, and all other entries are zero. The derivative in (3.1) is to be taken in the a.e. sense. The equivalence of (2.3) and (3.1) is in the sense that if y is a solution of (2.3), then $u = (y, y^{[1]}, \ldots, y^{[2n-1]})$ is a solution of (3.1) and if $u = (u_0, u_1, \ldots, u_{2n-1})$ is a solution of (3.1), then $y = u_0$ is a solution of (2.3) and $u_k = y^{[k]}$, $0 \leqslant k \leqslant 2n - 1$.

It is not uncommon in the theory of differential equations to make a change of the independent variable as well as other transformations of the differential operator in order, hopefully, to bring the differential operator into a more tractable form. This is what we shall do, first making a transformation of the independent variable.

Let Q_0 be a real, nonnegative, Lebesgue measurable function defined on $(0, \infty)$ so that Q_0^{-1} is integrable on every interval $(0, t)$. (We could replace the left endpoint by t_0, for t_0 as large as we please, but it would not increase the generality.) Further, we assume that

$$\int_0^\infty Q_0^{-1} = \infty.$$

Let us set

$$s = s(t) = \int_0^t Q_0^{-1}.$$

Clearly, $s(t)$ is a monotone increasing, absolutely continuous function and has a continuous, a.e. differentiable, monotone increasing inverse which we shall denote by $t = t(s)$. If u is a solution of (3.1) let us set $v(s, z) = u(t(s), z)$ and we get a vector equation

$$dv/ds = Q_0 B v, \qquad \text{a.e.} \tag{3.2}$$

Conversely, if v is a solution of (3.2), then $u(t, z) = v(s(t), z)$ is a solution of (3.1).

Next suppose that $Q_1, \ldots, Q_n$ are positive absolutely continuous functions of t [in each compact interval of $(0, \infty)$]. Set

$$C(s) = \begin{bmatrix} Q_n & & & & & \\ & \ddots & & & & \\ & & Q_1 & & & \\ & & & Q_1^{-1} & & \\ & & & & \ddots & \\ & & & & & Q_n^{-1} \end{bmatrix},$$

where each Q_k is evaluated at $t(s)$ and the entries of C which are not on the main diagonal are zero. Make the transformation

$$v = Cw,$$

and from (3.2) we get the differential equation

$$\frac{dw}{ds} = \left[Q_0 C^{-1} B C - C^{-1} \frac{dC}{ds}\right] w = Dw, \qquad \text{a.e.,} \tag{3.3}$$

where

$$D = \left[\begin{array}{cccc|cccc}
-b_n & d_{n-1} & & & & & & 0 \\
 & \ddots & \ddots & & & & \cdot & \\
 & & \cdot & d_1 & & \cdot & & \\
 & & & -b_1 & a_0 & & & \\
\hline
 & & & a_1 & b_1 & -d_1 & & \\
 & & \cdot & & & \ddots & \ddots & \\
 & a_{n-1} & & & & & \cdot & -d_{n-1} \\
a_n - zQ_0Q_n^2 & & & & & & & b_n
\end{array}\right], \tag{3.4}$$

$$\begin{aligned}
a_0 &= Q_0/Q_1^2 q_0 , \\
a_k &= Q_0 Q_k^2 q_k , \qquad 1 \leqslant k \leqslant n, \\
b_k &= \frac{dQ_k/ds}{Q_k} = \frac{Q_0(dQ_k/dt)}{Q_k}, \qquad 1 \leqslant k \leqslant n, \\
d_k &= \frac{Q_0 Q_k}{Q_{k+1}}, \qquad 1 \leqslant k \leqslant n-1,
\end{aligned} \tag{3.5}$$

the other entries of D are zero, and we have evaluated all of the functions in (3.5) at $t(s)$.

We can now state and prove a theorem in terms of the functions a_k, b_k, and d_k which will give the deficiency index of the differential operator (2.1). Before we do this we shall record several different instances of the matrix D of (3.4):

Case I. Let Q be positive and absolutely continuous on each compact interval $(0. \infty)$. Take

$$Q_0 = Q \qquad \text{and} \qquad Q_{n-k} = Q^{\rho-k}, \qquad \rho \text{ real}, \qquad 0 \leqslant k \leqslant n-1.$$

Then

$$
\begin{aligned}
a_0 &= 1/q_0 Q^{2(\rho-n)+1},\\
a_{n-k} &= Q^{2(\rho-k)+1} q_{n-k}, \qquad 0 \leqslant k \leqslant n-1,\\
b_k &= (\rho - n + k)(dQ/dt), \qquad 1 \leqslant k \leqslant n,\\
d_k &= 1.
\end{aligned}
$$

Case II. Take Q as in Case I and

$$Q_0 = Q^{-1}, \qquad Q_{n-k} = Q^{(\rho+k)}, \qquad \rho \text{ real}, \qquad 1 \leqslant k \leqslant n-1.$$

Then

$$
\begin{aligned}
a_0 &= 1/q_0 Q^{2(n+\rho)-1},\\
a_{n-k} &= Q^{2(\rho+k)-1} q_{n-k}, \qquad 0 \leqslant k \leqslant n-1,\\
b_k &= (\rho + n - k) Q^{-2}(dQ/dt), \qquad 1 \leqslant k \leqslant n,\\
d_k &= 1.
\end{aligned}
$$

Case III. Take Q as in Case I and $Q_0 = 1$, $Q_k = Q$, $1 \leqslant k \leqslant n$. Then

$$
\begin{aligned}
a_0 &= 1/Q^2 q_0, \qquad a_k = Q^2 q_k, \qquad 1 \leqslant k \leqslant n,\\
b_k &= \frac{dQ_k}{dt} \Big/ Q,\\
d_k &= 1.
\end{aligned}
$$

We have singled out these special cases since, as we shall see later, the theorems obtained in [5], [6] and [7] fall under these cases for appropriate Q.

4. The Main Theorem

In this section we shall prove a theorem on the deficiency index of certain formally self-adjoint differential operators. In what follows the functions Q_k, $0 \leqslant k \leqslant n$, shall satisfy the conditions given in the last section, the matrix D is given by (3.4) and the functions a_k, $0 \leqslant k \leqslant n$, b_k, $1 \leqslant k \leqslant n$, and d_k, $1 \leqslant k \leqslant n-1$, are given by (3.5).

THEOREM 1. *Suppose we can write*

$$D(t, z) = A(z) + V(t) + V_1(t, z) + R(t, z), \tag{4.1}$$

where, for fixed z, $A(z)$ is a constant matrix with real entries [except possibly in the position $(2n, 1)$] given by

$$A(z) = \left[\begin{array}{cccc|cccc} \beta_n & 1 & & & & & & \\ & \beta_{n-1} & 1 & & & & & \\ & & \ddots & \ddots & & & & \\ & & & \cdot\ 1 & & & & \\ & & & \beta_1 & 1/\alpha_0 & & & \\ \hline & & & \alpha_1 & -\beta_1 & & & \\ & & \cdot & & & \ddots & & \\ & \cdot & & & & & \ddots & \\ & \alpha_{n-1} & & & & & \cdot & -1 \\ \alpha_n - z\alpha & & & & & & & -\beta_n \end{array}\right], \tag{4.2}$$

where all unmarked entries are zero and α_n and α are real, $V(t)$ has real absolutely continuous [on every compact interval in $(0, \infty)$] entries, $V_1(t, z)$ has zero entries except in the position $(2n, 1)$ where it has the entry $zv(t)$, $v(t)$ real and absolutely continuous [on every compact interval in $(0, \infty)$], $R(t, z)$ has real entries except in the position $(2n, 1)$ where it has the entry $r_{2n,1}(t) - zr(t)$, $r_{2n,1}$ and r real and measurable, $V(t) + V_1(t, z) \to 0$ as $t \to \infty$, and $| V'(t) + V_1'(t, z)|$ and $| R(t, z)|/Q_0(t)$ are integrable.[1] *Suppose further that if $V = 0$ (see Footnote 2) and $\alpha = 0$ the n roots of the polynomial*

$$P(\mu) = \sum_{k=0}^{n-1} (-1)^k \alpha_k \prod_{j=k+1}^{n} (\mu - \beta_j^2) + (-1)^n \alpha_n \tag{4.3}$$

are all nonzero and simple; and if $\alpha = 0$ but $V(t) + V_1(t, z) \neq 0$[2]*, for $z \neq 0$, we have the added condition that those roots which lie in the half-plane $\{z : \operatorname{Im} z > 0\} \cup \{x : x \geqslant 0\}$ have square roots all of whose real parts are different. Then we have the following:*

(A) *If Q_n^2 is integrable let m be the number of roots of $P(\lambda^2)$ which lie in the half-plane*

$$\mathscr{P} = \bigcup_{\delta} \left\{ z : \operatorname{Re} z \leqslant \delta \text{ and } Q_n(t) \exp \delta \int_0^t Q_0^{-1} \in L^2 \right\}. \tag{4.4}$$

[1] If A is a matrix, by $| A |$ we mean any convenient norm on A. If a function is defined on $(0, \infty)$ we say that it is integrable if it is integrable in the usual sense in some interval (t_0, ∞).

[2] By this we mean zero, or not zero, a.e. in some interval (t_0, ∞) respectively.

Let $\mathscr{P}$ have the boundary line $\{z : \mathrm{Re}\, z = \delta_0\}$ (see Footnote 3) (including the case $\delta_0 = \infty$).

(h_1) *If $V \neq 0$, $\delta_0 > 0$, and $P(\lambda^2)$ has more than one root in $(\bar{\mathscr{P}})^c$ (the complement of the closure of $\mathscr{P}$), we suppose there is a γ, $0 \leqslant \gamma < 1$, so that $Q_0(t)\, Q_n{}^2(t) \exp\{2\gamma\delta_1 \int_0^t Q_0^{-1}\}$ has a positive nonincreasing minorant $m(t)$,* a.e., *so that $m(t)/Q_0(t)$ is not in L^1, where δ_1 is the smallest real part of all the real parts of those roots of $P(\lambda^2)$ which are in $(\bar{\mathscr{P}})^c$.*

(h_2) (i) *If $V = 0$ and $P(\lambda^2)$ has more than one root in $\mathscr{P}^c$, or*
(ii) *if $\delta_0 = 0$ and $P(\lambda^2)$ has more than one root in $\mathscr{P}^c$, we suppose that $Q_0(t)\, Q_n{}^2(t) \exp 2\delta_1 \int_0^t Q_0^{-1}$ has a positive nonincreasing minorant $m(t)$,* a.e., *so that $m(t)/Q_0(t)$ is not in L^1, where δ_1 is the smallest real part of all of the real parts of those roots $P(\lambda^2)$ which are in $\mathscr{P}^c$.*

We have the following conclusions:

(c_1) (i) *If the boundary line of $\mathscr{P}$ contains no roots of $P(\lambda^2)$, or*
(ii) *if $V = 0$,*

then the deficiency index of L_0 is (m, m).

(c_2) *If $V \neq 0$, $\delta_0 > 0$, and the boundary line of $\mathscr{P}$ contains roots of $P(\lambda^2)$, then the deficiency index of L_0 is one of the pairs*[4] *$(m + j, m + j)$, $j = \pm 1, \pm 2$.*

(c_3) *If $\delta_0 = 0$, then $\mathscr{P}$ is closed, and the deficiency index of L_0 is $(n + k, n + k)$ if and only if $P(\mu)$ has k real, negative roots.*

(B) *If $Q_n \notin L^2$, let m be the number of roots of $P(\lambda^2)$ in the half-plane given by* (4.4).[5]

(h_1') *If $\alpha = 0$, $v = 0$, and $V \neq 0$, then the statement after the third comma in* (h_1) *shall hold.*

(h_2') *If $\alpha = 0$, $v = 0$, and $V = 0$, then the statement after the second comma in* (h_2) *shall hold.*

(h_3') *If $\alpha = 0$ and $v \neq 0$, we suppose that v remains positive and $Q_0(t)\, v(t)$ has a positive nonincreasing minorant $m(t)$ so that $m(t)/Q_0(t) \notin L^1$. Further, setting $G(\lambda) = P(\lambda^2)$, we assume that if $\mathrm{Re}(\lambda_j - \lambda_k) = 0$, $j \neq k$, then $G'(\lambda_j) \neq G'(\lambda_k)$.*

[3] Since we are assuming $Q_n \in L^2$ it follows that $\delta_0 \geqslant 0$.

[4] We shall show by examples in Section 5 that every one of these pairs is possible.

[5] Since $Q_n \notin L^2$, it follows that $\delta_0 < 0$.

We have the following conclusions:

(c_1') (i) *If* $\alpha = 0$ *and the boundary line of* $\mathscr{P}$ *contains no roots of* $P(\lambda^2)$, *or*

(ii) *if* $V + V_1 = 0$,

then the deficiency index of L_0 *is* (m, m).

(c_2') *If* $\alpha = 0$, $V + V_1 \neq 0$, *and the boundary line of* $\mathscr{P}$ *contains roots of* $P(\lambda^2)$, *then the deficiency index of* L_0 *is one of the pairs* $(m + j, m + j)$, $j = 0, \pm 1, \pm 2$.

(c_3') (i) *If* $\alpha \neq 0$, *or*

(ii) *if* $v \neq 0$,

then the deficiency index of L_0 *is* (n, n). *If* $\alpha \neq 0$ *this is true even if* $P(\lambda^2)$ *has multiple roots and a root at zero.*

Proof. After a certain amount of preparation it will become apparent that part of the proof is a consequence of an asymptotic theorem of Levinson. We shall first prove part (A).

Let us first notice that

$$\int_{s_0}^{\infty} Q_0(t(s)) Q_n^2(t(s))\, ds = \int_{t_0}^{\infty} Q_n^2(t)\, dt < \infty, \qquad s_0 = s(t_0). \tag{4.5}$$

Hence, we must have $\alpha = 0$, and may take $V_1 = 0$ and incorporate the term $-zQ_0Q_n^2$ into the $R(t, z)$ part. This means that $A(z)$ is independent of z and is the reason that in part (A) our statements are concerned only with the term $V(t)$ instead of $V(t) + V_1(t, z)$. Hence, for simplicity, we shall suppress the z in our computations, except where it is essential.

An easy calculation shows the characteristic polynomial of A is $(1/\alpha_0) P(\lambda^2)$, where P is the polynomial of (4.3). Because $V(t) \to 0$ as $t \to \infty$, the characteristic polynomial of $A + V(t)$ can be written as

$$(1/\alpha_0) P(\lambda^2) + o(\lambda^2, t), \tag{4.6}$$

where $o(\lambda^2, t)$ is a polynomial in λ^2 of degree at most $(n - 1)$ with coefficients which are functions of t which go to zero as $t \to \infty$. We shall write

$$P(\lambda^2, s) = P(\lambda^2) + o(\lambda^2, t(s)), \tag{4.7}$$

where we recall that $s(t) = \int_0^t Q_0^{-1}$.

By assumption, all of the roots of $P(\mu)$ are simple and hence, from (4.7), all of the roots of $P(\mu, s)$ are simple, provided s is sufficiently large. This means that the number of real roots of $P(\mu, s)$ is the same as the number of real roots of $P(\mu)$, for all sufficiently large s. Indeed, let $\mu_1, \ldots, \mu_n$ be the roots of $P(\mu)$ and $\mu_1(s), \ldots, \mu_n(s)$ be the roots of $P(\mu, s)$. The latter roots can be chosen to be continuous functions of s in a neighborhood of infinity including the point at infinity. This is a simple consequence of the implicit function theorem. To see this, suppose we consider a polynomial of degree n as a function F of its $(n + 1)$ coefficients and the variable μ. This function F, of $(n + 2)$ variables, is certainly continuously differentiable (and even much more) and there is an $(n + 2)$-tuple $(\kappa_0, \kappa_1, \ldots, \kappa_n, \mu_j)$ at which F vanishes, where the κ_j are the coefficients of $P(\mu)$ in (4.3). Further, since we have postulated that the $P(\mu)$ of (4.3) has only simple roots, it follows that

$$(\partial F/\partial \mu)(\kappa_0, \kappa_1, \ldots, \kappa_n, \mu_j) \neq 0. \tag{4.8}$$

Thus in a neighborhood of $(\kappa_0, \kappa_1, \ldots, \kappa_n)$ we can solve for μ_j as a continuously differentiable function of the other variables, and if we replace these $(n + 1)$ variables by continuous functions of s, then μ_j becomes a continuous function of s. In particular $\mu_j(s) \to \mu_j$ as $s \to \infty$.

Now, if $\mu_1, \ldots, \mu_l$ are the real roots of $P(\mu)$, then for all sufficiently large s, $\mu_1(s), \ldots, \mu_l(s)$ must be real. Otherwise since the coefficients of $P(\mu, s)$ are real, its nonreal roots must come in conjugate pairs, and since $\mu_{l+1}(s), \ldots, \mu_n(s)$ are close to $\mu_{l+1}, \ldots, \mu_n$, they are not real and we would get to many roots for the n-th-order polynomial $P(\mu, s)$. Indeed this argument shows that $\mu_1(s), \ldots, \mu_l(s)$ are exactly the real roots of $P(\mu, s)$ for all sufficiently large s.

The $2n$ roots of $P(\lambda^2)$ are $\{\pm\sqrt{\mu_j} : 0 \leqslant j \leqslant n\}$, where some fixed branch of the square root has been chosen. We shall designate these $2n$ roots by $\{\lambda_{\pm j} : 1 \leqslant j \leqslant n\}$, where $\lambda_{-j} = -\lambda_j$. We label the roots of $P(\lambda^2, s)$ in a corresponding manner; the $2n$ roots of this polynomial are $\{\pm\sqrt{[\mu_j(s)]} : 1 \leqslant j \leqslant n\}$, where the same branch of the square root has been chosen as before, and we correspondingly designate these roots by $\{\lambda_{\pm j}(s) : 1 \leqslant j \leqslant n\}$.

We can suppose we have chosen a branch of the square root in such a way that these latter roots are continuous in a neighborhood of infinity, including the point at infinity. We also suppose we have labeled the roots in such a way so that $\operatorname{Re} \lambda_j(s) \geqslant 0$, $1 \leqslant j \leqslant n$.

We would now like to apply Levinson's asymptotic theorem to our situation. Before we do this we must make sure that all of the hypotheses of his theorem are fulfilled. In addition to the hypotheses on the matrices

$V(t)$, $V_1(t, z)$ and $R(t, z)$ the following two conditions must be fulfilled for the roots of $P(\lambda^2)$ and $P(\lambda^2, s)$:

(i) The $2n$ roots of $P(\lambda^2)$ are simple;[6]

(ii) For a given j let $d_{jm}(s) = \text{Re}(\lambda_j(s) - \lambda_m(s))$ and suppose that all m, $1 \leqslant |m| \leqslant n$, fall into one of two classes I_{1j} and I_{2j}:

$$m \in I_{1j} \quad \text{if} \quad \int_0^s d_{jm}(\sigma)\, d\sigma \to \infty \text{ as } s \to \infty, \text{ and}$$

$$\int_{s_1}^{s_2} d_{jm}(\sigma)\, d\sigma \geqslant -K \qquad (s_2 \geqslant s_1 \geqslant 0);$$

$$m \in I_{2j} \quad \text{if} \quad \int_{s_1}^{s_2} d_{jm}(\sigma)\, d\sigma \leqslant K \qquad (s \geqslant s_1 \geqslant 0),$$

where K is a constant.

Condition (i) is fulfilled by virtue of the hypothesis on the polynomial $P(\mu)$. We must therefore examine condition (ii). If $\text{Re}(\lambda_j - \lambda_m) \geqslant 2\eta > 0$, then for all sufficiently large s, $\text{Re}(\lambda_j(s) - \lambda_m(s) \geqslant \eta$ and hence $n \in I_{1j}$. If $\text{Re}(\lambda_j - \lambda_s) \leqslant 2\eta < 0$, then for all sufficiently large s, $\text{Re}(\lambda_j(s) - \lambda_m(s)) \leqslant \eta$ and $m \in I_{2j}$. If $\text{Re}(\lambda_j - \lambda_m) = 0$, and $V = 0$, then, for all sufficiently large s, $\lambda_j(s) = \lambda_j$ and $\lambda_m(s) = \lambda_m$ and hence $m \in I_{2j}$. If $V \neq 0$, then from the hypothesis concerning the simplicity of the roots of $P(\mu)$, and the hypothesis about the square roots of $P(\mu)$ we must have either $\lambda_j = \bar{\lambda}_m$, or, λ_j and λ_m are purely imaginary. In the first case we must have, for all sufficiently large s, $\lambda_j(s) = \bar{\lambda}_m(s)$ [since the coefficients of $P(\mu, s)$ are real], and hence $m \in I_{2j}$. In the second case, λ_j^2 and λ_m^2 are negative real numbers and hence by considering the discussion given several paragraphs back, of the real roots of $P(\mu)$ and $P(\mu, s)$ we see we must have that $\lambda_j(s)$ and $\lambda_m(s)$ are purely imaginary for all sufficiently large s. Hence, we must again have that $m \in I_{2j}$.

Finally we note that $V(t(s)) \to 0$ as $s \to \infty$,

$$\int_{s_0}^{\infty} \left| \frac{dV}{ds} \right| ds = \int_{t_0}^{\infty} \left| \frac{dV}{dt} \right| dt < \infty,$$

and (4.9)

$$\int_{s_0}^{\infty} |R(t(s), z)|\, ds = \int_{t_0}^{\infty} |R(t, z)|\, Q_0(t)^{-1}\, dt < \infty.$$

Hence all of the conditions for Levinson's theorem are satisfied.

[6] It is at this point we use the fact that $P(\mu)$ does not have a zero root.

Let $\{p_k: 1 \leqslant |k| \leqslant n\}$ be a linearly independent set of eigenvectors for the matrix A of (4.1) corresponding to the eigenvalues $\{\lambda_k: 1 \leqslant |k| \leqslant n\}$. Apply Levinson's theorem and we find $2n$ linearly independent solutions $\{w_j : 1 \leqslant |j| \leqslant n\}$ to (3.5) and an s_0, as large as we please, with $0 \leqslant s_0 < \infty$, so that

$$\lim_{s\to\infty} w_j(s) \exp\left[-\int_{s_0}^{s} \lambda_j(\sigma)\, d\sigma\right] = p_j\,. \tag{4.10}$$

This means, in particular, that if $w_{jk}(s)$ and p_{jk} are the k-th components of $w_j(s)$ and p_j, respectively, then if $p_{jk} \neq 0$,

$$w_{jk}(s) = p_{jk} \exp\left[\int_{s_0}^{s} \lambda_j(\sigma)\, d\sigma\right] [1 + o(1)]. \tag{4.10'}$$

Before we proceed further let us note that for any j we can always choose p_j so that $p_{j1} = 1$. Indeed, if we write out the set of linear equations corresponding to $Ap_j = \lambda_j p_j$ we see immediately that if $p_{j1} = 0$, then all of the other components p_{jk} must be zero. Hence, we shall suppose from now on that p_j is that eigenvector with $p_{j1} = 1$.

Returning to the considerations of Section 3, we recall that

$$v_j(s) = C(t(s))\, w_j(s), \qquad j = \pm 1,\ldots, \pm n.$$

Since the set $\{w_j : 1 \leqslant |j| \leqslant n\}$ is linearly independent and $C(t)$ is nonsingular for each t, the set $\{v_j : 1 \leqslant |j| \leqslant n\}$ is linearly independent. Consequently, if $u_j(t) = v_j(s(t))$, the set $\{u_{j1} : 1 \leqslant |j| \leqslant n\}$ is a set of linearly independent solutions to the equation (2.3).

Now,

$$v_{j1}(s(t)) = Q_n(t)\, w_{j1}(s(t)),$$

and hence from (4.10′) we get

$$|u_{j1}(t)| = |v_{j1}(s(t))| = \left\{|Q_n(t)| \exp \int_{s_0}^{s(t)} \operatorname{Re} \lambda_j(\sigma)\, d\sigma\right\} \{1 + o_j(1)\}. \tag{4.11}$$

We shall now break the proof into several parts corresponding to the different parts of the theorem listed under the conclusions of part (A).

Let us first prove the first case of the conclusion (c_1). Suppose, at first, that $V(t) \neq 0$ and $\delta_0 > 0$. [Recall that in the case (c_1) $P(\lambda^2)$ has no roots on the boundary of $\mathscr{P}$.] If $P(\lambda^2)$ has no roots in $(\mathscr{P})^c$, then it is an immediate consequence of (4.11) that the deficiency index of L_0 is $(2n, 2n) = (m, m)$. In case $P(\lambda^2)$ has only one root with real part greater than δ_0 it is again clear from (4.11) that the deficiency index is $(2n - 1, 2n - 1) = (m, m)$.

If $P(\lambda^2)$ has more than one root with real part greater than δ_0, we will suppose we have labeled the roots of $P(\lambda^2)$ so that $\operatorname{Re} \lambda_j > \delta_0$ for $1 \leqslant j \leqslant p$. Let us write $\lambda_j = \rho_j + i\nu_j$, $\lambda_j(\sigma) = \rho_j(\sigma) + i\nu_j(\sigma)$, where ρ_j, ν_j, $\rho_j(\sigma)$, and $\nu_j(\sigma)$ are real. Let $\{c_j : 1 \leqslant j \leqslant p\}$ be any set of complex numbers which are not all zero. We want to show that

$$\sum_{j=1}^{p} c_j u_{j1}(t) \notin L^2. \tag{4.12}$$

For the sake of convenience suppose that we have relabeled (if necessary) the complex numbers c_j (and correspondingly the eigenvalues) so that $c_j \neq 0$ for $1 \leqslant j \leqslant q$, and $c_j = 0$ for $q < j \leqslant p$.

If $q = 1$, then from (4.11) it is immediate that (4.12) is satisfied. Hence we shall suppose that $q \geqslant 2$. Let $\rho = \max\{\rho_j : 1 \leqslant j \leqslant q\}$, and, again for convenience, suppose $\rho = \rho_1$. We shall set $\lambda(\sigma) = \rho(\sigma) + i\nu(\sigma) = \lambda_1(\sigma)$. From (4.10′), with $k = 1$, we get

$$\sum_{j=1}^{q} c_j v_{j1}(s) = Q_n(t(s)) \sum_{j=1}^{q} \left\{c_j \exp \int_{s_0}^{s} \lambda_j(\sigma)\, d\sigma\right\} \{1 + o_j(1)\}. \tag{4.13}$$

Since we are supposing that $V(t) \neq 0$ and $\delta_0 > 0$, it follows from the hypotheses of the theorem that there are at most two roots of $P(\lambda^2)$ which have the same real part ρ. Let us suppose, at first, that there are two such roots. These two roots are conjugate complex to each other. For simplicity, we shall suppose we have labeled the c_j, and the corresponding eigenvalues so that $\bar{\lambda} = \lambda_2$ and $\overline{\lambda(\sigma)} = \lambda_2(\sigma)$. Thus from (4.13) and the fact that $\rho(s) > \operatorname{Re} \lambda_j(s)$, $3 \leqslant j \leqslant q$, we get that as $s \to \infty$,

$$\left| \sum_{j=1}^{q} c_j u_{j1}(t(s)) \right| = |c_2|\, |Q_n(t(s))| \exp \int_{s_0}^{s} \rho(\sigma)\, d\sigma$$

$$\times \left| 1 + \frac{c_1}{c_2} \exp 2i \int_{s_0}^{s} \nu(\sigma)\, d\sigma \right| |1 + o(1)|.$$

Squaring and integrating,

$$\int_{t_0}^{\infty} \left| \sum_{j=1}^{q} c_j u_{j1}(t) \right|^2 dt = \int_{s_0}^{\infty} \left| \sum_{j=1}^{q} c_j u_{j1}(t(s)) \right|^2 Q_0(t(s))\, ds$$

$$= | c_2 |^2 \int_{s_0}^{\infty} Q_0(t(s))\, Q_n^2(t(s)) \exp 2 \int_{s_0}^{s} \rho(\sigma)\, d\sigma$$

$$\times \left| 1 + \frac{c_2}{c_2} \exp 2i \int_{s_0}^{s} \nu(\sigma)\, d\sigma \right|^2 | 1 + o(1)|^2\, ds.$$

If s_0 is sufficiently large, $| 1 + o(1)|^2 \geqslant 1/2$ and $\rho(\sigma) \geqslant \gamma\, \delta_1$. Hence we get

$$\int_{t_0}^{\infty} \left| \sum_{j=1}^{q} c_j u_{j1}(t) \right|^2 dt$$

$$\geqslant \frac{| c_2 |^2}{2} \left| 1 - \left| \frac{c_2}{c_1} \right| \right|^2 \int_{t_0}^{\infty} Q_n^2(t) \exp 2\gamma\delta_1(s(t) - s_0)\, dt.$$

The integral on the right is divergent by the hypothesis (h_1). Hence, if $| c_2/c_1 | \neq 1$, the left-hand integral is divergent.

We shall now examine the case where $| c_1/c_2 | = 1$. In this case we may write

$$\left| \sum_{j=1}^{q} c_j u_j(t(s)) \right| = | c_2 |\, Q_n(t(s)) \exp \int_{s_0}^{s} \rho(\sigma)\, d\sigma$$

$$\times \left| 1 + \exp \left\{ 2i \int_{s_0}^{s} \nu(\sigma)\, d\sigma + 2i\beta \right\} \right| | 1 + o(1)|, \qquad (4.14)$$

where β is some real number. We may assume that $\nu > 0$, since otherwise we could work with the complex conjugate of the term involving the function $\nu(\sigma)$. For every positive integer k let us take $I_k = [s_k, s_k']$ and $J_k = [s_k', s_{k+1}]$, where we have taken s_k and s_k' so that

$$2 \int_{s_0}^{s_k} \nu(\sigma)\, d\sigma + 2\beta = 2k\pi + \frac{3\pi}{2},$$

$$2 \int_{s_0}^{s_k'} \nu(\sigma)\, d\sigma + 2\beta = 2k\pi + \frac{5\pi}{2}. \qquad (4.15)$$

This is always possible for all sufficiently large k since for s_0 sufficiently large, $\nu(\sigma) > \nu/2$ for $\sigma \geqslant s_0$, and hence $\int_{s_0}^{s} \nu(\sigma)\, d\sigma$ is a monotone-increasing continuous function of s which takes on all values in $[0, \infty)$. Consequently, it follows that $| 1 + \exp\{2i \int_{s_0}^{s} \nu(\sigma)\, d\sigma + 2i\beta\} |^2$ takes all values in $[2, 4]$ as s varies over I_k, and takes all values in $[0, 2]$ as s varies over J_k. Let us now set

$$\omega(\sigma) = 2[\nu(\sigma) - \nu];$$

for all sufficiently large k we get, from (4.15),

$$2 \int_{s_0}^{s_k'} \nu(\sigma)\, d\sigma - 2 \int_{s_0}^{s_k} \nu(\sigma)\, d\sigma = \int_{s_k}^{s_k'} \omega(\sigma)\, d\sigma + 2\nu(s_k' - s_k) = \pi.$$

From the mean-value theorem there exists an η_k so that

$$\int_{s_k}^{s_k'} \omega(\sigma)\, d\sigma = \eta_k(s_k' - s_k).$$

Thus we have, for all sufficiently large k,

$$s_k' - s_k = \pi/(2\nu + \eta_k). \tag{4.16}$$

Since $\omega(\sigma) \to 0$ as $\sigma \to \infty$, it follows that $\eta_k \to 0$ as $k \to \infty$. In the same way there exists a sequence $\{\zeta_k\}$ so that $\zeta_k \to 0$ as $k \to \infty$ and

$$s_{k+1} - s_k' = \pi/(2\nu + \zeta_k). \tag{4.16'}$$

From (4.16) and (4.16′) we see that there exist two positive constants b and B so that for all sufficiently large k,

$$b \mid J_k \mid \leqslant \mid I_k \mid \leqslant B \mid J_{k-1} \mid,$$

where $| I_k |$ and $| J_k |$ are the lengths of I_k and J_k, respectively. From this it follows that both series

$$\sum_{k=k_0}^{\infty} \int_{I_k} m(t(s))\, ds, \quad \sum_{k=k_0}^{\infty} \int_{J_k} m(t(s))\, ds$$

converge or diverge simultaneously, where we recall that $m(t)$ is a positive nonincreasing minorant of $Q_0(t)\, Q_n^2(t) \exp 2\gamma\, \delta_1 \int_{t_0}^{t} Q_0^{-1}$. Indeed, it is clear that for all sufficiently large k,

$$b \int_{J_k} m(t(s))\, ds \leqslant \int_{I_k} m(t(s))\, ds \leqslant B \int_{J_{k-1}} m(t(s))\, ds.$$

Since $m(t(s)) \notin L^1$, we see that both series must diverge. Hence, we have established (4.12) in the case where $P(\lambda^2)$ has two conjugate complex roots with real part ρ. If $\nu = 0$, i.e., there is only one real root ρ, then we think it is clear that (4.12) is still valid.

Let us now prove the second case of the conclusion (c_1). In this case $V = 0$. If there is no root of $P(\lambda^2)$ in $\mathscr{P}^c$, then clearly the deficiency index of L_0 is $(m, m) = (2n, 2n)$. If there is only one root of $P(\lambda^2)$ in $\mathscr{P}^c$, then again it is clear that L_0 has the deficiency index $(m, m) = (2n - 1, 2n - 1)$. If there is more than one root of $P(\lambda^2)$ in $\mathscr{P}^c$ let us label the eigenvalues so that $\lambda_j \in \mathscr{P}^c$ for $1 \leqslant j \leqslant p$. Let $\{c_j : 1 \leqslant j \leqslant p\}$ be any set of complex numbers, not all zero, and suppose c_j, $1 \leqslant j \leqslant q$, are those which are not zero. Let ρ be the largest number in the set $\{\operatorname{Re} \lambda_j : 1 \leqslant j \leqslant q\}$, and suppose we have arranged the labeling so that λ_j, $1 \leqslant j \leqslant r$, be those eigenvalues with the same real part ρ; i.e., $\lambda_j = \rho + i\nu_j$, $1 \leqslant j \leqslant r$.

Working in the same way as in the first part of the proof, but noting that $\lambda_j(s) \equiv \lambda_j$, we find that

$$\int_{t_0}^{\infty} \left| \sum_{j=1}^{q} c_j u_{j1}(t) \right|^2 dt$$

$$= \int_{s_0}^{\infty} Q_0(t(s))\, Q_n^2(t(s)) \exp 2\rho(s - s_0) \left| \sum_{j=1}^{r} c_j e^{i\nu_j(s-s_0)} \right|^2 |\, 1 + o(1)|^2\, ds. \quad (4.17)$$

The summation term in the right-hand integral is a nonzero, almost periodic function which we will designate by $p(s)$. Let a be a real number so that $|\,p(a)| > 0$. Let us take $\epsilon = |\,p(a)|/4$; then there is a positive number M so that in every interval of length M, there is a number τ so that for every real s

$$|\,p(s + \tau) - p(s)| < \epsilon.$$

Let $\tau_k \in [kM, (k + 1)M]$, $k = 0, 1,\ldots$, for which this last inequality is true, and let I be a closed interval containing a of length less than $M/2$, and so that $s \in I$ implies $|\,p(s)| > |\,p(a)|/2$. If we set $I_k = I + \tau_{2k}$, then for all sufficiently large k, $I_k \subset [s_0, \infty)$, and $I_j \cap I_k = \varnothing$, unless $j = k$. Further, there are two positive constants, b and B, so that if J_k is the interval which lies between I_k and I_{k+1}, then

$$b\, |\, J_k \,| \leqslant |\, I_k \,| \leqslant B\, |\, J_{k-1} \,|,$$

where, as before, $|\, I_k \,|$ and $|\, J_k \,|$ are the lengths of I_k and J_k, respectively.

Now, if $s \in I_k$ we have

$$|p(s)| \geqslant |p(a)|/4.$$

Hence, if $m(t)$ is a nonincreasing positive minorant of

$$Q_0(t) Q_n^2(t) \exp 2\delta_1 \int_{t_0}^{t} Q_0^{-1},$$

then the series

$$\sum_{k=k_0}^{\infty} \int_{I_k} m(t(s))\, ds, \qquad \sum_{k=k_0}^{\infty} \int_{J_k} m(t(s))\, ds$$

both converge or diverge simultaneously. However, since $m(t(s)) \notin L^1$ it follows that they both diverge. Thus from (4.17) it follows that (4.12) is satisfied.

We now turn to the proof of (c_2) of part (A). In this case we are assuming that $V \neq 0$, $\delta_0 > 0$, and $\mathscr{P}$ has roots on its boundary line. From the general hypothesis of the theorem, when $V \neq 0$, and $\delta_0 > 0$, there can at most be two roots lying on the boundary line of $\mathscr{P}$. (Recall, that in part (A) of the theorem we have $\alpha = 0$ and $V_1(t, z) \equiv 0$).

Suppose, at first that $\mathscr{P}$ is closed. If there is only one root on the boundary line of $\mathscr{P}$, then it must be real, say λ_1. Suppose $\lambda_1(s) \to \lambda_1$ in such a way that the corresponding solution $u_{11} \in L^2$. If $P(\lambda^2)$ has roots with real parts greater than δ_0, then working in the same way as in the proof of (c_1) we see that no nontrivial linear combination of the corresponding solutions can be in L^2. Hence, the deficiency index is (m, m). On the other hand, suppose $\lambda_1(s) \to \lambda_1$ in such a way that the corresponding solution $u_{11} \notin L^1$. Then the deficiency index is $(m - 1, m - 1)$.

If there are two (conjugate complex) roots of $P(\lambda^2)$ on the boundary line of $\mathscr{P}$ ($\mathscr{P}$ still closed), say λ_1 and $\lambda_2 = \bar{\lambda}_1$, then, of course, $\lambda_2(s) = \overline{\lambda_1(s)}$. If $\lambda_1(s) \to \lambda_1$ in such a way that $u_{11} \in L^2$, then $u_{21} \in L^2$ and the deficiency index is (m, m). If $\lambda_1(s) \to \lambda_1$ in such a way that the corresponding $u_{11} \notin L^2$, then $u_{21} \notin L^2$. If no nontrivial linear combination of these solutions is in L^2, then the deficiency index is $(m - 2, m - 2)$. On the other hand if some nontrivial linear combination of u_{11} and u_{21} is in L^2, then the deficiency index is $(m - 1, m - 1)$. Of course, if $P(\lambda^2)$ has roots with real parts greater than δ_0, then the same argument as used in (c_1) shows that no nontrivial linear combination of these roots is in L^2.

Let us now look at the situation when $V \neq 0$, $\delta_0 > 0$, $\mathscr{P}$ has roots of $P(\lambda^2)$ on its boundary line, but $\mathscr{P}$ is open. If we have one real root, say

λ_1, on this boundary line and $\lambda_1(s) \to \lambda_1$ in such a way that $u_{11} \notin L^2$, then the deficiency index is $(m - 1, m - 1)$; if $u_{11} \in L^2$, then the deficiency index is $(m + 1, m + 1)$.

If the boundary line of $\mathscr{P}$ (still open) contains two (conjugate complex) roots of $P(\lambda^2)$, then, of course $\overline{\lambda_1(s)} = \lambda_2(s)$. If the corresponding solution $u_{11} \in L^2$, then $u_{21} \in L^2$, and the deficiency index is $(m + 2, m + 2)$. If $u_{11} \notin L^2$, then $u_{21} \notin L^2$, and if a nonzero linear combination of u_{11} and u_{21} belongs to L^2, the deficiency index is $(m + 1, m + 1)$; if no nontrivial linear combination of u_{11} and u_{21} belongs to L^2, then the deficiency index is (m, m). Of course, here again, if $P(\lambda^2)$ has roots with real parts greater than δ_0, then the method of proof of (c_1) shows that no nontrivial linear combination of these solutions is in L^2.

Finally we come to the last conclusion (c_3) in the proof of part (A), namely, $\delta_0 = 0$. From the discussion of the real roots of $P(\mu)$ and $P(\mu, s)$, it follows that if λ_j is a purely imaginary root of $P(\lambda^2)$, then $\lambda_j(s)$ is purely imaginary for all sufficiently large s. From formula (4.11) it follows that $\mathscr{P}$ is closed. If $P(\mu)$ has k negative real roots, then $P(\lambda^2)$ has $2k$ purely imaginary roots, $(n - k)$ roots with negative real parts, and $(n - k)$ roots with positive real parts. [Recall that we assumed that $P(\mu)$ has no zero root.] Hence, the deficiency index is at least $(n + k, n + k)$. The proof that this is exactly the deficiency index follows *mutatis mutandis* the proof of (c_1), using (h_2), in either case, $V = 0$ or $V \neq 0$.

Conversely, if the deficiency index of L_0 is $(n + k, n + k)$, then $P(\mu)$ must have exactly k real negative roots. For if $P(\mu)$ has more or less than k negative roots, the first part of the proof [of (c_3)] shows that the deficiency index of L_0 is not $(n + k, n + k)$.

Let us now turn our attention to the proof of part (B) of the theorem. The proofs of (c_1') and (c_2') follow the proofs given for part (A). Hence we proceed to the proof of (c_3'). Because of the way we have assumed we can decompose $Q_0 Q_n^2$ [Eq. (4.1)] we must now compute the characteristic roots of

$$D(t, z) = A(z) + V(t) + V_1(t, z); \tag{4.18}$$

i.e., we are interested in finding the solutions to

$$P(\lambda^2, s) = z[\alpha + v(t(s))]. \tag{4.19}$$

If $\alpha \neq 0$ we may choose a nonreal z_0, as close to zero as we please, so that the roots of $P(\lambda^2) - z_0\alpha$ are all simple and have different real parts regardless of the multiplicity of the roots of $P(\lambda^2)$. Moreover, since P has real coefficients, $P(\lambda^2) - z_0\alpha$ can have no purely imaginary roots. Hence, there are n roots of $P(\lambda^2) - z_0\alpha$ each of which has a positive real part, and n-roots each of which has a negative real part. Designate those roots having positive real part by $\lambda_j(z_0)$, $1 \leqslant j \leqslant n$, and the corresponding solutions to (4.19), with z replaced by z_0, by $\lambda_j(s, z_0)$, $1 \leqslant j \leqslant n$, where we have taken these roots so that $\lambda_j(s, z_0) \to \lambda_j(z_0)$ as $s \to \infty$.

Clearly, the hypotheses of Levinson's theorem are satisfied for the functions $\lambda_j(s, z_0)$, $1 \leqslant j \leqslant n$, so that there are n linearly independent solutions to the equation $Lf = z_0 f$, say $u_{j1}(t, z_0)$, $1 \leqslant j \leqslant n$, so that

$$| u_{j1}(t(s), z_0)| = | Q_n(t(s))| \left\{\exp \int_{s_0}^{s} \operatorname{Re} \lambda_j(\sigma, z_0)\, d\sigma\right\} \{1 + o_j(1)\}.$$

We claim that no nontrivial linear combination of the set $\{u_{j1}(t, z_0) : 1 \leqslant j \leqslant n\}$ can belong to L^2. Indeed, suppose $\{c_j : 1 \leqslant j \leqslant p\}$ is a set of complex numbers, all not zero. We have

$$\sum_{j=1}^{p} c_j u_{j1}(t, z_0) = \sum_{j=1}^{p} c_j \left\{Q_n(t) \exp \int_{s_0}^{s(t)} \lambda_j(\sigma, z_0)\, d\sigma\right\} \{1 + o_j(1)\}.$$

Suppose we have arranged the labeling of the $\lambda_j(z_0)$ so that $\lambda_1(z_0)$ has the largest real part, and $c_1 \neq 0$. There exists an $\eta > 0$ so that for all sufficiently large s, and for all $2 \leqslant j \leqslant p$, we have

$$\operatorname{Re}[\lambda_j(s, z_0) - \lambda_1(s, z_0)] < -\eta$$

and thus, if s_0 is sufficiently large, there exists an $s_1 > s_0$ so that for all $s(t) \geqslant s_1$,

$$\left| \sum_{j=2}^{n} c_j \left\{\exp \int_{s_0}^{s(t)} [\lambda_j(\sigma, z_0) - \lambda_1(\sigma, z_0)]\, d\sigma\right\} \{1 + o_j(1)\} \right| < \frac{|c_1|}{2}.$$

Consequently, for $s(t) \geqslant s_1$,

$$\left| \sum_{j=1}^{n} c_j \left\{\exp \int_{s_0}^{s(t)} [\lambda_j(\sigma, z_0) - \lambda_1(\sigma, z_0)]\, d\sigma\right\} \{1 + o_j(1)\} \right| \geqslant \frac{|c_1|}{2}.$$

Also, since $\operatorname{Re} \lambda_1(z_0) > 0$ we have for all $s(t) \geqslant s_0$,

$$\left| \exp \int_{s_0}^{s(t)} \lambda_1(\sigma, z_0)\, d\sigma \right| \geqslant c > 0.$$

Thus, for sufficiently large s_0, there exists an $s_1 > s_0$ so that for all $s(t) \geqslant s_1$ we have

$$\left| Q_n(t) \exp \int_{s_0}^{s(t)} \lambda_1(\sigma)\, d\sigma \right| \left| \sum_{j=1}^{n} c_j \left\{ \exp \int_{s_0}^{s(t)} [\lambda_j(\sigma, z_0) - \lambda_1(\sigma, z_0)]\, d\sigma \right\} \{1 + o_j(1)\} \right|$$

$$\geqslant \frac{c \mid c_1 \mid}{2} \mid Q_n(t) \mid .$$

However, $Q_n(t) \notin L^2$, and this establishes our claim.

If $\alpha = 0$, we must look at solutions to the equation

$$P(\lambda^2, s) = zv(t(s)), \qquad \operatorname{Im} z \neq 0. \tag{4.20}$$

Recall that in this case we are supposing that v remains positive. Because $P(\lambda^2)$ has only simple roots, and since $v(t) \to 0$ as $t \to \infty$, for all sufficiently large s we have $2n$ simple solutions to the equation (4.20). Also, since $v(t) > 0$, for all sufficiently large t, it follows that (4.20) can have no purely imaginary solutions. Hence, for $s \neq \infty$, we have n solutions each with a positive real part, and n solutions each with a negative real part.

From the fact that the roots of $P(\lambda^2)$ are simple it follows from the implicit function theorem that we can choose $2n$ distinct solutions, $\lambda_j(s, z)$, $1 \leqslant |j| \leqslant n$, to (4.20), each of which is continuous in a neighborhood of $(\infty, 0)$ in the (s, z) space, and for a fixed s is an analytic function of z. If we set $G(\lambda, s) = P(\lambda^2, s)$, and if we differentiate

$$G(\lambda_j(s, z), s) = P(\lambda_j^2(s, z), s) = zv(t(s))$$

with respect to z we get

$$\frac{\partial \lambda_j(s, z)}{\partial z} = v(t(s)) \Big/ \frac{\partial G(\lambda_j(s, z), s)}{\partial \lambda}. \tag{4.21}$$

Note that $\partial G(\lambda_j(s, z), s)/\partial \lambda \neq 0$ if z remains in a bounded neighborhood of the origin and s is sufficiently large (depending, of course, on the bounded neighborhood we allow for z), since the roots of $P(\lambda^2)$ are simple.

We can use (4.21) to find all of the derivatives of $\lambda_j(s, z)$, with respect to z, in some neighborhood of $(\infty, 0)$ in the (s, z) space. Indeed, since in some neighborhood of $(\infty, 0)$ in the (s, z) space, $[\partial G(\lambda_j(s, z), s)/\partial \lambda]^{-1}$ is analytic in z for every fixed s, we have from (4.21),

$$\frac{\partial \lambda_j(s, z)}{\partial z} = \frac{v(t(s))}{2\pi i} \int_\Gamma \left[\frac{\partial G(\lambda_j(s, \zeta)\ s)}{\partial \lambda} \right]^{-1} \frac{d\zeta}{(\zeta - z)},$$

and hence,

$$\frac{\partial^k \lambda_j(s, z)}{\partial z^k} = \frac{v(t(s))\, k!}{2\pi i} \int_\Gamma \left[\frac{\partial G(\lambda_j(s, \zeta), s)}{\partial \lambda}\right]^{-1} \frac{d\zeta}{(\zeta - z)^{k+1}},$$

where Γ is a sufficiently small circle about $z = 0$. Actually, we think it is clear that if the radius of Γ is small enough it can be chosen independent of s, in the fixed neighborhood of $(\infty, 0)$. Consequently, for fixed s we may expand $\lambda_j(s, z)$ in a Taylor series around $z = 0$ and get [recall $\lambda_j(s, 0) = \lambda_j(s)$]

$$\begin{aligned} \lambda_j(s, z) &= \lambda_j(s) + z v(t(s)) \left\{ \sum_{k=1}^{\infty} \frac{1}{k!} \frac{\partial^k \lambda_j(s, 0)}{\partial z^k} z^{k-1} \right\} \\ &= \lambda_j(s) + z v(t(s)) \left\{ \sum_{k=1}^{\infty} z^{k-1} \frac{1}{2\pi i} \int_\Gamma \left[\frac{\partial G(\lambda_j(s, \zeta), s)}{\partial \lambda}\right]^{-1} \frac{d\zeta}{\zeta^{k+1}} \right\}. \end{aligned} \tag{4.22}$$

Now, since $P(\lambda^2)$ has only simple zeros, it follows that $[\partial P(\lambda_j^2)/\partial \lambda] \neq 0$. Hence for all sufficiently large s and all sufficiently small ζ we get

$$\left| \frac{\partial G(\lambda_j(s, \zeta), s)}{\partial \lambda} \right| = \left| \frac{\partial P(\lambda_j^2(s, \zeta), s)}{\partial \lambda} \right| \geqslant c > 0.$$

Hence, if r is the radius of Γ we get from (4.22)

$$\lambda_j(s, z) = \lambda_j(s) + z v(t(s)) \left\{ \left[\frac{\partial G(\lambda_j(s), s)}{\partial \lambda}\right]^{-1} + o(s, z) \right\}, \tag{4.23}$$

where for $|z| < r/2$ we have,

$$|o(s, z)| \leqslant \frac{|z|}{cr^2} \sum_{k=2}^{\infty} \left| \frac{z}{r} \right|^{k-2} < \frac{2|z|}{cr^2}.$$

Thus we see that $o(s, z) \to 0$ as $z \to 0$, uniformly in s, provided s remains in some suitable neighborhood of infinity.

From (4.23) we get

$$\begin{aligned} \operatorname{Re}[\lambda_j(s, z) - \lambda_k(s, z)] &= \operatorname{Re}[\lambda_j(s) - \lambda_k(s)] + v(t(s)) \\ &\times \operatorname{Re} z \left\{ \left[\frac{\partial G(\lambda_j(s), s)}{\partial \lambda}\right]^{-1} - \left[\frac{\partial G(\lambda_k(s), s)}{\partial \lambda}\right]^{-1} + o(s, z) \right\}. \end{aligned} \tag{4.24}$$

Using this formula we see that if $\text{Re}[\lambda_j - \lambda_k] \neq 0$, then for fixed z, $\text{Re}[\lambda_j(s, z) - \lambda_k(s, z)]$ always has the same sign, provided that s is sufficiently large, since $v(t(s)) \to 0$, $\lambda_j(s) \to \lambda_j$, and $\lambda_k(s) \to \lambda_k$ as $s \to \infty$. On the other hand, if $\text{Re}[\lambda_j - \lambda_k] = 0$, then we claim that for all sufficiently small z, except on a finite number of straight lines through the origin, and for all sufficiently large s, $\text{Re}[\lambda_j(s, z) - \lambda_k(s, z)]$ remains positive or negative, for every j, k, $j \neq k$, $1 \leqslant |j|$, $|k| \leqslant n$. Indeed since $G'(\lambda_j) - G'(\lambda_k) \neq 0$ we see that $\text{Re}\, z\{G'(\lambda_j)^{-1} - G'(\lambda_k)^{-1}\} = 0$ only for those z which lie on the straight line

$$x \,\text{Re}\{G'(\lambda_j) - G'(\lambda_k)\} - y \,\text{Im}\{G'(\lambda_j)^{-1} - G'(\lambda_k)^{-1}\} = 0.$$

Take z_0 sufficiently small and not on any of these lines, so that for all sufficiently large s,

$$\text{Re}\, z_0\{G'(\lambda_j(s))^{-1} - G'(\lambda_k(s))^{-1} + o(s, z_0)\}$$

maintains its sgn. Since $v(t(s))$ remains positive in the interval $[s_0, \infty)$, s_0 sufficiently large, it follows from (4.24) that $\text{Re}[\lambda_j(s, z_0) - \lambda_k(s, z_0)]$ maintains its sgn for all sufficiently large s, which establishes our claim. From this it follows easily that we may apply Levinson's asymptotic theorem.

Let $\lambda_j(s, z_0)$, $1 \leqslant j \leqslant n$, $s \neq \infty$, be the solutions to Eq. (4.20), whose real parts lie in the right half of the complex plane. Let $v_j(s, z_0)$, $1 \leqslant j \leqslant n$, be the corresponding solutions to (3.3) and $v_{j1}(s, z_0)$ their first components. We want to show that no nontrivial linear combination of $u_j(t, z_0) = v_{j1}(s(t), z_0)$, $1 \leqslant j \leqslant n$, lies in L^2. Using (4.11), which is a consequence of Levinson's theorem, we may write

$$\sum_{j=1}^{n} c_j v_{j1}(s, z_0) = Q_n(t(s)) \sum_{j=1}^{n} c_j \left\{\exp \int_{s_0}^{s} \lambda_j(\sigma, z_0)\, d\sigma\right\} \{1 + o_j(1)\}.$$

If $c_k \neq 0$ and $c_j = 0$ for $j \neq k$, then we have

$$|c_k|^2 \,|v_{k1}(s, z_0)|^2 = |c_k|^2 Q_n{}^2(t(s)) \left\{\exp 2 \int_{s_0}^{s} \lambda_k(\sigma, z_0)\, d\sigma\right\} \{1 + o_k(1)\}.$$

We may take s_0 so large that if $s \geqslant s_0$ we have

$$|1 + o_k(1)| \geqslant 1/2,$$

$$\exp 2 \int_{s_0}^{s} \text{Re}\, \lambda_k(\sigma, z_0)\, d\sigma \geqslant 1/2.$$

Consequently,

$$\int_{t_0}^{\infty} | c_k u_{k1}(t, z_0)|^2 \, dt \geqslant \frac{| c_k |^2}{4} \int_{t_0}^{\infty} Q_n^2(t) \, dt = \infty,$$

where, of course, we have taken $u_{j1}(t, z_0) = v_{j1}(s(t), z_0)$ and $t_0 = t(s_0)$.

Let us now consider the case where more than one c_j is different from zero. Since $v(t(s)) \neq 0$, for all sufficiently large s, $s \neq \infty$, the hypotheses of the theorem require that there can be at most two roots of $P(\lambda^2)$ which have the same real part. For the sake of convenience we shall suppose we have labeled the roots of $P(\lambda^2)$ so that $\operatorname{Re} \lambda_1 \geqslant \operatorname{Re} \lambda_2 \geqslant \cdots \geqslant \operatorname{Re} \lambda_n \geqslant 0$, and we have labeled the complex numbers c_j so that $c_1 \neq 0$. If $\operatorname{Re} \lambda_1 > \operatorname{Re} \lambda_2$, in which case λ_1 is real, we have

$$\Big| \sum_{j=1}^{n} c_j \Big\{ \exp \int_{s_0}^{s} \lambda_j(\sigma, z_0) \, d\sigma \Big\} \{1 + o_j(1) \Big|$$

$$= \exp \int_{s_0}^{s} \operatorname{Re} \lambda_1(\sigma, z_0) \, d\sigma \, \Big| \sum_{j=1}^{n} c_j \Big\{ \exp \int_{s_0}^{s} [\lambda_j(\sigma, z_0) - \lambda_1(\sigma, z_0)] \, d\sigma \Big\} \{1 + o_j(1)\} \Big|.$$

For s_0 sufficiently large there exists an $s_1 > s_0$ so that $s \geqslant s_1$ implies

$$\Big| \sum_{j=2}^{n} c_j \Big\{ \exp \int_{s_0}^{s} [\lambda_j(\sigma, z_0) - \lambda_1(\sigma, z_0) \, d\sigma] \Big\} \{1 + o_j(1)\} \Big| < \frac{| c_1 |}{4},$$

$$\Big| \exp 2 \int_{s_0}^{s} \lambda_1(\sigma, z_0) \, d\sigma \Big| > \frac{1}{2},$$

$$| c_1\{1 + o_1(1)\}| > \frac{| c_1 |}{2}.$$

Hence,

$$\int_{t_0}^{\infty} | Q_n(t)|^2 \Big| \sum_{j=1}^{n} c_j \exp u_{j1}(t, z_0) \Big|^2 \, dt$$

$$\geqslant \int_{t_0}^{\infty} Q_n^2(t) \exp 2 \int_{t_0}^{s(t)} \lambda_1(\sigma, z_0) \, d\sigma \Big[| c_1\{1 + o_1(1)\}|$$

$$- \Big| \sum_{j=2}^{n} c_j \Big\{ \exp \int_{t_0}^{s(t)} (\lambda_j(\sigma, z_0) - \lambda_1(\sigma, z_0)) \, d\sigma \Big\} \{1 + o_j(1)\} \Big| \Big]^2 \, dt$$

$$\geqslant \frac{| c_1 |}{8} \int_{t_0}^{\infty} | Q_n(t)|^2 \, dt = \infty.$$

Now let us suppose that $\operatorname{Re}\lambda_1 = \operatorname{Re}\lambda_2$, $c_1 \neq 0$, and without loss of generality we may take $\operatorname{Im}\lambda_2 > 0$. Again using Levinson's theorem we may write

$$\int_{t_0}^{\infty} \Big| \sum_{j=1}^{n} c_j u_{j1}(t, z_0) \Big|^2 dt$$

$$= \int_{t_0}^{\infty} Q_n^2(t) \Big| \sum_{j=1}^{n} c_j \Big\{\exp \int_{s_0}^{s(t)} \lambda_j(\sigma, z_0)\, d\sigma\Big\} \{1 + o_j)1)\} \Big|^2 dt$$

$$= \int_{t_0}^{\infty} Q_n^2(t) \exp 2 \int_{s_0}^{s(t)} \operatorname{Re}\lambda_1(\sigma, z_0)\, d\sigma$$

$$\times \Big| \sum_{j=1}^{n} c_j \Big\{\exp \int_{s_0}^{s(t)} [\lambda_j(\sigma, z_0) - \lambda_1(\sigma, z_0)]\, d\sigma\Big\} \{1 + o_j(1)\} \Big|^2 dt.$$

Clearly, for $k = 1, 2$,

$$\sum_{j=3}^{n} c_j \Big\{\exp \int_{s_0}^{s} [\lambda_j(\sigma, z_0) - \lambda_k(\sigma, z_0)]\, d\sigma\Big\}\{1 + o_j(1)\} = o(1) \qquad \text{as} \quad s \to \infty.$$

Hence,

$$\int_{t_0}^{\infty} \Big| \sum_{j=1}^{n} c_j u_{j1}(t, z_0) \Big|^2 dt$$

$$= | c_1 |^2 \int_{t_0}^{\infty} Q_n^2(t) \exp 2 \int_{s_0}^{s(t)} \operatorname{Re}\lambda_1(\sigma, z_0)\, d\sigma$$

$$\times \Big| \Big\{1 + \frac{c_2}{c_1} \exp \int_{s_0}^{s(t)} [\lambda_2(\sigma, z_0) - \lambda_1(\sigma, z_0)]\, d\sigma\Big\} \{1 + o(1)\} \Big|^2 dt.$$

Now, as $\sigma \to \infty$, $\operatorname{Re}\lambda_1(\sigma, z_0) \to \operatorname{Re}\lambda_1 \geqslant 0$ and $\operatorname{Re}\lambda_2(\sigma, z_0) \to \operatorname{Re}\lambda_2 = \operatorname{Re}\lambda_1 \geqslant 0$. Hence, if t_0, and thus $s_0 = s(t_0)$ is sufficiently large, there is a positive constant c so that

$$\int_{t_0}^{\infty} \Big| \sum_{j=1}^{n} c_j u_{j1}(t) \Big|^2 dt$$

$$\geqslant c \, | c_1 |^2 \int_{t_0}^{\infty} Q_n^2(t) \Big| 1 - \Big| \frac{c_2}{c_1} \Big| \exp \int_{s_0}^{s(t)} \operatorname{Re}[\lambda_2(\sigma, z_0) - \lambda_1(\sigma, z_0)]\, d\sigma \Big|^2 dt. \tag{4.25}$$

As we have shown previously, if z_0 is taken sufficiently small (and not on a finite number of certain straight lines through the origin) and for all σ sufficiently large, $\mathrm{Re}[\lambda_2(\sigma, z_0) - \lambda_1(\sigma, z_0)]$ maintains its sgn. Without any loss of generality we may suppose its sgn is -1. Hence, for s_0 sufficiently large,

$$\exp \int_{s_0}^{s(t)} \mathrm{Re}[\lambda_2(\sigma, z_0) - \lambda_1(\sigma, z_0)]\, d\sigma$$

is monotone decreasing. If this goes to zero, (4.25) shows that $\sum_1^n c_j u_{j1} \notin L^2$. On the other hand, if this goes to a positive constant d, and $|c_2/c_1| \neq 1/d$, then we again have the same conclusion. Thus we are led to consider the integral

$$\int_{t_0}^{\infty} Q_n^2(t) \left| 1 + \frac{c_2}{c_1} \exp \int_{s_0}^{s(t)} [\lambda_2(\sigma, z_0) - \lambda_1(\sigma, z_0)]\, d\sigma \right|^2 dt,$$

where $c_2/c_1 = e^{i\beta}/d$, β real. We can rewrite this integral as

$$\int_{t_0}^{\infty} Q_n^2(t) \Big| 1 + \exp \Big\{ i \int_{s_0}^{s(t)} \mathrm{Im}[\lambda_2(\sigma, z_0) - \lambda_1(\sigma, z_0)]\, d\sigma + i\beta$$

$$\times (1/d) \exp \int_{s_0}^{s(t)} \mathrm{Re}[\lambda_2(\sigma, z_0) - \lambda_1(\sigma, z_0)]\, d\sigma \Big\} \Big|^2 dt.$$

The last term of this integral is $\{1 + o(1)\}$ so that if t_0 is taken large enough, the last integral is greater than some positive constant times the integral

$$\int_{t_0}^{\infty} Q_n^2(t) \left| 1 + \exp \left\{ i \int_{s_0}^{s(t)} \mathrm{Im}[\lambda_2(\sigma, z_0) - \lambda_1(\sigma, z_0)]\, d\sigma + i\beta \right\} \right|^2 dt$$

$$= \int_{s_0}^{\infty} Q_0(t(s)) Q_n^2(t(s)) \left| 1 + \exp \left\{ i \int_{s_0}^{s} \mathrm{Im}[\lambda_2(\sigma, z_0) - \lambda_1(\sigma, z_0)]\, d\sigma + i\beta \right\} \right|^2 ds$$

$$= \int_{s_0}^{\infty} \{v(t(s)) + r(t(s))\} \left| 1 + \exp \left\{ i \int_{s_0}^{s} \mathrm{Im}[\lambda_2(\sigma, z_0) - \lambda_1(\sigma, z_0)]\, d\sigma + i\beta \right\} \right|^2 ds.$$

The integrand

$$r(t(s)) \left| 1 + \exp \left\{ i \int_{s_0}^{s} \mathrm{Im}[\lambda_2(\sigma, z_0) - \lambda_1(\sigma, z_0)]\, d\sigma + i\beta \right\} \right|^2$$

belongs to L^1. Hence, if we can show that

$$\int_{s_0}^{\infty} v(t(s)) \left| 1 + \exp \left\{ i \int_{s_0}^{s} \operatorname{Im}[\lambda_2(\sigma, z_0) - \lambda_1(\sigma, z_0)] \, d\sigma + i\beta \right\} \right|^2 ds$$

is infinite we will have shown that the deficiency index of L_0 is (n, n), since we have that (2.1) is regular at the origin and hence its deficiency index is at least (n, n).

Working in the same way as in the proof of part (A), for every positive integer k let us take $I_k = [s_k, s_k']$ and $J_k = [s_k', s_{k+1}]$, where we have taken s_k and s_k' so that

$$\beta + \int_{s_0}^{s_k} \operatorname{Im}[\lambda_2(\sigma, z_0) - \lambda_1(\sigma, z_0)] \, d\sigma = 2k\pi + \frac{3}{2}\pi,$$
$$\beta + \int_{s_0}^{s_k'} \operatorname{Im}[\lambda_2(\sigma, z_0) - \lambda_1(\sigma, z_0)] \, d\sigma = 2k\pi + \frac{5}{2}\pi. \tag{4.26}$$

This is always possible, for sufficiently large k, since

$$\int_{s_0}^{s} \operatorname{Im}[\lambda_2(\sigma, z_0) - \lambda_1(\sigma, z_0)] \, d\sigma$$

is a monotone-increasing continuous function of s which takes all values in the interval $[0, \infty)$. Note that the integral diverges since

$$\operatorname{Im}[\lambda_2(\sigma, z_0) - \lambda_1(\sigma, z_0)] \to 2 \operatorname{Im} \lambda_2 > 0, \qquad \text{as} \quad \sigma \to \infty.$$

From (4.26) it follows that

$$\left| 1 + \exp \left\{ i \int_{s_0}^{s_k} \operatorname{Im}[\lambda_2(\sigma, z_0) - \lambda_1(\sigma, z_0)] \, d\sigma + i\beta \right\} \right|^2$$

takes all values in $[2, 4]$ as s varies over I_k, and takes on all values in $[0, 2]$ as s varies over J_k.

Let us now set

$$\omega(\sigma, z_0) = \operatorname{Im}[\lambda_2(\sigma, z_0) - \lambda_1(\sigma, z_0)] - 2 \operatorname{Im} \lambda_2 .$$

Then for k sufficiently large

$$\left(\int_{s_0}^{s_k'} - \int_{s_0}^{s_k}\right) \operatorname{Im}[\lambda_2(\sigma, z_0) - \lambda_1(\sigma, z_0)]\, d\sigma$$

$$= \int_{s_k}^{s_k'} \omega(\sigma, z_0)\, d\sigma + 2 \operatorname{Im} \lambda_2(s_k' - s_k) = \pi.$$

From the mean-value theorem there exists an η_k so that

$$\int_{s_k}^{s_k'} \omega(\sigma, z_0)\, d\sigma = \eta_k(s_k' - s_k).$$

Thus we have

$$s_k' - s_k = \pi/(2 \operatorname{Im} \lambda_2 + \eta_k). \tag{4.27}$$

Since $\omega(\sigma, z_0) \to 0$ as $\sigma \to \infty$, it follows that $\eta_k \to 0$ as $k \to \infty$. In the same way there exists a sequence $\{\mu_k\}$ so that $\mu_k \to 0$ as $k \to \infty$ and

$$s_{k+1} - s_k' = \pi/(2 \operatorname{Im} \lambda_2 + \mu_k). \tag{4.28}$$

From (4.27) and (4.28) we see that there exist two positive constants b and B so that for all sufficiently large k,

$$b \mid J_k \mid \leqslant \mid I_k \mid \leqslant B \mid J_{k-1} \mid,$$

where $\mid I_k \mid$ and $\mid J_k \mid$ are the lengths of I_k and J_k, respectively. From this it follows that both series

$$\sum_{k=k_0}^{\infty} \int_{I_k} m(t(s))\, ds, \quad \sum_{k=k_0}^{\infty} \int_{J_k} m(t(s))\, ds$$

both diverge or converge simultaneously. However, since $m(t(s)) \notin L^1$ we see they both diverge. Hence, every nontrivial linear combination of the u_{j1}, $1 \leqslant j \leqslant n$, does not belong to L^2. The proof is complete.

5. Some Special Cases

We now want to show that by properly choosing $Q_0, Q_1, \ldots, Q_n$ we can obtain from Theorem 1 a number of theorems which have appeared in the literature. We start with a theorem of M. A. Naimark [5, p. 195].

THEOREM. *Let the following conditions be fulfilled*:

1. $|q_n(t)| \to \infty$ *as* $t \to \infty$;
2. q_n', q_n'' *maintain their* sgn *in a neighborhood of infinity*;
3. *for* $t \to \infty$,

$$q_n' = 0(|q_n|^{\gamma}), \qquad 0 < \gamma < 1 + 1/2n;$$

4. q_0'/q_0, $q_1 |q_n|^{-1/2n}$, $q_2 |q_n|^{-3/2n}, \ldots, q_{n-1} |q_n|^{-(2n-3)/2n}$ *are integrable*;
5. $\lim_{t\to\infty} q_0(t) > 0$.

If $q_n(t) \to \infty$ *for* $t \to \infty$, *then* L_0 *has deficiency index* (n, n).

If $q_n(t) \to -\infty$ *for* $t \to \infty$, *then* L_0 *has deficiency index* (n, n) *or* $(n + 1, n + 1)$ *if the integral*

$$\int^{\infty} |q_n|^{-1+1/2n}\, dt$$

diverges or converges, respectively.

To show this theorem is a special case of Theorem 1 we take $Q = |q_n|^{-1/2n}$, $Q_0 = Q$ and $Q_{n-k} = Q^{\rho-k}$, where $\rho = (2n - 1)/2$. Since $|q_n(t)| \to \infty$ as $t \to \infty$, we have $\int^{\infty} Q_0^{-1} = \infty$. We are also implicitly assuming here that the differential operator is regular at the origin. This then falls under Case I in Section 3. Recalling the terminology used in Sections 3 and 4 we have

$$a_0 = 1/q_0, \qquad a_k = q_k |q_n|^{-k/n}, \qquad 1 \leqslant k \leqslant n,$$

$$b_k = \frac{2k-1}{2} \frac{d(|q_n|^{-1/2n})}{dt}, \qquad 1 \leqslant k \leqslant n,$$

$$d_k = 1, \qquad 1 \leqslant k \leqslant n - 1.$$

By hypothesis 4 of this theorem we have that the function

$$Q_0^{-1}(t)\, a_k(t) = q_k(t) |q_n(t)|^{-(2k-1)/2n} \tag{5.1}$$

is integrable for $1 \leqslant k \leqslant n - 1$, and hence we must take $\alpha_k = 0$ for these values of k. Further, from (5.1), if $q_n(t) \to \infty$ or $q_n(t) \to -\infty$, then $a_n = 1$ or $a_n = -1$ for all sufficiently large t, respectively, and thus we must take $\alpha_n = 1$ or $\alpha_n = -1$, respectively. From the fact that q_0'/q_0 is integrable it follows that $\lim_{t\to\infty} \ln q_0(t)$ exists as a finite number and hence $\lim_{t\to\infty} q_0(t)$ exists, and by hypothesis 5 this is a positive number; we label it $1/\alpha_0$. If we write

$$1/q_0(t) = \alpha_0 + v_0(t),$$

then by hypotheses 4 and 5, $v_0(t) \to 0$ as $t \to \infty$ and $v_0'(t) = q_0'/q_0^2$ is integrable.

It remains to examine the functions $b_k(t)$. Now,

$$b_k = (k - \tfrac{1}{2}) \frac{dQ}{dt} = (k - \tfrac{1}{2}) \frac{d \mid q_n \mid^{-1/2n}}{dt}.$$

Since q_n ultimately maintains its sgn, and the derivative of $\mid q_n \mid^{-1/2n}$ exists, we have

$$b_k = C_k \mid q_n \mid^{-(1+1/2n)} q_n', \qquad C_k = \pm(2k-1)/2n. \tag{5.2}$$

From hypotheses 1 and 3 it follows that $b_k(t) \to 0$ as $t \to \infty$. Further

$$b_k' = C_k\{(1 + 1/2n) \mid q_n \mid^{-(2+1/2n)} (q_n')^2 \pm \mid q_n \mid^{-(1+1/2n)} q_n''\}.$$

Now use hypotheses 1, 2, and 3; if $\gamma = 1 + 1/2n - \epsilon$, $\epsilon > 0$, then

$$\int_{t_0}^{\infty} \mid q_n \mid^{-(2+1/2n)} \mid q_n' \mid \leqslant C \int_{t_0}^{\infty} \mid q_n \mid^{-1-\epsilon} \mid q_n' \mid$$

$$= C \int_{t_0}^{\infty} \mid q_n \mid^{-1-\epsilon} \mid q_n \mid',$$

where C is some positive constant and t_0 is sufficiently large so that $q_n(t_0) \neq 0$, and $\mid q_n'(t) \mid = \mid q_n(t) \mid'$ for $t \geqslant t_0$. Note that the very last statement of the last sentence is possible since from condition 2, $q_n(t)$ ultimately monotonically increases to ∞ or monotonically decreases to $-\infty$. Further,

$$\pm \int_{t_0}^{\infty} \mid q_n \mid^{-(1+1/2n)} q_n'' = \int_{t_0}^{\infty} \mid q_n \mid^{-(1+1/2n)} \mid q_n \mid''$$

$$= - \mid q_n(t_0) \mid^{-(1+1/2n)} \mid q_n \mid'$$

$$+ \left(1 + \frac{1}{2n}\right) \int_{t_0}^{\infty} \mid q_n \mid^{-(2+1/2n)} \{\mid q_n \mid'\}^2 < \infty.$$

The last integral is finite since by hypothesis 3,

$$| q_n |^{-(2+1/2n)}\{| q_n |'\}^2 \leqslant C | q_n |^{-1-\epsilon} | q_n |',$$

where ϵ has been taken as previously. Hence $b_k' \in L^1$ and thus we must take $\beta_k = 0$ in (4.2).

We claim that the function

$$Q_n^2(t) \exp \delta \int_{t_0}^t Q_0^{-1} = Q^{2n-1}(t) \exp \delta \int_{t_0}^t Q^{-1}$$

is not integrable for every $\delta > 0$, regardless of whether Q_n^2 is integrable or not. Actually, we can show that

$$f(t) = Q^\rho(t) \exp \delta \int_{t_0}^t \frac{d\tau}{Q(\tau)}$$

is not summable for every $\rho \geqslant 0$ and every $\delta > 0$. Indeed, recalling that $s(t) = \int_0^t Q^{-1}(\tau)\, d\tau$ we get for $\rho > 0$,

$$f^{-1/\rho}(t) = \frac{ds(t)}{dt} \exp\{(-\delta/\rho)[s(t) - s_0]\}.$$

Integrating we get

$$\int_{t_0}^t f^{-1/\rho}(t)\, dt = \int_{s_0}^s \exp\{(-\delta/\rho)[\sigma - s_0]\, d\sigma.$$

Hence, the set of points in (t_0, ∞) where $f^{-1/\rho}(t) \leqslant 1$ is of infinite Lebesgue measure, and thus the set where $f(t) \geqslant 1$ is of infinite Lebesgue measure so that f is not integrable. If $\rho = 0$, then our claim is immediate.

We also claim that the function

$$Q_0(t) Q_n^2(t) \exp \delta \int_{t_0}^t \frac{d\tau}{Q_0(\tau)} = Q^{2n}(t) \exp \delta \int_{t_0}^t \frac{d\tau}{Q(\tau)}$$

has, for every $\delta > 0$, a positive nonincreasing minorant $m(t)$ so that $m(t)/Q(t)$ is not integrable. Indeed, we have

$$Q^{2n}(t) \exp \delta \int_{t_0}^t \frac{d\tau}{Q(\tau)} = | q_n(t)|^{-1} \exp \delta \int_{t_0}^t | q_n(\tau)|^{1/2n}\, d\tau$$

$$\geqslant | q_n(t)|^{-1} \exp c\delta \int_{t_0}^t | q_n(\tau)|^{(1/2n)-\gamma} | q_n(\tau)|'\, d\tau,$$

where c is a positive constant and the inequality comes from condition 3, with γ as given there. Notice that here again we have used the fact that $| q_n'(t)| = | q_n(t)|'$ for all sufficiently large t. Hence, we get

$$Q^{2n}(t) \exp \delta \int_{t_0}^{t} \frac{d\tau}{Q(\tau)} \geqslant | q_n(t)|^{-1} \exp \left\{ \frac{c\delta}{\epsilon} (| q_n(t)|^{\epsilon} - | q_n(t_0)|^{\epsilon}) \right\},$$

where $\epsilon = 1 + (1/2n) - \gamma > 0$. Since $| q_n(t)| \to \infty$ as $t \to \infty$ we see that if t_0 is sufficiently large we may take as a minorant for the left side, any positive constant m. Since $1/Q_0(t)$ is not integrable, it follows that the same is true for $m/Q_0(t)$.

If $Q_n^2(t) = | q_n(t)|^{-1+1/2n}$ is integrable, then we have just proved above that the boundary line of the plane $\mathscr{P}$ is $\{z : \operatorname{Re} z \leqslant 0\}$ and that hypothesis (h_2-ii) holds.

In the case we are considering here, the polynomial (4.3) becomes

$$P(\mu) = \alpha_0 \mu^n + (-1)^n \alpha_n , \qquad \alpha_0 > 0. \tag{5.3}$$

If $q_n(t) \to \infty$ as $t \to \infty$, then recall that $\alpha_n = 1$, and we are considering solutions to the equation

$$\alpha_0 \mu^n = (-1)^{n+1}.$$

This equation has no negative real solutions. Further, $\alpha_0 \lambda^{2n} = (-1)^{n+1}$ can have at most two solutions with the same real part. If $Q_n^2 = |q_n|^{-1+1/2n}$ is integrable we may apply Theorem 1(A), conclusion (c_3). We see that the deficiency index is (n, n).

If $Q_n^2 = | q_n |^{-1+1/2n}$ is not integrable we wish to apply the conclusion (c_3') of part (B) of Theorem 1. However, we must be sure that the hypothesis (h_3') is satisfied. Now,

$$v = Q_0 Q_n^2 = Q^{2n} = | q_n |^{-1}$$

is ultimately monotonically decreasing to zero. Further, it is clear that

$$dv/dt = - | q_n |^{-2} | q_n |'$$

is integrable, but, by assumption $v/Q_0 = | q_n |^{-1+1/2n}$ is not integrable. Finally,

$$G(\lambda) = \alpha_0 \lambda^{2n} + (-1)^{n+1},$$

$$G'(\lambda) = 2n\alpha_0 \lambda^{2n-1}.$$

If λ_i and λ_j are distinct roots of $G(\lambda)$, and $\mathrm{Re}(\lambda_i - \lambda_j) = 0$, then $\lambda_i = \bar{\lambda}_j$. Hence, we have

$$G'(\lambda_i) - G'(\bar{\lambda}_i) = i4n\alpha_0 \operatorname{Im} \lambda_i^{2n-1} \neq 0.$$

Further since the differential operator of this theorem is regular at the origin all of the conditions of (c_3'–ii) are fulfilled and the deficiency index is (n, n).

If $q_n(t) \to -\infty$ as $t \to \infty$, then we must have $\alpha_n = 1$. In this case we are considering the solutions to

$$\alpha_0 \mu^n = (-1)^n.$$

This equation has one and only one negative real solution. If $Q_n^2 = |q_n|^{-1+1/2n}$ is integrable we may apply (c_3) of part (A) of Theorem 1 and in this case the deficiency index is $(n + 1, n + 1)$. If $|q_n|^{-1+1/2n}$ is not integrable we may apply (c_3'–ii) of part (B) of Theorem 1, and in this case the deficiency index is (n, n).

Let us now give an improved version of a theorem originally given by S. A. Orlov [7] and then improved by F. A. Neimark [6].

THEOREM. *Suppose*

$$q_0(t) = t^{2n+\nu}[1/\alpha_0 + r_0(t)]^{-1} \quad and \quad q_k(t) = t^{2(n-k)+\nu}[\alpha_k + r_k(t)], \quad 1 \leqslant k \leqslant n,$$

where $t^{-1}r_k(t)$ is integrable, $0 \leqslant k \leqslant n$, and ν is any real number. Set

$$F(\lambda, \nu) = \sum_{k=1}^{n} (-1)^k \alpha_{n-k} \prod_{j=0}^{k-1} \left[\left(\lambda + \frac{\nu}{2}\right)^2 - \left(\frac{\nu+1}{2} + j\right)^2\right] + \alpha_n,$$

and suppose all roots of this polynomial are simple. Then the number of linearly independent L^2 solutions to (2.3) is, for $\nu > 0$, the number of roots of $F(\lambda, \nu)$ with Re $\lambda < \nu$, *and for $\nu \leqslant 0$ is n.*

This theorem is a special case of Theorem 1. To see this let us, at first, suppose that $\nu > 0$ and take $Q = 1/t$, $Q_0 = t$, and $Q_0 Q_k^2 = t^{-2(n-k)-\nu}$. It is clear that $Q_0 Q_k^2 = Q^{2(\rho+n-k)-1}$, where $\rho = (\nu + 1)/2$. Thus, we are in the situation under Case II given at the end of Section 3.

We have

$$a_0(t) = (1/\alpha_0) + r_0(t),$$
$$a_k(t) = a_k + r_k(t), \qquad 1 \leqslant k \leqslant n,$$
$$b_k(t) = -[(\nu + 1)/2 + n - k] = \beta_k, \qquad 1 \leqslant k \leqslant n.$$

Further, by hypothesis, $r_k(t)/Q_0(t) = r_k(t)/t \in L^1$. Hence we may take $V = 0$ in the decomposition (4.1) of $D(t, z)$. The polynomial (4.3) becomes

$$P(\lambda^2) = \sum_{k=1}^{n} (-1)^k \alpha_{n-k} \prod_{j=0}^{k-1} (\lambda^2 - [(\nu + 1)/2 + j]^2) + \alpha_n$$

$$= F(\lambda - \nu/2, \nu).$$

Now

$$Q_n{}^2 \exp 2\delta \int_1^t Q_0^{-1} = t^{2\delta-\nu-1},$$

and this belongs to L^1 for $\delta < \nu/2$. Since $\nu > 0$, it also follows that $Q_n{}^2(t) \in L^1$. Further,

$$Q_0(t) Q_n{}^2(t) \exp \nu \int_1^t Q_0^{-1} = 1,$$

so that this function has a constant minorant. Consequently, we get one part of the theorem from the conclusion (c_1-ii) of part (A). If $\nu = 0$, then $Q_n{}^2(t) \notin L^1$, but $Q_0 Q_n{}^2 = 1$, so that the conclusion follows from (c_3'-i) of part (B).

Now, let us suppose $\nu < 0$. Let us take $Q_0(t) = t^{1+(\nu/2n)}$ and $Q_k(t) = Q_0(t)^{(2k-1)/2} t^{-(2n+\nu)/2}$, $1 \leqslant k \leqslant n$. Then

$$a_0(t) = \frac{Q_0(t)}{Q_1{}^2(t)\, q_0(t)} = 1/\alpha_0 + r_0(t).$$

$$a_k(t) = Q_0(t) Q_k{}^2(t)\, q_k(t)$$

$$= Q_0^{2k}(t)\, t^{-(2n+\nu)} t^{2(n-k)+\nu} [\alpha_k + r_k(t)]$$

$$= t^{(k/n)\nu} [\alpha_k + r_k(t)], \qquad 1 \leqslant k \leqslant n.$$

Hence,

$$\frac{a_k(t)}{Q_0(t)} = t^{-1+[(k/n)-(1/2n)]\nu} [\alpha_k + r_k(t)].$$

Since $[(k/n) - (1/2n)]\nu < 0$, we see that $a_k(t)/Q_0(t) \in L^1$. Further we have

$$b_k(t) = \frac{Q_0(t)}{Q_k(t)} \frac{dQ_k(t)}{dt} = \left[\left(\frac{2k-1}{2}\right)\left(1 + \frac{\nu}{2n}\right) - \frac{2n+\nu}{2}\right] t^{\nu/2n}.$$

Clearly $b_k(t) \to 0$ as $t \to \infty$, and $b_k'(t) \in L^1$. Finally

$$d_k(t) = Q_0(t) Q_k(t)/Q_{k+1}(t) = 1.$$

and

$$Q_0(t) Q_n^2(t) = 1.$$

This shows that $Q_n^2(t) \notin L^1$, and our result now follows from conclusion (c_3'-i) of part (B).

The following two results given by Naimark [5, pp. 192, 193] fall under Case III at the end of Section 3. The proofs are immediate consequences of (c_3'-i) of part (B).

(a) *If* $(1/q_0)'$, $q_1, \ldots, q_n$ *are integrable and* $\lim_{t\to\infty} q_0(t) > 0$, *then* L_0 *has the deficiency index* (n, n).

(b) *If there are constants* $\alpha_0 \neq 0, \alpha_1, \ldots, \alpha_n$ *such that*

$$\frac{1}{q_0} - \frac{1}{\alpha_0}, q_1 - \alpha_1, \ldots, q_n - \alpha_n$$

are integrable, then L_0 *has the deficiency index* (n, n).

6. Examples

In this section we shall give some examples which show that all of the possibilities for the deficiency index listed in (c_2) of part (A) can occur.

I. We shall consider a matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & \alpha_1 & -1 & -1 \\ \alpha_2 & 0 & 0 & -1 \end{bmatrix}.$$

The polynomial (4.3) is given by

$$P(\mu) = \sum_{k=0}^{1} (-1)^k \alpha_k \prod_{j=k+1}^{2} (\mu - 1) + \alpha_2 \qquad (\alpha_0 = 1)$$
$$= (\mu - 1)^2 - \alpha_1(\mu - 1) + \alpha_2 . \tag{6.1}$$

Pick α_1 and α_2 so that if μ_1 and μ_2 are the roots of (6.1), then

$$\begin{aligned} \mu_1 - 1 &= 2i - 1, \\ \mu_2 - 1 &= -2i - 1. \end{aligned} \tag{6.2}$$

From this we can compute α_1 and α_2 . Indeed,

$$\begin{aligned} \alpha_1 &= (\mu_1 - 1) + (\mu_2 - 1) = -2, \\ \alpha_2 &= (\mu_1 - 1)(\mu_2 - 1) = 5. \end{aligned} \tag{6.3}$$

Let us now take $Q_0(t) = 1$ so that $s(t) = t = t(s) = s$. Also let us take $a_0 = 1, Q_1(t) = Q_2(t) = \exp(-t)$. Consequently we must have

$$a_0 = Q_0/Q_1^2 q_0 = 1 \tag{6.4}$$

so that

$$q_0 = \exp 2t.$$

Further, we have,

$$\begin{aligned} a_j(t) &= Q_0(t) Q_1^2(t)\, q_j(t) = q_j(t) \exp(-2t), \qquad j = 1, 2, \\ b_j(t) &= \frac{Q_0(t)}{Q_j(t)} \frac{dQ_j(t)}{dt} = -1, \qquad j = 1, 2, \\ d_1(t) &= Q_0(t) Q_1(t)/Q_2(t) = 1. \end{aligned} \tag{6.5}$$

Let us now take

$$q_j(t) = a_j(t) \exp(2t), \qquad j = 1, 2, \tag{6.6}$$

and set $a_j(t) = \alpha_j + v_j(t)$, $j = 1, 2$, where the α_j are given by (6.3). Hence, we get

$$a_j(t) = Q_0(t) Q_1^2(t)\, q_j(t) = \alpha_j + v_j(t), \qquad j = 1, 2. \tag{6.7}$$

From (6.2) we get

$$\begin{aligned} \lambda_{\pm 1} &= \pm \sqrt{\mu_1} = \pm(1 + i), \\ \lambda_{\pm 2} &= \pm \sqrt{\mu_2} = \pm(1 - i). \end{aligned} \tag{6.8}$$

Now

$$Q_2^2(t) \exp 2\delta \int_0^t \frac{d\tau}{Q_0(\tau)} = Q_2^2(t) \exp 2\delta t$$

$$= \exp 2(\delta - 1)t.$$

This belongs to L^1 if and only if $\delta < 1$ and hence the half-plane $\mathscr{P}$ is the set $\{z : \operatorname{Re} z < 1\}$. Remember we have that $Q_2 \in L^2$. From (6.8) we see that $\mathscr{P}$ contains two roots of the characteristic polynomial of the matrix A. Hence in this case we have $m = 2$.

Let us now set

$$V(t) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & v_1(t) & 0 & 0 \\ v_2(t) & 0 & 0 & 0 \end{bmatrix}$$

and we get

$$A + V(t) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -2 + v_1(t) & -1 & -1 \\ 5 + v_2(t) & 0 & 0 & -1 \end{bmatrix}. \tag{6.9}$$

The characteristic polynomial of $A + V(t)$, with μ replacing λ^2, is

$$P(\mu, t) = (\mu - 1)^2 - (-2 + v_1(t))(\mu - 1) + 5 + v_2(t). \tag{6.10}$$

The roots $\mu_1(t)$ and $\mu_2(t)$ of this polynomial satisfy the equations

$$\begin{aligned} (\mu_1(t) - 1) + (\mu_2(t) - 1) &= -2 + v_1(t) = a_1(t), \\ (\mu_1(t) - 1)(\mu_2(t) - 1) &= 5 + v_2(t) = a_2(t). \end{aligned} \tag{6.11}$$

Designate the four roots of the characteristic polynomial of (6.9) by $\lambda_1(t)$, $\lambda_{-1}(t) = -\lambda_1(t)$, $\lambda_2(t)$, and $\lambda_{-2}(t) = -\lambda_2(t)$. If we label $\lambda_2(t)$ so that $\lambda_2(t) = \overline{\lambda_1(t)}$, then from (6.11) we get

$$\begin{aligned} \operatorname{Re} \lambda_1^2(t) &= v_1(t)/2, \\ |\lambda_1^2(t) - 1|^2 &= 5 + v_2(t). \end{aligned} \tag{6.12}$$

Set $\lambda_1(t) = \rho(t) + i\sigma(t)$, $\rho(t)$ and $\sigma(t)$ real. Then $\operatorname{Re} \lambda_1^2(t) = \rho^2(t) - \sigma^2(t)$. Choose $\rho(t)$ and $\sigma(t)$ to be differentiable so that $\rho(t)\,\rho'(t) - \sigma(t)\,\sigma'(t)$

belongs to L^1 and $\rho^2(t) - \sigma^2(t) \to 0$ as $t \to \infty$. Hence from the first equation in (6.12) we see that $v_1(t) \to 0$ as $t \to \infty$ and $v_1'(t) \in L^1$. From the second equation in (6.12) we get $|\lambda_1^4(t)| - 2 \operatorname{Re} \lambda_1^2(t) - 4 = v_2(t)$. Since we have chosen $\lambda_1(t)$ so that $2 \operatorname{Re} \lambda_1^2(t) \to 0$, if we want $v_2(t) \to 0$ as $t \to \infty$ we must have $|\lambda_1(t)|^4 \to 4$ as $t \to \infty$. Consequently, we must have

$$\rho^2(t) - \sigma^2(t) \to 0,$$

$$\rho^4(t) + 2\rho^2(t)\,\sigma^2(t) + \sigma^4(t) \to 4, \qquad t \to \infty.$$

From these two equations we see that we must have $\rho^2(t) \to 1$ and $\sigma^2(t) \to 1$ as $t \to \infty$. Without loss of generality we may suppose that $\rho(t) \geqslant 0$ so that $\rho(t) \to 1$ and $\sigma(t) \to 1$ as $t \to \infty$.

Now let us choose

$$\rho(t) = 1 - 1/t, \qquad \sigma(t) = 1.$$

Clearly $\rho(t)\,\rho'(t) - \sigma(t)\,\sigma'(t)$ is in L^1. Also, it is very easy to check that $v_1(t) = 2 \operatorname{Re} \lambda_1^2(t) \to 0$ as $t \to \infty$ and $v_1'(t) \in L^1$. An equally simple computation shows that $v_2(t) \to 0$ as $t \to \infty$ and $v_2'(t) \in L^1$. Further $Q_2^2(t) \exp 2 \int_1^t [\rho(\tau) \pm i\sigma(\tau)]\, d\tau \in L^1$. This shows that the deficiency index of the corresponding self-adjoint differential operator is (4, 4), while we have seen before that $m = 2$. Hence, the deficiency index is $(m + 2, m + 2)$.

II. Now let us change Q_1 and Q_2 but keep $Q_0 = 1$. We shall take $Q_1(t) = Q_2(t) = \{\exp(-t)\}/t$. Then by choosing $a_0(t) = 1$, we see we must take $q_0(t) = t^2 \exp 2t$. Further, referring to the first equation in (6.5) we must take $q_j(t) = a_j(t)\, t^2 \exp 2t$, $j = 1, 2$. In this case we find

$$b_j(t) = \frac{Q_0(t)}{Q_j(t)} \frac{dQ_j(t)}{dt} = -1 - (1/t),$$

$$d_1(t) = \frac{Q_0(t)\,Q_1(t)}{Q_2(t)} = 1.$$

The matrix A remains the same as in the first example with α_1 and α_2 taken the same as before. But now the matrix $A + V(t)$ becomes

$$A + V(t) = \begin{bmatrix} 1 + 1/t & 1 & 0 & 0 \\ 0 & 1 + 1/t & 1 & 0 \\ 0 & -2 + v_1(t) & -1 - 1/t & -1 \\ 5 + v_2(t) & 0 & 0 & -1 - 1/t \end{bmatrix}. \tag{6.13}$$

The polynomial corresponding to (6.10) is

$$P(\mu, t) = \left(\mu - \left(1 + \frac{1}{t}\right)^2\right)^2 - (-2 + v_1(t))\left(\mu - \left(1 + \frac{1}{t}\right)^2\right) + 5 + v_2(t). \quad (6.14$$

The roots $\mu_1(t)$ and $\mu_2(t)$ of this equation now satisfy

$$\begin{aligned} \mu_1(t) - \left(1 + \frac{1}{t}\right)^2 + \mu_2(t) - \left(1 + \frac{1}{t}\right)^2 &= -2 + v_1(t), \\ \left\{\mu_1(t) - \left(1 + \frac{1}{t}\right)^2\right\}\left\{\mu_2(t) - \left(1 + \frac{1}{t}\right)^2\right\} &= 5 + v_2(t). \end{aligned} \quad (6.15)$$

Working in the same way as before we now get

$$\begin{aligned} \operatorname{Re} \lambda_1^2(t) &= v_1(t)/2 + (1 + 1/t)^2, \\ | \lambda_1^2(t) - (1 + 1/t)^2|^2 &= 5 + v_2(t). \end{aligned} \quad (6.16)$$

We again set $\lambda_1(t) = \rho(t) + i\sigma(t)$, but this time we take

$$\rho(t) = 1 + (1/t), \qquad \sigma(t) = 1.$$

Performing the same kind of computations as in the first example we see that $v_j(t) \to 0$ as $t \to \infty$ and $v_j' \in L^1, j = 1, 2$. However, now we get However, now we get

$$Q_2^2(t) \exp \left\{2\delta \int_0^t \frac{d\tau}{Q_0(\tau)}\right\} = \{\exp(\delta - 1)\, 2t\}/t^2.$$

This belongs to L^1 if and only if $\delta \leqslant 1$. Hence, in this case the plane $\mathscr{P}$ becomes $\{z : x \leqslant 1\}$, and thus $m = 4$. Further,

$$Q_2^2(t) \exp \left\{2 \int_{t_0}^t \rho(\tau)\, d\tau\right\} = C \notin L^1,$$

where C is a nonzero constant. Thus we see that $P(\lambda^2)$ has two roots on its boundary whose corresponding solutions are not in L^2. If we look at a linear combination of these two solutions, then by the same type of argument as we used in the proof of Theorem 1 we see that we are reduced to considering a linear combination of the form

$$Q_2(t) \left\{\exp \int_{t_0}^t \rho(\tau)\, d\tau\right\}\left[1 + \exp \left\{2i \int_{t_0}^t \sigma(\tau)\, d\tau + i\beta\right\}\right].$$

A simple computation shows that for every real β this does not belong to L^2. Hence in this case the deficiency index is $(m - 2, m - 2)$.

III. If in examples I and II we would have taken $\rho(t) = 1 = \sigma(t)$, then in both cases, we would have the deficiency index as (m, m).

IV. Let us now take $\lambda_1 = 1$, $\lambda_2 = 2i$, $Q_0(t) = 1$, and $Q_1(t) = Q_2(t) = \exp(-t)$. In this case we have $\mu_1 = \lambda_1^2 = 1$, $\mu_2 = \lambda_2^2 = -4$, and hence from (6.3) we get

$$\alpha_1 = (\mu_1 - 1) + (\mu_2 - 1) = -5,$$

$$\alpha_2 = (\mu_1 - 1)(\mu_2 - 1) = 0.$$

Hence, the matrix A becomes

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -5 & -1 & -1 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

As before the plane $\mathscr{P}$ is $\{z : x < 1\}$ so that $m = 3$. Now take $\rho(t) = 1 - (1/t)$, $\sigma(t) = 0$ and $\lambda_2(t) = 2i$; then we get

$$Q_2^2(t) \exp 2 \int_{t_0}^{t} \rho(\tau)\, d\tau = C/t^2, \qquad C = \text{const.}$$

Since this is in L^2, the deficiency index is $(m + 1, m + 1)$.

V. Finally, let us take $Q_0(t) = 1$, $Q_1(t) = Q_2(t) = \{\exp(-t)\}/t$, and $\rho(t) = 1 + (1/t)$, $\sigma(t) = 0$. Then $\mathscr{P}$ is the closed half-plane $\{z : x \leqslant 1\}$, so that $m = 4$. However, now we see that

$$Q_2^2(t) \exp 2 \int_{t_0}^{t} \rho(\tau)\, d\tau = C > 0,$$

so that the deficiency index is $(m - 1, m - 1)$.

It is also possible to construct a self-adjoint differential operator where the characteristic equation of the matrix A has two conjugate complex roots, whose corresponding solutions are not in L^2 but a nontrivial linear combination is in L^2. This can be done by proceeding as in Titchmarsh [8, p. 19]. We shall leave the details for the interested reader.

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