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Integral Representations for Some Classes of Functions Holomorphic in a Siegel Domain

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0. INTRODUCTION

In the articles [1, 2] the classes $H^p(\alpha)$, $1 \leqslant p < \infty$, $\alpha > -1$, of functions holomorphic in the unit disk and satisfying the condition

$$\iint_{|\zeta|<1} |f(\zeta)|^p \cdot (1-|\zeta|^2)^\alpha \, du \, dv < +\infty\,^1 \qquad (\zeta = u + iv) \tag{1}$$

were introduced for the first time. Simultaneously it was proved that any function from the class $H^p(\alpha)$ admits an integral representation

$$f(z) \equiv \frac{\alpha+1}{\pi} \cdot \iint_{|\zeta|<1} \frac{f(\zeta)\cdot(1-|\zeta|^2)^\alpha \, du \, dv}{(1-z\cdot\bar{\zeta})^{2+\alpha}}, \qquad |z|<1 \quad (\zeta = u+iv). \tag{2}$$

This basic result had become a starting point for a whole series of articles devoted to the integral representations for certain classes of holomorphic functions. Before we give the brief review of these researches, let us introduce some notation.

[1] In the last few years some authors call the classes $H^p(\alpha)$ Bergman spaces or weighted Bergman spaces, without serious scientific basis.

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For arbitrary $n \geqslant 1$, let $\mathbb{C}^n$ denote the ordinary n-dimensional coordinate space of complex numbers. If $z = (z_1, ..., z_n) \in \mathbb{C}^n$, $w = (w_1, ..., w_n) \in \mathbb{C}^n$, then put

$$\langle z, w \rangle = \sum_{k=1}^{n} z_k \cdot \bar{w}_k, \qquad |z| = \langle z, z \rangle^{1/2}. \tag{3}$$

Let us introduce the following notation for $0 < r < \infty$:

$$B_{n,r} = \{z \in \mathbb{C}^n, |z| < r\}. \tag{4}$$

When $r = 1$, the standard notation B_n will be used instead of $B_{n,1}$ (the unit ball in $\mathbb{C}^n$). Next, put

$$\Pi_+ = \{z \in \mathbb{C}^1, \operatorname{Im} z > 0\}. \tag{5}$$

The space of all functions holomorphic in an arbitrary open set $\mathscr{D} \subset \mathbb{C}^n$ will be denoted by $H(\mathscr{D})$. For $z \in \mathbb{C}^n$ let $m(z)$ denote the $2n$-dimensional Lebesgue measure in the space $\mathbb{C}^n \cong \mathbb{R}^{2n}$. Finally, if $n \geqslant 1$ and $\beta \in \mathbb{C}^1$, then put

$$c_{n,\beta} = \frac{(\beta+1) \cdot \cdots \cdot (\beta+n)}{\pi^n}. \tag{6}$$

The following result holds.

THEOREM A. *Let* $1 \leqslant p < \infty$, $\alpha > -1$, $0 < r < +\infty$, $f \in H(B_{n,r})$ *and*

$$\int_{B_{n,r}} |f(\zeta)|^p \cdot (r^2 - |\zeta|^2)^\alpha \, dm(\zeta) < +\infty. \tag{7}$$

Then

$$f(z) \equiv c_{n,\beta} \cdot \int_{B_{n,r}} \frac{r^2 \cdot f(\zeta) \cdot (r^2 - |\zeta|^2)^\beta}{(r^2 - \langle z, \zeta \rangle)^{n+1+\beta}} \, dm(\zeta), \qquad z \in B_{n,r}, \tag{8}$$

where $\beta > (\alpha+1)/p - 1$ *for* $1 < p < \infty$ *and* $\beta \geqslant \alpha$ *for* $p = 1$.

When $n = 1$, that is for the disk $B_{1,r} = \{z \in \mathbb{C}^1, |z| < r\}$ this theorem is a direct consequence of the above-mentioned integral representation (2) established in the papers [1, 2]. It should be mentioned that for the case $n > 1$ Theorem A can be proved by literal repetition of the proof given in the original investigations [1, 2] for one-dimensional case. For the special case $\alpha = 0$ ($n > 1$) Theorem A was proved by Forelli and Rudin [3] for the first time.

The natural continuation of these researches were the attempts to obtain the analogues of the integral representation (8) for unbounded domains. On the basis of the integral representation (2) M. M. Dzhrbashyan and A. E. Dzhrbashyan [4] by means of a fine limiting passage strictly proved for the first time the following theorem.

THEOREM B. *Let* $1 \leqslant p < \infty$, $\alpha > -1$, $f \in H(\Pi_+)$ *and*

$$\int_{\Pi_+} |f(\omega)|^p \cdot (\operatorname{Im} \omega)^\alpha \, dm(\omega) < +\infty. \tag{9}$$

Then

$$f(w) \equiv -\frac{(\alpha+1)\cdot 2^\alpha \cdot e^{-i(\pi/2)\alpha}}{\pi} \cdot \int_{\Pi_+} \frac{f(\omega)\cdot(\operatorname{Im}\omega)^\alpha \, dm(\omega)}{(\bar{\omega}-w)^{2+\alpha}} \qquad (w \in \Pi_+). \tag{10}$$

Our object consists in the generalization of the Theorem B for a Siegel domain in $\mathbb{C}^n$, which represents an n-dimensional analogue of upper half-plane $\Pi_+ \subset \mathbb{C}^1$. For wider classes of unbounded multidimensional domains S. G. Gindikin [5] derived integral representations for holomorphic functions whose squares are integrable without weight.

In their article Coifman and Rochberg [6] briefly noted only that on the basis of the methods developed by S. G. Gindikin in [5] one can obtain integral representations for the same classes of holomorphic functions but this time from the weighted L^2-spaces. But in both of these articles it was supposed that $p = 2$, because the proofs were essentially based on the techniques of the Fourier-Plancherel transform.

1. PRELIMINARIES

1.1. We shall frequently use the following natural notations. If $z = (z_1, z_2, ..., z_n) \in \mathbb{C}^n$, then put $z' = (z_2, ..., z_n) \in \mathbb{C}^{n-1}$, so $z = (z_1, z')$. Besides, if $z, w \in \mathbb{C}^n$, we put

$$\langle z', w' \rangle = \sum_{k=2}^{n} z_k \cdot \bar{w}_k, \qquad |z'| = \langle z', z' \rangle^{1/2}. \tag{1.1}$$

The main object of our investigation is the Siegel domain defined as follows:

$$\Omega_n = \{ w = (w_1, w') \in \mathbb{C}^n, \operatorname{Im} w_1 > |w'|^2 \} \tag{1.2}$$

Note that for $n=1$, $\Omega_1 = \Pi_+ \subset \mathbb{C}^1$.

Let us consider now the following mappings (so-called Cayley transforms):

$$\varphi\colon (z_1, ..., z_n) \to \left(i\frac{1+z_1}{1-z_1}, i\frac{z_2}{1-z_1}, ..., i\frac{z_n}{1-z_1} \right) \qquad (z \in B_n), \quad (1.3)$$

$$\varphi^{-1}\colon (w_1, ..., w_n) \to \left(\frac{w_1 - i}{w_1 + i}, \frac{2 \cdot w_2}{w_1 + i}, ..., \frac{2 \cdot w_n}{w_1 + i} \right) \qquad (w \in \Omega_n). \quad (1.4)$$

It is well known that $\varphi\colon B_n \to \Omega_n$ and $\varphi^{-1}\colon \Omega_n \to B_n$ are biholomorphic isomorphisms (see, for instance, [7]). Let Δ and Δ^{-1} be complex Jacobians of the mappings φ and φ^{-1} respectively.

LEMMA 1.1. 1°. *If* $z \in B_n$, $w \in \Omega_n$, *then*

$$\Delta(z) = \frac{2 \cdot i^n}{(1-z_1)^{n+1}}, \qquad \Delta^{-1}(w) = \frac{2^n \cdot i}{(w_1 + i)^{n+1}}. \quad (1.5)$$

2°. *If* $w, \omega \in \Omega_n$, *then*

$$1 - \langle \varphi^{-1} w, \varphi^{-1} \omega \rangle = \frac{2i(\bar{\omega}_1 - w_1) - 4\langle w', \omega' \rangle}{(w_1 + i) \cdot (\bar{\omega}_1 - i)} \quad (1.6)$$

$$1 - |\varphi^{-1}\omega|^2 = \frac{4 \cdot (\operatorname{Im} \omega_1 - |\omega'|^2)}{(\omega_1 + i) \cdot (\bar{\omega}_1 - i)}. \quad (1.7)$$

Since all the assertions could be verified by direct computations, we omit the proof of the lemma.

In addition to Ω_n we shall consider the domain

$$\tilde{\Omega}_n = \left\{ \tilde{w} = (\tilde{w}_1, ..., \tilde{w}_n) \in \mathbb{C}^n, \operatorname{Im} \tilde{w}_1 > \sum_{k=2}^{n} (\operatorname{Im} \tilde{w}_k)^2 \right\}. \quad (1.8)$$

For $n=1$ we have again $\tilde{\Omega}_1 = \Pi_+ \subset \mathbb{C}^1$. Let us introduce the following mappings:

$$\psi\colon (w_1, ..., w_n) \to (\tilde{w}_1, ..., \tilde{w}_n), \qquad w \in \Omega_n,$$

where

$$\tilde{w}_1 = w_1 - i \cdot \sum_{k=2}^{n} (w_k)^2, \qquad \tilde{w}_k = 2^{1/2} \cdot w_k \quad (2 \leqslant k \leqslant n); \quad (1.9)$$

$$\psi^{-1}\colon (\tilde{w}_1, ..., \tilde{w}_n) \to (w_1, ..., w_n), \qquad \tilde{w} \in \tilde{\Omega}_n,$$

where

$$w_1 = \tilde{w}_1 + \frac{i}{2} \cdot \sum_{k=2}^{n} (\tilde{w}_k)^2, \qquad w_k = 2^{-1/2} \cdot \tilde{w}_k \quad (2 \leqslant k \leqslant n). \tag{1.10}$$

One can prove (see [5]) that $\psi: \Omega_n \to \tilde{\Omega}_n$ and $\psi^{-1}: \tilde{\Omega}_n \to \Omega_n$ are biholomorphic isomorphisms. If δ and δ^{-1} are complex Jacobians of the mappings ψ and ψ^{-1} respectively, it is easy to verify the identities:

$$\delta(w) \equiv 2^{(n-1)/2} \ (w \in \Omega_n), \qquad \delta^{-1}(\tilde{w}) \equiv (\tfrac{1}{2})^{(n-1)/2} \ (\tilde{w} \in \tilde{\Omega}_n). \tag{1.11}$$

1.2. Let $\mathscr{D}$ be one of the domains B_n, Ω_n, $\tilde{\Omega}_n$. Then put

$$K_{\mathscr{D}}(z) = \begin{cases} 1 - |z|^2, \quad z \in \mathscr{D}, & \text{if } \mathscr{D} = B_n \\ \operatorname{Im} z_1 - |z'|^2, \quad z \in \mathscr{D}, & \text{if } \mathscr{D} = \Omega_n \\ \operatorname{Im} z_1 - \sum_{k=2}^{n} (\operatorname{Im} z_k)^2, \quad z \in \mathscr{D}, & \text{if } \mathscr{D} = \tilde{\Omega}_n. \end{cases} \tag{1.12}$$

Suppose also that $p \in (0, +\infty)$, $\alpha \in (-\infty; +\infty)$. For arbitrary measurable complex function f defined in $\mathscr{D}$ we put

$$\|f\|_{p,\alpha}^p = \int_{\mathscr{D}} |f(z)|^p \cdot [K_{\mathscr{D}}(z)]^\alpha \, dm(z). \tag{1.13}$$

Then introduce the corresponding spaces:

$$L_\alpha^p(\mathscr{D}) = \{f, \|f\|_{p,\alpha} < +\infty\}, \tag{1.14}$$

$$H_\alpha^p(\mathscr{D}) = H(\mathscr{D}) \cap L_\alpha^p(\mathscr{D}).^2 \tag{1.15}$$

Note that for $p \in [1; +\infty)$ and $\alpha \in (-\infty; +\infty)$, $L_\alpha^p(\mathscr{D})$ are Banach spaces with respect to the norm $\|\cdot\|_{p,\alpha}$.

LEMMA 1.2. *Assume that* $p \in (0; +\infty)$ *and* $\alpha \in (-\infty; +\infty)$.

1°. *If* $g \in L_\alpha^p(\Omega_n)$, *then*

$$\frac{g(\varphi(\zeta))}{(1-\zeta_1)^{2(n+1+\alpha)/p}} \in L_\alpha^p(B_n), \qquad \zeta = (\zeta_1, ..., \zeta_n) \in B_n. \tag{1.16}$$

2°. *If* $f \in L_\alpha^p(B_n)$, *then*

$$\frac{f(\varphi^{-1}(\omega))}{(\omega_1 + i)^{2(n+1+\alpha)/p}} \in L_\alpha^p(\Omega_n), \qquad \omega = (\omega_1, ..., \omega_n) \in \Omega_n. \tag{1.17}$$

[2] These spaces are more frequently denoted by A_α^p, but we prefer the notation H_α^p which is closer to the original one [2].

Proof. We shall prove only 1°, as 2° is obtained by the same way. Let $g \in L^p_\alpha(\Omega_n)$, then

$$J = \int_{\Omega_n} |g(\omega)|^p \cdot (\operatorname{Im} \omega_1 - |\omega'|^2)^\alpha \, dm(\omega) < +\infty. \tag{1.18}$$

Next, make the change of variable in the integral (1.18): $\omega = \varphi(\zeta)$, $\zeta \in B_n$. Then

$$\operatorname{Im} \omega_1 - |\omega'|^2 = \frac{1 - |\zeta|^2}{|1 - \zeta_1|^2}, \qquad dm(\omega) = |\Delta(\zeta)|^2 \cdot dm(\zeta), \tag{1.19}$$

where

$$|\Delta(\zeta)|^2 = \frac{4}{|1 - \zeta_1|^{2n+2}}. \tag{1.20}$$

Consequently, we have:

$$J = 4 \cdot \int_{B_n} \frac{|g(\varphi(\zeta))|^p \cdot (1 - |\zeta|^2)^\alpha \, dm(\zeta)}{|1 - \zeta_1|^{2(n+1+\alpha)}} < +\infty. \tag{1.21}$$

Finally, one can easily deduce from (1.21) that

$$\frac{g(\varphi(\zeta))}{(1 - \zeta_1)^{2(n+1+\alpha)/p}} \in L^p_\alpha(B_n)$$

and the proof is complete.

The following assertion is proved in the same manner.

LEMMA 1.3. *Let* $p \in (0; +\infty)$ *and* $\alpha \in (-\infty; +\infty)$.

1°. *If* $g \in L^p_\alpha(\Omega_n)$, *then* $g \circ \psi^{-1} \in L^p_\alpha(\tilde{\Omega}_n)$.

2°. *If* $\varphi \in L^p_\alpha(\tilde{\Omega}_n)$, *then* $\varphi \circ \psi \in L^p_\alpha(\Omega_n)$.

It is well known that for arbitrary $p \in (0; +\infty)$, $H^p_\alpha(B_n) \neq \{0\}$ if and only if $\alpha > -1$, therefore Lemmas 1.2 and 1.3 imply immediately that for any $p \in (0; +\infty)$, $H^p_\alpha(\Omega_n) \neq \{0\}$ and $H^p_\alpha(\tilde{\Omega}_n) \neq \{0\}$ if and only if $\alpha > -1$.

2. MAIN INTEGRAL REPRESENTATIONS

2.1. We shall suppose from now on that $1 \leq p < \infty$, $\alpha > -1$. Let $\beta \in \mathbb{C}^1$, then we shall write $\beta > (p, \alpha)$ if and only if β satisfies the following condition:

$$\begin{aligned} &\operatorname{Re}\beta > (\alpha+1)/p - 1 && \text{for} \quad 1 < p < \infty \\ &\operatorname{Re}\beta \geqslant \alpha && \text{for} \quad p = 1. \end{aligned} \tag{2.1}$$

LEMMA 2.1. *Let* $f \in L^p_\alpha(\Omega_n)$ *and* $\beta > (p, \alpha)$. *Then*

$$F_\beta(\omega) \equiv \frac{f(\omega)\cdot(\operatorname{Im}\omega_1 - |\omega'|^2)^\beta}{(\bar{\omega}_1 - i)^{n+1+\beta}} \in L^1(\Omega_n). \tag{2.2}$$

Proof. Note that for $\omega \in \Omega_n$

$$\begin{aligned} |F_\beta(\omega)| &= \frac{|f(\omega)|\cdot(\operatorname{Im}\omega_1 - |\omega'|^2)^{\operatorname{Re}\beta}}{|\bar{\omega}_1 - i|^{n+1+\operatorname{Re}\beta}\cdot\exp\{-\operatorname{Im}\beta\cdot\arg(\bar{\omega}_1 - i)\}} \\ &\leqslant \exp\{\pi\cdot|\operatorname{Im}\beta|\}\cdot\tilde{F}_\beta(\omega), \end{aligned} \tag{2.3}$$

where

$$\tilde{F}_\beta(\omega) = \frac{|f(\omega)|\cdot(\operatorname{Im}\omega_1 - |\omega'|^2)^{\operatorname{Re}\beta}}{|\bar{\omega}_1 - i|^{n+1+\operatorname{Re}\beta}} \qquad (\omega \in \Omega_n). \tag{2.4}$$

It suffices to prove that $\tilde{F}_\beta(\omega) \in L^1(\Omega_n)$. We shall distinguish two cases.

Case $p = 1$. Then $\operatorname{Re}\beta \geqslant \alpha$, so we have

$$\begin{aligned} \tilde{F}_\beta(\omega) &= \frac{|f(\omega)|\cdot(\operatorname{Im}\omega_1 - |\omega'|^2)^\alpha}{|\bar{\omega}_1 - i|^{n+1+\alpha}}\cdot\frac{(\operatorname{Im}\omega_1 - |\omega'|^2)^{\operatorname{Re}\beta-\alpha}}{|\bar{\omega}_1 - i|^{\operatorname{Re}\beta-\alpha}} \\ &\leqslant |f(\omega)|\cdot(\operatorname{Im}\omega_1 - |\omega'|^2)^\alpha \qquad (\omega \in \Omega_n), \end{aligned} \tag{2.5}$$

if we take into account the following inequalities:

$$\frac{(\operatorname{Im}\omega_1 - |\omega'|^2)^{\operatorname{Re}\beta-\alpha}}{|\bar{\omega}_1 - i|^{\operatorname{Re}\beta-\alpha}} \leqslant 1, \qquad \frac{1}{|\bar{\omega}_1 - i|^{n+1+\alpha}} \leqslant 1. \tag{2.6}$$

Since $f \in L^1_\alpha(\Omega_n)$, we deduce from (2.5) that $\tilde{F}_\beta(\omega) \in L^1(\Omega_n)$.

Case $1 < p < \infty$, i.e., $\operatorname{Re}\beta > (\alpha+1)/p - 1$. Then choose $1 < q < \infty$ so that $1/p + 1/q = 1$ and rewrite $\tilde{F}_\beta(\omega)$ as follows:

$$\tilde{F}_\beta(\omega) = |f(\omega)|\cdot(\operatorname{Im}\omega_1 - |\omega'|^2)^{\alpha/p}\cdot\frac{(\operatorname{Im}\omega_1 - |\omega'|^2)^{\operatorname{Re}\beta-\alpha/p}}{|\bar{\omega}_1 - i|^{n+1+\operatorname{Re}\beta}}. \tag{2.7}$$

Next, applying Hölder's inequality we get

$$\begin{aligned} \int_{\Omega_n}\tilde{F}_\beta(\omega)\,dm(\omega) &\leqslant \|f\|_{p,\alpha}\left\{\int_{\Omega_n}\frac{(\operatorname{Im}\omega_1 - |\omega'|^2)^{q(\operatorname{Re}\beta-\alpha/p)}\,dm(\omega)}{|\bar{\omega}_1 - i|^{q(n+1+\operatorname{Re}\beta)}}\right\}^{1/q} \\ &= \|f\|_{p,\alpha}\cdot I^{1/q}. \end{aligned} \tag{2.8}$$

So all what remains to show is that the integral I converges. By Fubini's theorem,

$$I=\int_{\Pi_+}\frac{dm(\omega_1)}{|\bar{\omega}_1-i|^{q(n+1+\operatorname{Re}\beta)}}$$
$$\times\int_{|\omega'|<(\operatorname{Im}\omega_1)^{1/2}}(\operatorname{Im}\omega_1-|\omega'|^2)^{q(\operatorname{Re}\beta-\alpha/p)}\,dm(\omega'). \tag{2.9}$$

Introducing polar coordinates we find that the inner integral in (2.9) is equal to const $\cdot\,(\operatorname{Im}\omega_1)^{n-1+q(\operatorname{Re}\beta-\alpha/p)}$. Thus, we obtain

$$I=\text{const}\cdot\int_{\Pi_+}\frac{(\operatorname{Im}\omega_1)^{n-1+q(\operatorname{Re}\beta-\alpha/p)}\,dm(\omega_1)}{|\bar{\omega}_1-i|^{q(n+1+\operatorname{Re}\beta)}}. \tag{2.10}$$

Put in (2.10) $\omega_1=u+iv$, $dm(\omega_1)=du\,dv$, then

$$I=\text{const}\cdot\int_0^{+\infty}\int_{-\infty}^{+\infty}\frac{v^{n-1+q(\operatorname{Re}\beta-\alpha/p)}\,du\,dv}{([u^2+(v+1)^2]^{1/2})^{q(n+1+\operatorname{Re}\beta)}}$$
$$=\text{const}\cdot\int_0^{+\infty}\frac{v^{n-1+q(\operatorname{Re}\beta-\alpha/p)}\,dv}{(v+1)^{q(n+1+\operatorname{Re}\beta)-1}}$$
$$\cdot\int_{-\infty}^{+\infty}\frac{dt}{(1+t^2)^{q(n+1+\operatorname{Re}\beta)/2}}. \tag{2.11}$$

So, the integral I can be written as the product of two integrals. Consequently, it should be shown that both of them converge. The second integral in (2.11) converges if and only if

$$q(n+1+\operatorname{Re}\beta)>1. \tag{2.12}$$

The first integral in (2.11) converges if and only if

$$n-1+q(\operatorname{Re}\beta-\alpha/p)>-1, \tag{2.13}$$
$$q(n+1+\operatorname{Re}\beta)-1-(n-1)-q(\operatorname{Re}\beta-\alpha/p)>1. \tag{2.14}$$

Recalling the original choice of β we see that all the conditions (2.12)–(2.14) are satisfied and this completes the proof.

2.2. For each $\beta\in\mathbb{C}^1$ consider the following integral operator:

$$T_\beta f(w)=2^{n-1+\beta}\cdot c_{n,\beta}\cdot\int_{\Omega_n}\frac{f(\omega)\cdot(\operatorname{Im}\omega_1-|\omega'|^2)^\beta\,dm(\omega)}{[i(\bar{\omega}_1-w_1)-2\langle w',\omega'\rangle]^{n+1+\beta}}$$
$$(w\in\Omega_n), \tag{2.15}$$

where f is an arbitrary function defined in Ω_n. As it follows from Lemma 1.1 the operator T_β can be written in a different way:

$$T_\beta f(w) = \frac{4^{n+\beta} \cdot c_{n,\beta}}{(w_1+i)^{n+1+\beta}} \cdot \int_{\Omega_n} \frac{f(\omega) \cdot (\operatorname{Im} \omega_1 - |\omega'|^2)^\beta \, dm(\omega)}{[1 - \langle \varphi^{-1} w, \varphi^{-1}\omega \rangle]^{n+1+\beta} \cdot (\bar{\omega}_1 - i)^{n+1+\beta}}. \tag{2.16}$$

Therefore it seems natural to introduce the following function:

$$G(w, \omega, \beta, f) = \frac{f(\omega) \cdot (\operatorname{Im} \omega_1 - |\omega'|^2)^\beta}{[1 - \langle \varphi^{-1} w, \varphi^{-1}\omega \rangle]^{n+1+\beta} \cdot (\bar{\omega}_1 - i)^{n+1+\beta}} \quad (w, \omega \in \Omega_n, \beta \in \mathbb{C}^1). \tag{2.17}$$

LEMMA 2.2. *Assume that* $f \in L^p_\alpha(\Omega_n)$ *and* $G(w, \omega, \beta, f)$ *is defined by* (2.17).

1°. *If* $p = 1$, *the compact set* $K \subset \Omega_n$ *and* $0 < A < \infty$, $\alpha < a < \infty$, *then there exists a function* $\Psi \in L^1(\Omega_n)$ *such that the inequality*

$$|G(w, \omega, \beta, f)| \leqslant \Psi(\omega), \qquad \omega \in \Omega_n \tag{2.18}$$

holds uniformly in $w \in K$ *and* $\beta \in \mathbb{C}^1$ *with* $|\operatorname{Im} \beta| \leqslant A$, $\alpha \leqslant \operatorname{Re} \beta \leqslant a$.

2°. *If* $1 < p < \infty$, *the compact set* $K \subset \Omega_n$ *and* $0 < A < \infty$, $(\alpha+1)/p - 1 < a_1 < a_2 < \infty$, *then there exists a function* $\Psi \in L^1(\Omega_n)$ *such that the inequality*

$$|G(w, \omega, \beta, f)| \leqslant \Psi(\omega), \qquad \omega \in \Omega_n \tag{2.19}$$

holds uniformly in $w \in K$ *and* $\beta \in \mathbb{C}^1$ *with* $|\operatorname{Im} \beta| \leqslant A$, $a_1 \leqslant \operatorname{Re} \beta \leqslant a_2$.

Proof. Fix number $\rho \in (0; 1)$ such that $|\varphi^{-1} w| \leqslant \rho$ whenever $w \in K$. Then for arbitrary $w \in K$ and $\omega \in \Omega_n$ we have:

$$\begin{aligned} &\left| \frac{1}{[1 - \langle \varphi^{-1} w, \varphi^{-1}\omega \rangle]^{n+1+\beta}} \right| \\ &= \frac{1}{|1 - \langle \varphi^{-1} w, \varphi^{-1}\omega \rangle|^{n+1+\operatorname{Re}\beta}} \\ &\quad \times \exp\{\operatorname{Im} \beta \cdot \arg(1 - \langle \varphi^{-1} w, \varphi^{-1}\omega \rangle)\} \\ &\leqslant \frac{\exp\{\pi/2 \cdot |\operatorname{Im} \beta|\}}{(1 - |\varphi^{-1} w|)^{n+1+\operatorname{Re}\beta}} \leqslant \frac{\exp\{\pi/2 \cdot A\}}{(1-\rho)^{n+1+\operatorname{Re}\beta}}. \end{aligned} \tag{2.20}$$

If $p=1$, the following estimates are valid:

$$|G(w,\omega,\beta,f)| \leqslant \frac{\exp\{(\pi/2)A\}}{(1-\rho)^{n+1+\operatorname{Re}\beta}} \cdot \frac{|f(\omega)|\cdot(\operatorname{Im}\omega_1-|\omega'|^2)^{\operatorname{Re}\beta}}{|\bar{\omega}_1-i|^{n+1+\operatorname{Re}\beta}}$$
$$\times \exp\{\operatorname{Im}\beta\cdot\arg(\bar{\omega}_1-i)\}$$
$$\leqslant \frac{\exp\{3\pi/2\cdot A\}}{(1-\rho)^{n+1+\operatorname{Re}\beta}} \frac{|f(\omega)|\cdot(\operatorname{Im}\omega_1-|\omega'|^2)^{\alpha}}{|\bar{\omega}_1-i|^{n+1+\alpha}}$$
$$\cdot\frac{(\operatorname{Im}\omega_1-|\omega'|^2)^{\operatorname{Re}\beta-\alpha}}{|\bar{\omega}_1-i|^{\operatorname{Re}\beta-\alpha}}$$
$$\leqslant \frac{\exp\{(3\pi/2)A\}}{(1-\rho)^{n+1+a}} \cdot \frac{|f(\omega)|\cdot(\operatorname{Im}\omega_1-|\omega'|^2)^{\alpha}}{|\bar{\omega}_1-i|^{n+1+\alpha}}$$
$$\equiv \Psi(\omega). \tag{2.21}$$

It follows from the Lemma 2.1 that $\Psi\in L^1(\Omega_n)$.

If $1<p<\infty$, then

$$|G(w,\omega,\beta,f)| \leqslant \frac{\exp\{(\pi/2)A\}}{(1-\rho)^{n+1+\operatorname{Re}\beta}} \cdot \frac{|f(\omega)|\cdot(\operatorname{Im}\omega_1-|\omega'|^2)^{\operatorname{Re}\beta}}{|\bar{\omega}_1-i|^{n+1+\operatorname{Re}\beta}}$$
$$\times \exp\{\operatorname{Im}\beta\cdot\arg(\bar{\omega}_1-i)\}$$
$$\leqslant \frac{\exp\{(3\pi/2)A\}}{(1-\rho)^{n+1+\operatorname{Re}\beta}} \frac{|f(\omega)|\cdot(\operatorname{Im}\omega_1-|\omega'|^2)^{a_1}}{|\bar{\omega}_1-i|^{n+1+a_1}}$$
$$\cdot\frac{(\operatorname{Im}\omega_1-|\omega'|^2)^{\operatorname{Re}\beta-a_1}}{|\bar{\omega}_1-i|^{\operatorname{Re}\beta-a_1}}$$
$$\leqslant \frac{\exp\{(3\pi/2)A}{(1-\rho)^{n+1+a_2}} \cdot \frac{|f(\omega)|\cdot(\operatorname{Im}\omega_1-|\omega'|^2)^{a_1}}{|\bar{\omega}_1-i|^{n+1+a_1}}$$
$$\equiv \Psi(\omega). \tag{2.22}$$

It follows from the Lemma 2.1 that $\Psi\in L^1(\Omega_n)$. Thus, the Lemma 2.2 is proved.

COROLLARY (a). *Let* $f\in L^p_\alpha(\Omega_n)$ *and* $\beta>(p,\alpha)$. *Then* $T_\beta f(w)$ *as the function of variable* $w\in\Omega_n$ *belongs to* $H(\Omega_n)$.

COROLLARY (b). *Let* $f\in L^p_\alpha(\Omega_n)$ *and* $w\in\Omega_n$. *If* $1<p<\infty$, *then* $T_\beta f(w)$ *as the function of variable* β *is holomorphic in the domain* $\operatorname{Re}\beta>(\alpha+1)/p-1$. *If* $p=1$, *then* $T_\beta f(w)$ *as the function of variable* β *is holomorphic inside and continuous on the closed domain* $\operatorname{Re}\beta\geqslant\alpha$.

2.3. Now we prove the main theorem of the present article.

THEOREM 2.1. *Suppose* $1 \leqslant p < \infty$, $\alpha > -1$, $f \in H^p_\alpha(\Omega_n)$ *and* $\beta > (p, \alpha)$. *Then*

$$f(w) \equiv T_\beta f(w), \qquad w \in \Omega_n. \tag{2.23}$$

Proof. Fix an arbitrary $w \in \Omega_n$, then $T_\beta f(w)$ is the holomorphic function of the variable β as it follows from the Corollary (b) of the Lemma 2.2. Next, put $\beta_0 = \max\{0; (\alpha+1)/p - 1\}$. By the uniqueness theorem for holomorphic functions, it suffices to establish (2.23) only for real $\beta > \beta_0$.

Put $z = \varphi^{-1} w \in B_n$ and choose $r_0 \in (0; 1)$ so that $z \in B_{n,r}$ for $r_0 \leqslant r < 1$. Then

$$w \in \Omega_{n,r} = \varphi(B_{n,r}) \qquad (r_0 \leqslant r < 1). \tag{2.24}$$

Let us consider the following function,

$$g(\zeta) = \frac{f(\varphi(\zeta))}{(1-\zeta_1)^{n+1+\beta}}, \tag{2.25}$$

where $\zeta = (\zeta_1, ..., \zeta_n) \in B_n$. Obviously $g \in H(B_n)$. Consequently, we have

$$\int_{B_{n,r}} |g(\zeta)|^p \cdot (r^2 - |\zeta|^2)^\beta \, dm(\zeta) < +\infty \quad (r_0 \leqslant r < 1). \tag{2.26}$$

Since $z \in B_{n,r}$ for $r_0 \leqslant r < 1$, Theorem A implies

$$g(z) = c_{n,\beta} \cdot r^2 \cdot \int_{B_{n,r}} \frac{g(\zeta) \cdot (r^2 - |\zeta|^2)^\beta \, dm(\zeta)}{(r^2 - \langle z, \zeta \rangle)^{n+1+\beta}}. \tag{2.27}$$

Substituting (2.25) into (2.27) we obtain

$$\frac{f(\varphi(z))}{(1-z_1)^{n+1+\beta}} = c_{n,\beta} \cdot r^2 \cdot \int_{B_{n,r}} \frac{f(\varphi(\zeta)) \cdot (r^2 - |\zeta|^2)^\beta \, dm(\zeta)}{(1-\zeta_1)^{n+1+\beta} \cdot (r^2 - \langle z, \zeta \rangle)^{n+1+\beta}}. \tag{2.28}$$

Next recall that $z = \varphi^{-1} w$, make the change of variable $\zeta = \varphi^{-1}\omega$, $\omega \in \Omega_{n,r}$ in the integral (2.28) and take into account that

$$\frac{1}{(1-\zeta_1)^{n+1+\beta}} = \frac{(\omega_1 + i)^{n+1+\beta}}{(2i)^{n+1+\beta}}, \qquad \frac{1}{(1-z_1)^{n+1+\beta}} = \frac{(w_1 + i)^{n+1+\beta}}{(2i)^{n+1+\beta}}. \tag{2.29}$$

As a result we obtain that for $r_0 \leqslant r < 1$ the following equality holds:

$$f(w)\cdot(w_1+i)^{n+1+\beta} = c_{n,\beta}\cdot r^2 \int_{\Omega_{n,r}} \frac{f(\omega)\cdot(r^2-|\varphi^{-1}\omega|^2)^\beta\cdot(\omega_1+i)^{n+1+\beta}}{(r^2-\langle\varphi^{-1}w,\varphi^{-1}\omega\rangle)^{n+1+\beta}} \cdot|\Delta^{-1}(\omega)|^2\,dm(\omega). \tag{2.30}$$

Since we intend to let $r\uparrow 1$ in the right side of (2.30), let us introduce the following function:

$$F_r(\omega)=\frac{f(\omega)\cdot(r^2-|\varphi^{-1}\omega|^2)^\beta\cdot(\omega_1+i)^{n+1+\beta}}{(r^2-\langle\varphi^{-1}w,\varphi^{-1}\omega\rangle)^{n+1+\beta}} \cdot|\Delta^{-1}(\omega)|^2\cdot\chi_{n,r}(\omega), \tag{2.31}$$

where $r_0\leqslant r<1$, $\omega\in\Omega_n$ and $\chi_{n,r}$ is characteristic function of $\Omega_{n,r}$. By (2.31) the formula (2.30) can be rewritten as follows:

$$f(w)\cdot(w_1+i)^{n+1+\beta}=c_{n,\beta}\cdot r^2\cdot\int_{\Omega_n} F_r(\omega)\,dm(\omega). \tag{2.32}$$

Now we are going to apply Lebesgue's dominated convergence theorem to (2.32). Let $r_0\leqslant r<1$, $\omega\in\Omega_{n,r}$. Then due to the fact that $\beta>\beta_0\geqslant 0$ we have

$$(r^2-|\varphi^{-1}\omega|^2)^\beta\leqslant(1-|\varphi^{-1}\omega|^2)^\beta. \tag{2.33}$$

Furthermore, the following inequalities hold:

$$\left|\frac{1}{(r^2-\langle\varphi^{-1}w,\varphi^{-1}\omega\rangle)^{n+1+\beta}}\right| = \frac{1}{|r^2-\langle\varphi^{-1}w,\varphi^{-1}\omega\rangle|^{n+1+\beta}} \leqslant\frac{1}{(r^2-r\cdot|\varphi^{-1}w|)^{n+1+\beta}} \leqslant\frac{1}{(r_0^2-r_0\cdot|\varphi^{-1}w|)^{n+1+\beta}}. \tag{2.34}$$

Hence, for $\omega\in\Omega_n$ we have

$$|F_r(\omega)|\leqslant\frac{|f(\omega)|\cdot(1-|\varphi^{-1}\omega|^2)^\beta\cdot|\omega_1+i|^{n+1+\beta}\cdot|\Delta^{-1}(\omega)|^2}{(r_0^2-r_0\cdot|\varphi^{-1}w|)^{n+1+\beta}}. \tag{2.35}$$

Using the formula (1.5), (1.7), we obtain

$$|F_r(\omega)|\leqslant\frac{4^{n+\beta}\cdot|f(\omega)|\cdot(\operatorname{Im}\omega_1-|\omega'|^2)^\beta}{(r_0^2-r_0\cdot|\varphi^{-1}w|)^{n+1+\beta}\cdot|\bar\omega_1-i|^{n+1+\beta}}, \tag{2.36}$$

for $r_0 \leqslant r < 1$, $\omega \in \Omega_n$. It follows from the Lemma 2.1, that the right side of (2.36) belongs to $L^1(\Omega_n)$, so above-mentioned Lebesgue's theorem can be applied. Consequently, we obtain:

$$f(w)\cdot(w_1+i)^{n+1+\beta} = c_{n,\beta}\cdot\int_{\Omega_n}\frac{f(\omega)\cdot(1-|\varphi^{-1}\omega|^2)^\beta}{(1-\langle\varphi^{-1}w,\varphi^{-1}\omega\rangle)^{n+1+\beta}}$$
$$\times(\omega_1+i)^{n+1+\beta}\cdot|\Delta^{-1}(\omega)|^2\,dm(\omega). \qquad (2.37)$$

Finally, using the formula (1.5)–(1.7) it can be shown that (2.37) coincides with (2.23). This completes the proof.

2.4. Now we briefly consider the case of the domain $\tilde{\Omega}_n$. Let us introduce the following integral operator:

$$\tilde{T}_\beta\varphi(\tilde{w}) = 2^\beta\cdot c_{n,\beta}\cdot\int_{\tilde{\Omega}_n}\frac{\varphi(\tilde{\omega})\cdot[\operatorname{Im}\tilde{\omega}_1-\sum_{k=2}^n(\operatorname{Im}\tilde{\omega}_k)^2]^\beta\,dm(\tilde{\omega})}{[i(\bar{\tilde{\omega}}_1-\tilde{w}_1)+\frac{1}{2}\cdot\sum_{k=2}^n(\bar{\tilde{\omega}}_k-\tilde{w}_k)^2]^{n+1+\beta}}, \qquad (2.38)$$

where $\tilde{w}=(\tilde{w}_1, ..., \tilde{w}_n)\in\tilde{\Omega}_n$, $\beta\in\mathbb{C}^1$ and φ is an arbitrary function defined in $\tilde{\Omega}_n$. Notice that $\tilde{T}_\beta$ is the analogue of the integral operator T_β.

THEOREM 2.2. *Suppose* $1\leqslant p<\infty$, $\alpha>-1$, $\varphi\in H^p_\alpha(\tilde{\Omega}_n)$ *and* $\beta>(p,\alpha)$. *Then*

$$\varphi(\tilde{w})\equiv\tilde{T}_\beta\varphi(\tilde{w}), \qquad \tilde{w}\in\tilde{\Omega}_n. \qquad (2.39)$$

Proof. Recall that ψ and ψ^{-1} denote the mappings (1.9) and (1.10) respectively. By the Lemma 1.3, $\varphi\in H^p_\alpha(\tilde{\Omega}_n)$ implies $f=\varphi\circ\psi\in H^p_\alpha(\Omega_n)$. Therefore the assertion of the Theorem 2.1 can be applied to the function f:

$$f(w)\equiv 2^{n+1+\beta}\cdot c_{n,\beta}\cdot\int_{\Omega_n}\frac{f(\omega)\cdot(\operatorname{Im}\omega_1-|\omega'|^2)^\beta\,dm(\omega)}{[i(\bar{\omega}_1-w_1)-2\langle w',\omega'\rangle]^{n+1+\beta}}$$
$$(w\in\Omega_n). \qquad (2.40)$$

Then put $f=\varphi\circ\psi$, $w=\psi^{-1}\tilde{w}$ and make the change of variable: $\omega=\psi^{-1}\tilde{\omega}$, $\tilde{\omega}\in\tilde{\Omega}_n$, in (2.40). Recall also that $\delta^{-1}(\tilde{\omega})\equiv(\frac{1}{2})^{(n-1)/2}$, $\tilde{\omega}\in\tilde{\Omega}_n$. Furthermore, the following relations are easily verified:

$$\operatorname{Im}\omega_1-|\omega'|^2=\operatorname{Im}\tilde{\omega}_1-\sum_{k=2}^n(\operatorname{Im}\tilde{\omega}_k)^2, \qquad (2.41)$$

$$i(\bar{\omega}_1-w_1)-2\langle w',\omega'\rangle = i(\bar{\tilde{\omega}}_1-\tilde{w}_1)+\tfrac{1}{2}\sum_{k=2}^n(\bar{\tilde{\omega}}_k-\tilde{w}_k)^2. \qquad (2.42)$$

Taking all these into account we get the desired formula (2.39).

Remark. In further consideration all the main results obtained for Ω_n will be formulated for the domain $\tilde{\Omega}_n$ without detailed explanation.

3. Continuous Projections

For $t > -1$ and $c > 0$ let us consider the following expression:

$$J^n_{t,c}(w) = \int_{\Omega_n} \frac{(\operatorname{Im} \omega_1 - |\omega'|^2)^t \, dm(\omega)}{|i(\bar{\omega}_1 - w_1) - 2\langle w', \omega' \rangle|^{n+1+t+c}} \qquad (w \in \Omega_n). \tag{3.1}$$

Lemma 3.1.

$$J^n_{t,c}(w) \equiv \frac{\text{const}}{(\operatorname{Im} w_1 - |w'|^2)^c}, \qquad w \in \Omega_n \tag{3.2}$$

where constant depends only on n, t, c.

Using the methods borrowed from [5], more general integrals are computed in [6, Lemma 2.2], so we omit the proof of the Lemma 3.1.

Theorem 3.1. *Suppose* $1 \leqslant p < \infty$, $\alpha > -1$ *and complex number* β *satisfies the condition* $\operatorname{Re} \beta > (\alpha + 1)p - 1$. *Then the operator* T_β *is a continuous projection from* $L^p_\alpha(\Omega_n)$ *to* $H^p_\alpha(\Omega_n)$.

Proof. Assume that $f \in L^p_\alpha(\Omega_n)$, then $T_\beta f \in H(\Omega_n)$ as it follows from the Corollary (a) of Lemma 2.2. Furthermore, we have

$$|T_\beta f(w)| \leqslant |2^{n-1+\beta} \cdot c_{n,\beta}| \cdot \int_{\Omega_n} \frac{|f(\omega)| \cdot (\operatorname{Im} \omega_1 - |\omega'|^2)^{\operatorname{Re}\beta}}{|i(\bar{\omega}_1 - w_1) - 2\langle w', \omega' \rangle|^{n+1+\operatorname{Re}\beta}}$$
$$\times \exp\{\operatorname{Im} \beta \cdot \arg[i(\bar{\omega}_1 - w_1) - 2\langle w', \omega' \rangle]\} \, dm(\omega). \tag{3.3}$$

Since $\operatorname{Re}[i(\bar{\omega}_1 - w_1) - 2\langle w', \omega' \rangle] > 0$, (3.3) yields

$$|T_\beta f(w)| \leqslant A(n, \beta) \cdot \int_{\Omega_n} \frac{|f(\omega)| \cdot (\operatorname{Im} \omega_1 - |\omega'|^2)^{\operatorname{Re}\beta} \, dm(\omega)}{|i(\bar{\omega}_1 - w_1) - 2 < w', \omega' > |^{n+1+\operatorname{Re}\beta}} \tag{3.4}$$

where

$$A(n, \beta) = |2^{n-1+\beta} \cdot c_{n,\beta}| \cdot \exp\left\{\frac{\pi}{2} \cdot |\operatorname{Im} \beta|\right\}. \tag{3.5}$$

First we consider the case $p = 1$.

$$\|T_\beta f\|_{1,\alpha} = \int_{\Omega_n} |T_\beta f(w)| \cdot (\operatorname{Im} w_1 - |w'|^2)^\alpha \, dm(w) \leqslant A(n, \beta)$$

$$\times \int_{\Omega_n} (\operatorname{Im} w_1 - |w'|^2)^\alpha \, dm(w)$$

$$\times \int_{\Omega_n} \frac{|f(\omega)| \cdot (\operatorname{Im} \omega_1 - |\omega'|^2)^{\operatorname{Re} \beta} \, dm(\omega)}{|i(\bar{\omega}_1 - w_1) - 2\langle w', \omega' \rangle|^{n+1+\operatorname{Re} \beta}}$$

$$= A(n, \beta) \cdot \int_{\Omega_n} |f(\omega)| \cdot (\operatorname{Im} \omega_1 - |\omega'|^2)^{\operatorname{Re} \beta} \, dm(\omega)$$

$$\times \int_{\Omega_n} \frac{(\operatorname{Im} w_1 - |w'|^2)^\alpha \, dm(w)}{|i(\bar{\omega}_1 - w_1) - 2\langle w', \omega' \rangle|^{n+1+\operatorname{Re} \beta}}. \tag{3.6}$$

Since $\operatorname{Re} \beta > \alpha$, one can use the Lemma 3.1 to compute the inner integral in (3.6). So we obtain:

$$\|T_\beta f\|_{1,\alpha} \leqslant \text{const} \cdot A(n, \beta) \cdot \int_{\Omega_n} \frac{|f(\omega)| \cdot (\operatorname{Im} \omega_1 - |\omega'|^2)^{\operatorname{Re} \beta} \, dm(\omega)}{(\operatorname{Im} \omega_1 - |\omega'|^2)^{\operatorname{Re} \beta - \alpha}}$$

$$= \text{const} \cdot A(n, \beta) \cdot \|f_{1,\alpha},$$

the theorem is proved for $p = 1$.

If $1 < p < \infty$, then choose $1 < q < \infty$ so that $1/p + 1/q = 1$ and put

$$d\mu(\omega) = (\operatorname{Im} \omega_1 - |\omega'|^2)^\alpha \, dm(\omega), \tag{3.7}$$

$$Q(w, \omega) = A(n, \beta) \cdot \frac{(\operatorname{Im} \omega_1 - |\omega'|^2)^{\operatorname{Re} \beta - \alpha}}{|i(\bar{\omega}_1 - w_1) - 2\langle w', \omega' \rangle|^{n+1+\operatorname{Re} \beta}}$$

$$(w, \omega \in \Omega_n). \tag{3.8}$$

Then (3.4) becomes

$$|T_\beta f(w)| \leqslant \int_{\Omega_n} |f(\omega)| \cdot Q(w, \omega) \, d\mu(\omega). \tag{3.9}$$

We intend to prove the boundedness of the operator T_β in the space $L^p_\alpha(\Omega_n) = L^p(\Omega_n, \mu)$ by means of Forelli-Rudin's lemma (see [3]). Hence, it is enough to find a positive function g defined in Ω_n such that

$$\int_{\Omega_n} Q(w, \omega) \cdot g^p(w) \, d\mu(w) \leqslant \text{const} \cdot g^p(\omega), \qquad \omega \in \Omega_n, \tag{3.10}$$

$$\int_{\Omega_n} Q(w, \omega) \cdot g^q(\omega) \, d\mu(\omega) \leqslant \text{const} \cdot g^q(w), \qquad w \in \Omega_n. \tag{3.11}$$

Let us consider the function of the form

$$g(\omega)=(\operatorname{Im}\omega_1-|\omega'|^2)^{-\delta}, \qquad 0<\delta<\infty, \quad \omega\in\Omega_n. \tag{3.12}$$

It follows from the Lemma 3.1 that for this function the inequalities (3.10) and (3.11) hold whenever δ satisfy the following conditions:

$$\operatorname{Re}\beta-\delta\cdot q>-1, \qquad \alpha-\delta\cdot p>-1, \quad \operatorname{Re}\beta-\alpha+\delta\cdot p>0. \tag{3.13}$$

Under the hypotheses of the theorem such choice of a positive number δ is possible, so the proof is complete.

Remark. For $n\geqslant 1$, $1<p<\infty$, $\alpha>-1$ and for $\beta>\alpha$ the Theorem 3.1 is the special case of more general assertion (see [6, Lemma 2.8]). For $n=1$, $\alpha>-1$ and for β satisfying the condition

$$\begin{aligned} &\beta>\alpha, \qquad \text{if} \quad p=1 \\ &\beta=\alpha, \qquad \text{if} \quad 1<p<\infty \end{aligned} \tag{3.14}$$

Theorem 3.1 was established for the first time in [4].

The following assertion is also true.

THEOREM 3.2. *Suppose $1\leqslant p<\infty$, $\alpha>-1$ and complex number β satisfies the condition $\operatorname{Re}\beta>(\alpha+1)/p-1$. Then the operator $\tilde{T}_\beta$ is a continuous projection from $L^p_\alpha(\tilde{\Omega}_n)$ to $H^p_\alpha(\tilde{\Omega}_n)$.*

4. APPLICATIONS

Now we give some applications of the obtained results.

4.1. From now on we shall assume, that $1<p<\infty$, $1<q<\infty$, $1/p+1/q=1$ and $\alpha>-1$. For arbitrary functions $f\in L^p_\alpha(\Omega_n)$ and $g\in L^q_\alpha(\Omega_n)$ put

$$[f,g]=\int_{\Omega_n} f(\omega)\cdot\overline{g(\omega)}\cdot(\operatorname{Im}\omega_1-|\omega'|^2)^\alpha\,dm(\omega). \tag{4.1}$$

PROPOSITION 4.1. (a) $[\cdot,\cdot]$ *is linear relative to the first argument and anti-linear relative to the second argument.*

(b) *If $f\in L^p_\alpha(\Omega_n)$ and $g\in L^q_\alpha(\Omega_n)$, then*

$$|[f,g]|\leqslant\|f\|_{p,\alpha}\cdot\|g\|_{q,\alpha} \tag{4.2}$$

(c) *If* $\{f_m\}_1^\infty$ *tends to* f *in the space* $L_\alpha^p(\Omega_n)$ *and* $\{g_m\}_1^\infty$ *tends to* g *in the space* $L_\alpha^q(\Omega_n)$, *then*

$$\lim_{m\to\infty}\,[f_m, g_m] = [f, g]. \tag{4.3}$$

Proof. (a) is readily verified, (b) is exactly the Hölder's integral inequality, and (c) follows at once from (b).

PROPOSITION 4.2. *For arbitrary functions* $f \in L_\alpha^p(\Omega_n)$ *and* $g \in L_\alpha^q(\Omega_n)$ *the following relation holds*:

$$[T_\alpha f, g] = [f, T_\alpha g]. \tag{4.4}$$

Proof. Recall that for $0<r<1$, $\Omega_{n,r} = \varphi(B_{n,r})$, where φ denote the Cayley transform. Let us consider the functions

$$f_r(\omega) = f(\omega)\cdot\chi_{n,r}(\omega), \qquad g_r(\omega) = g(\omega)\cdot\chi_{n,r}(\omega), \tag{4.5}$$

where $\chi_{n,r}$ denote the characteristic function of $\Omega_{n,r}$. Obviously, f_r tends to f in the space $L_\alpha^p(\Omega_n)$ and g_r tends to g in the space $L_\alpha^q(\Omega_n)$ whenever $r\uparrow 1$. Consequently, it is enough to show that

$$[T_\alpha f_r, g_r] = [f_r, T_\alpha g_r], \qquad 0<r<1, \tag{4.6}$$

as it follows from the Theorem 3.1 and the Proposition 4.1(c). Finally, Fubini's theorem immediately imlies that (4.6) holds, so the proof is complete.

DEFINITION. Assume, that $f \in L_\alpha^p(\Omega_n)$ and $g \in L_\alpha^q(\Omega_n)$. We shall write $f \perp g$ if and only if $[f, g] = 0$.

PROPOSITION 4.3. *Suppose* $g \in L_\alpha^q(\Omega_n)$, *then the following conditions are equivalent*:

$$\text{(a)}\quad H_\alpha^p(\Omega_n) \perp g \qquad \text{(b)}\quad T_\alpha g \equiv 0 \quad \textit{in } \Omega_n.$$

Proof. Let g satisfy (a), that is $[f, g] = 0$ for all $f \in H_\alpha^p(\Omega_n)$. Since $T_\alpha f \in H_\alpha^p(\Omega_n)$ for arbitrary function $f \in L_\alpha^p(\Omega_n)$, then $[f, T_\alpha g] = [T_\alpha f, g] = 0$. Thus, we obtain $L_\alpha^p(\Omega_n) \perp T_\alpha g$, which means that $T_\alpha g \equiv 0$. Hence, (a) implies (b). By the same argument one can prove the implication (b) $\Rightarrow$ (a).

Next, the Proposition 4.3 together with the Theorem 2.1 yield:

PROPOSITION 4.4. *Suppose* $g \in H^q_\alpha(\Omega_n)$, *the following conditions are equivalent*:

$$\text{(a)}\quad H^p_\alpha(\Omega_n) \perp g. \qquad \text{(b)}\quad T_\alpha g \equiv 0 \quad \text{in } \Omega_n. \qquad \text{(c)}\quad g \equiv 0 \quad \text{in } \Omega_n.$$

4.2. Let $H^{p*}_\alpha(\Omega_n)$ denote the Banach space of all continuous linear functionals defined on $H^p_\alpha(\Omega_n)$. Next, if g is any function from $H^q_\alpha(\Omega_n)$, $G = \pi(g)$ will denote the following element of $H^{p*}_\alpha(\Omega_n)$:

$$G(f) \equiv [f, g] \qquad \text{for any function} \quad f \in H^p_\alpha(\Omega_n). \tag{4.7}$$

Proposition 4.1 implies that $G = \pi(g) \in H^{p*}_\alpha(\Omega_n)$ and

$$\|G\| = \|\pi(g)\| \leqslant \|g\|_{q,\alpha}. \tag{4.8}$$

Thus, π is a certain correspondence between the spaces $H^q_\alpha(\Omega_n)$ and $H^{p*}_\alpha(\Omega_n)$.

THEOREM 4.1. *The correspondence* π *is a linear isomorphism of the spaces* $H^q_\alpha(\Omega_n)$ *and* $H^{p*}_\alpha(\Omega_n)$. *Moreover,* $\|\pi\| \leqslant 1$, $\|\pi^{-1}\| \leqslant \|T_\alpha\|$, *where* $\|T_\alpha\|$ *denote the norm of the operator* T_α *as a continuous projection from* $L^q_\alpha(\Omega_n)$ *to* $H^q_\alpha(\Omega_n)$.

Proof. Evidently, π is a linear correspondence. Moreover, (4.8) gives $\|\pi\| \leqslant 1$. Assume next that for $i = 1, 2$, $g_i \in H^q_\alpha(\Omega_n)$, $G_i = \pi(g_i) \in H^{p*}_\alpha(\Omega_n)$. If $G_1 = G_2$, then $[f, g_1] = [f, g_2]$ for all functions $f \in H^p_\alpha(\Omega_n)$. It follows from Proposition 4.4 that $g_1 = g_2$, so π is one-to-one. Now let $G \in H^{p*}_\alpha(\Omega_n)$. According to the Hahn-Banach theorem, functional G admits a norm-preserving continuation $\tilde{G} \in L^{p*}_\alpha(\Omega_n)$. Here $L^{p*}_\alpha(\Omega_n)$ denote the space dual to $L^p_\alpha(\Omega_n)$. Since $L^{p*}_\alpha(\Omega_n)$ can be identified with $L^q_\alpha(\Omega_n)$, one can find a function $\varphi \in L^q_\alpha(\Omega_n)$, $\|\varphi\|_{q,\alpha} = \|\tilde{G}\| = \|G\|$, such that for all $f \in H^p_\alpha(\Omega_n)$ we have:

$$G(f) = [f, \varphi] = [T_\alpha f, \varphi] = [f, T_\alpha \varphi] = [f, g],$$

where $g = T_\alpha \varphi \in H^q_\alpha(\Omega_n)$. Thus, π maps $H^q_\alpha(\Omega_n)$ onto $H^{p*}_\alpha(\Omega_n)$. At the same time:

$$\begin{aligned}\|\pi^{-1} G\|_{q,\alpha} = \|g\|_{q,\alpha} = \|T_\alpha \varphi\|_{q,\alpha} &\leqslant \|T_\alpha\| \cdot \|\varphi\|_{q,\alpha} \\ &= \|T_\alpha\| \cdot \|G\|.\end{aligned}$$

Hence, $\|\pi^{-1}\| \leqslant \|T_\alpha\|$, this completes the proof.

Similarly, one can show that the spaces $H^q_\alpha(\tilde{\Omega}_n)$ and $H^{p*}_\alpha(\tilde{\Omega}_n)$ are isomorphic.

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409 179 1-8