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On the Eneström–Kakeya Theorem

K. K. DEWAN

Department of Mathematics, Faculty of Natural Sciences, Jamia Millia Islamia, New Delhi-110025, India

AND

M. BIDKHAM

Department of Mathematics, Faculty of Natural Sciences, Naval Science University, Nouushar, Iran

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A classical result of Eneström and Kakeya (If $a_n \geqslant a_{n-1} \geqslant \cdots \geqslant a_0 > 0$, then, for $|z| > 1$, $\sum_{k=0}^{n} a_k z^k \neq 0$) is extended to polynomials whose coefficients satisfy the condition

$$t^n a_n \leqslant t^{n-1} a_{n-1} \leqslant \cdots \leqslant t^{\lambda+1} a_{\lambda+1} \leqslant t^{\lambda} a_{\lambda} \geqslant t^{\lambda-1} a_{\lambda-1} \geqslant \cdots \geqslant t a_1 \geqslant a_0,$$

for some $t > 0$ and $0 < \lambda \leqslant n$.

1. INTRODUCTION AND STATEMENT OF RESULTS

The following result due to Eneström and Kakeya [5] is well known in the theory of distribution of zeros of polynomials.

THEOREM A. *If $p(z) = \sum_{k=0}^{n} a_k z^k$ be a polynomial of degree n such that*

$$a_n \geqslant a_{n-1} \geqslant \cdots \geqslant a_1 \geqslant a_0 > 0, \tag{1.1}$$

then $p(z)$ does not vanish in $|z| > 1$.

This is a very elegant result but it is equally limited in scope as the hypothesis is very restrictive. Joyal *et al.* [4] extended this theorem to polynomials whose coefficients were monotonic but not necessarily non-negative, which was further improved upon by Dewan and Govil [2].

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In this paper, we consider the class of polynomials $\sum_{k=0}^{n} a_k z^k$, $a_n \neq 0$ whose coefficients satisfy the condition

$$t^n a_n \leqslant t^{n-1} a_{n-1} \leqslant \cdots \leqslant t^{\lambda+1} a_{\lambda+1} \leqslant t^{\lambda} a_{\lambda} \geqslant t^{\lambda-1} a_{\lambda-1} \geqslant \cdots \geqslant t a_1 \geqslant a_0, \quad 0 < \lambda \leqslant n; \tag{1.2}$$

for some $t > 0$ and obtain the following generalisation of Theorem A.

THEOREM 1. *Let $p(z) = \sum_{k=0}^{n} a_k z^k$ be a polynomial of degree n such that for some $t > 0$ and $0 < \lambda \leqslant n$,*

$$t^n a_n \leqslant t^{n-1} a_{n-1} \leqslant \cdots \leqslant t^{\lambda} a_{\lambda} \geqslant t^{\lambda-1} a_{\lambda-1} \geqslant \cdots \geqslant t a_1 \geqslant a_0.$$

Then $p(z)$ has all its zeros in the circle

$$|z| \leqslant \frac{t}{|a_n|} \left\{ \left(\frac{2a_{\lambda}}{t^{n-\lambda}} - a_n \right) + \frac{1}{t^n} (|a_0| - a_0) \right\}. \tag{1.3}$$

In particular, for $t = 1$, the bound obtained in (1.3) can be considerably improved. In the next result we obtain a ring shaped region containing all the zeros of $p(z)$ for the special case, when $t = 1$. The outer radius of the ring obtained being smaller than

$$\frac{t}{|a_n|} \left\{ \left(\frac{2a_{\lambda}}{t^{n-\lambda}} - a_n \right) + \frac{1}{t^n} (|a_0| - a_0) \right\}.$$

More precisely, we prove the following:

THEOREM 2. *Let $p(z) = \sum_{k=0}^{n} a_k z^k$ be a polynomial of degree n such that*

$$a_n \leqslant a_{n-1} \leqslant \cdots \leqslant a_{\lambda+1} \leqslant a_{\lambda} \geqslant a_{\lambda-1} \geqslant \cdots \geqslant a_0,$$

for some λ, $0 < \lambda \leqslant n$. Then $p(z)$ has all its zeros in the annulus (perhaps degenerate)

$$R_2 \leqslant |z| \leqslant R_1.$$

Here

$$R_1 = \frac{c}{2} \left(\frac{1}{|a_n|} - \frac{1}{M_1} \right) + \left\{ \frac{c^2}{4} \left(\frac{1}{|a_n|} - \frac{1}{M_1} \right)^2 + \frac{M_1}{|a_n|} \right\}^{1/2}$$

and

$$R_2 = \frac{1}{2M_2^2} \left[-R_1^2 b(M_2 - |a_0|) + \left\{ R_1^4 b^2 (M_2 - |a_0|)^2 + 4\, |a_0|\, R_1^2 M_2^3 \right\}^{1/2} \right],$$

where

$$\begin{aligned} M_1 &= -a_n + 2a_\lambda - a_0 + |a_0|, \\ M_2 &= R_1^n(|a_n|\, R_1 + 2a_\lambda - a_n - a_0), \\ c &= |a_n - a_{n-1}|, \\ b &= a_1 - a_0. \end{aligned} \tag{1.4}$$

For $\lambda = n$, Theorem 2 reduces to a theorem due to Dewan and Govil [2] and refines the result of Joyal *et al.* [4]. In the case $\lambda = n$ and $a_0 > 0$, Theorem 2 refines Theorem A due to Eneström and Kakeya [5].

Regarding the number of zeros in $|z| \leqslant 1/2$ of the polynomial $\Sigma_{k=0}^{n} a_k z^k$, $a_n \neq 0$, we have been able to prove the following:

THEOREM 3. *Let $p(z) = \sum_{k=0}^{n} a_k z^k$ be a polynomial of degree n such that*

$$a_n \leqslant a_{n-1} \leqslant \cdots \leqslant a_{\lambda+1} \leqslant a_\lambda \geqslant a_{\lambda-1} \geqslant \cdots \geqslant a_0,$$

for some λ, $0 < \lambda \leqslant n$. Then the number of zeros of $p(z)$ in $|z| \leqslant 1/2$ does not exceed

$$\frac{1}{\log 2}\left\{\log \frac{|a_n| + |a_0| - a_n - a_0 + 2a_\lambda}{|a_0|}\right\}.$$

For $\lambda = n$ and $a_0 > 0$, the above theorem reduces to a result due to Mohammad [6].

2. LEMMAS

The following result is a well-known generalisation of Schwarz's lemma (see [7, p. 112]).

LEMMA 1. *If $p(z)$ is analytic inside and on the unit circle, $|p(z)| \leqslant M$ on $|z| = 1$, and $p(0) = a$, where $0 < |a| < M$, then for $|z| < 1$,*

$$|p(z)| \leqslant M \frac{M\,|z| + |a|}{|a|\,|z| + M}. \tag{2.1}$$

From a lemma due to Govil *et al.* [3, p. 325], one can easily prove

LEMMA 2. *If $p(z)$ is analytic in $|z| \leqslant R$, $p(0) = 0$, $p'(0) = b$ and $|p(z)| \leqslant M$ for $|z| = R$, then, for $|z| \leqslant R$,*

$$|p(z)| \leqslant \frac{M\,|z|}{R^2} \quad \frac{M\,|z| + R^2\,|b|}{M + |z|\,|b|}. \tag{2.2}$$

LEMMA 3. *Let $f(z)=a_nz^n+a_pz^p+\cdots+a_1z+a_0$, $0\leqslant p\leqslant n-1$, be a polynomial of degree n with complex coefficients. Then for every positive real number r, all the zeros of $f(z)$ lie in the circle*

$$|z|\leqslant \max\left\{r, \sum_{k=0}^{p} \frac{|a_k|}{|a_n|\, r^{n-k-1}}\right\}.$$

The above lemma is due to Aziz and Mohammad [1].

3. PROOFS OF THE THEOREMS

Proof of Theorem 1. Consider

$$\begin{aligned} g(z) &= (t-z)\,p(z) \\ &= -a_nz^{n+1}+\sum_{k=0}^{n}(ta_k-a_{k-1})\,z^k, \qquad a_{-1}=0. \end{aligned}$$

Now since $g(z)$ is a polynomial of degree $n+1$, hence applying Lemma 3 to the polynomial $g(z)$ with $p=n$ and $r=t$, it follows that all the zeros of $g(z)$ lie in the circle

$$\begin{aligned} |z| &\leqslant \max\left\{t, \sum_{k=0}^{n}\frac{|ta_k-a_{k-1}|}{t^{n-k}\,|a_n|}\right., \qquad a_{-1}=0 \\ &= \sum_{k=0}^{n}\frac{|ta_k-a_{k-1}|}{t^{n-k}\,|a_n|}, \end{aligned}$$

since

$$t=\left|\sum_{k=0}^{n}\frac{(ta_k-a_{k-1})}{t^{n-k}a_n}\right|\leqslant \sum_{k=0}^{n}\frac{|ta_k-a_{k-1}|}{t^{n-k}\,|a_n|}.$$

Now

$$\begin{aligned} \sum_{k=0}^{n}\frac{|ta_k-a_{k-1}|}{t^{n-k}\,|a_n|} &= \sum_{k=0}^{\lambda}\frac{|ta_k-a_{k-1}|}{t^{n-k}\,|a_n|}+\sum_{k=\lambda+1}^{n}\frac{|ta_k-a_{k-1}|}{t^{n-k}\,|a_n|} \\ &= \frac{t}{|a_n|}\left\{\left(\frac{2a_\lambda}{t^{n-\lambda}}-a_n\right)+\frac{1}{t^n}(|a_0|-a_0)\right\}. \end{aligned}$$

Hence all the zeros of $g(z)$ lie in

$$|z|\leqslant\frac{t}{|a_n|}\left\{\left(\frac{2a_\lambda}{t^{n-\lambda}}-a_n\right)+\frac{1}{t^n}(|a_0|-a_0)\right\}.$$

Since all the zeros of $p(z)$ are also the zeros of $g(z)$, the theorem is proved.

Proof of Theorem 2. Consider

$$\begin{aligned} g(z) &= (1-z)\,p(z) \\ &= -a_n z^{n+1} + \sum_{k=1}^{n} (a_k - a_{k-1})\, z^k + a_0 \\ &= -a_n z^{n+1} + P(z), \qquad \text{say.} \end{aligned} \tag{3.1}$$

If by $Q(z)$ we denote the polynomial $z^n P(1/z)$, then

$$Q(z) = \sum_{k=1}^{n} (a_k - a_{k-1})\, z^{n-k} + a_0 z^n.$$

For $|z| = 1$, we have

$$\begin{aligned} |Q(z)| &\leqslant \sum_{k=1}^{n} |a_k - a_{k-1}| + |a_0| \\ &= \sum_{k=1}^{\lambda} |a_k - a_{k-1}| + \sum_{k=\lambda+1}^{n} |a_k - a_{k-1}| + |a_0| \\ &= -a_n + 2a_\lambda - a_0 + |a_0| \\ &= M_1. \end{aligned}$$

Hence by the Maximum Modulus Principle

$$\begin{aligned} |Q(0)| &= |a_n - a_{n-1}| \\ &< M_1. \end{aligned}$$

Applying Lemma 1 to the function $Q(z)$, we get for $|z| \leqslant 1$

$$|Q(z)| \leqslant M_1 \frac{M_1 |z| + |a_n - a_{n-1}|}{|a_n - a_{n-1}|\, |z| + M_1},$$

which implies that

$$\left| z^n P\left(\frac{1}{z}\right) \right| \leqslant M_1 \frac{M_1 |z| + |a_n - a_{n-1}|}{|a_n - a_{n-1}|\, |z| + M_1}, \qquad |z| \leqslant 1. \tag{3.2}$$

If $R > 1$, then $(1/R)\, e^{-i\theta}$ lies inside the unit circle for every real θ and from (3.2) it follows that

$$|P(Re^{i\theta})| \leqslant M_1 R^n \frac{M_1 + |a_n - a_{n-1}|\, R}{|a_n - a_{n-1}| + M_1 R} \tag{3.3}$$

for every $R \geqslant 1$ and θ real.

Thus for $|z| = R > 1$,

$$\begin{aligned} |g(Re^{i\theta})| &= |-a_n R^{n+1} e^{i(n+1)\theta} + P(Re^{i\theta})| \\ &\geqslant |a_n|\, R^{n+1} - |P(Re^{i\theta})| \\ &\geqslant |a_n|\, R^{n+1} - M_1 R^n \frac{M_1 + R\,|a_n - a_{n-1}|}{M_1 R + |a_n - a_{n-1}|} \qquad \text{(by (3.3))} \\ &= |a_n|\, R^{n+1} - M_1 R^n \frac{M_1 + cR}{M_1 R + c} \qquad \text{(by (1.4))} \\ &= \frac{R^n}{M_1 R + c} [M_1 |a_n|\, R^2 - cR(M_1 - |a_n|) - M_1^2] \\ &> 0, \end{aligned}$$

if

$$\begin{aligned} R &> \frac{c}{2}\left(\frac{1}{|a_n|} - \frac{1}{M_1}\right) + \left\{\frac{c^2}{4}\left(\frac{1}{|a_n|} - \frac{1}{M_1}\right)^2 + \frac{M_1}{|a_n|}\right\}^{1/2} \\ &= R_1. \end{aligned}$$

Hence $p(z)$ has all its zeros in

$$|z| \leqslant R_1. \tag{3.4}$$

Next we show that $p(z)$ has no zeros in $|z| < R_2$.
We have by (3.1)

$$\begin{aligned} g(z) &= a_0 + \sum_{k=1}^{n} (a_k - a_{k-1})\, z^k - a_n z^{n+1} \\ &= a_0 + f(z), \qquad \text{say.} \end{aligned} \tag{3.5}$$

Clearly if $|z| \leqslant R_1$, $(R_1 \geqslant 1)$, then

$$\begin{aligned} |f(z)| &\leqslant |a_n|\, R_1^{n+1} + \sum_{k=1}^{n} |a_k - a_{k-1}|\, R_1^k \\ &\leqslant |a_n|\, R_1^{n+1} + R_1^n \left\{ \sum_{k=1}^{\lambda} |a_k - a_{k-1}| + \sum_{k=\lambda+1}^{n} |a_k - a_{k-1}| \right\} \\ &= R_1^n(|a_n|\, R_1 + 2a_\lambda - a_n - a_0) \\ &= M_2. \end{aligned} \tag{3.6}$$

Thus for $|z| \leqslant R_1$,

$$|f(z)| \leqslant M_2.$$

Further, since $f(0)=0$, $f'(0)=a_1-a_0=b$, by Lemma 2 we have for $|z|\leqslant R_1$

$$|f(z)|\leqslant\frac{M_2\,|z|}{R_1^2}\quad\frac{M_2\,|z|+R_1^2b}{M_2+|z|\,b}. \tag{3.7}$$

Combining (3.5) and (3.7), we get, for $|z|\leqslant R_1$

$$\begin{aligned}|g(z)|&\geqslant|a_0|-\frac{M_2\,|z|}{R_1^2}\quad\frac{M_2\,|z|+R_1^2b}{M_2+|z|\,b}\\&=-\frac{1}{R_1^2(M_2+|z|\,b)}\left[|z|^2M_2^2+R_1^2b\,|z|\,(M_2-|a_0|)-|a_0|\,R_1^2M_2\right]\\&>0,\end{aligned}$$

if

$$\begin{aligned}|z|&<\frac{-R_1^2b(M_2-|a_0|)+\{R_1^4b^2(M_2-|a_0|)^2+4\,|a_0|\,R_1^2M_2^3\}^{1/2}}{2M_2^2}\\&=R_2\end{aligned}$$

(since $M_2-|a_0|=M_2-|f(1)|\geqslant 0$ by (3.6), which implies that $p(z)$ has no zeros in

$$|z|<R_2, \tag{3.8}$$

and the theorem follows.

Proof of Theorem 3. Consider

$$\begin{aligned}g(z)&=(1-z)\,p(z)\\&=-a_nz^{n+1}+\sum_{k=1}^{n}(a_k-a_{k-1})\,z^k+a_0.\end{aligned}$$

For $|z|\leqslant 1$

$$\begin{aligned}|g(z)|&\leqslant|a_n|+|a_0|+\sum_{k=1}^{n}|a_k-a_{k-1}|\\&=|a_n|+|a_0|+\sum_{k=1}^{\lambda}|a_k-a_{k-1}|+\sum_{k=\lambda+1}^{n}|a_k-a_{k-1}|\\&=|a_n|+|a_0|+2a_\lambda-a_0-a_n,\end{aligned}$$

which implies

$$\left|\frac{g(z)}{g(0)}\right| \leqslant \frac{|a_n| + |a_0| + 2a_\lambda - a_0 - a_n}{|a_0|}.$$

Now it is known (see [7, p. 171]) that if $f(z)$ is regular, $f(0) \neq 0$ and $|f(z)| \leqslant M$ in $|z| \leqslant 1$, then the number of zeros of $f(z)$ in $|z| \leqslant 1/2$ does not exceed

$$\frac{1}{\log 2}\left(\log \frac{M}{|f(0)|}\right).$$

Thus, if $n(1/2)$ denotes the number of zeros of $g(z)$ in $|z| \leqslant 1/2$, then

$$n\left(\frac{1}{2}\right) \leqslant \frac{1}{\log 2}\left\{\log \frac{|a_n| + |a_0| - a_n - a_0 + 2a_\lambda}{|a_0|}\right\}.$$

As the number of zeros of $p(z)$ in $|z| \leqslant 1/2$ is also equal to $n(1/2)$, the theorem follows.

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