Fractional derivatives of Weierstrass-type functions

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Abstract

For a special type of fractional differentiation, formulas for the Weierstrass and the Weierstrass–Mandelbrot functions are shown. For integer valued parameters \( \lambda \), a well conditioned numerical procedure for computing the derivatives in the mean of these functions is derived and used to compute some values.

Keywords: Fractional derivatives; Weierstrass functions

1. Introduction

Let \( h: \mathbb{R} \to \mathbb{R} \) be a Hölder continuous function of order \( \beta \leq 1 \) with Hölder constant \( H(h) \) and period 1, and let \( 0 < \gamma < \beta \). For fixed integral parameter \( \lambda = 2, 3, \ldots \) define the Weierstrass-type function

\[ W(x) = W_h^\lambda(x) = \sum_{k=1}^{\infty} \lambda^{-k\gamma} h(\lambda^k x). \]  

(1)

For \( h(0) = 0 \), the corresponding Weierstrass–Mandelbrot function is given by

\[ M(x) = M_h^\lambda(x) = \sum_{k=-\infty}^{\infty} \lambda^{-k\gamma} h(\lambda^k x). \]  

(2)

It has the scaling property

\[ M(x) = \lambda^{-\gamma} M(\lambda x). \]  

(3)

If \( h(u) = \sin(2\pi u) \) and \( \lambda = 2 \) one obtains the classical Weierstrass function with exponent \( \gamma \). In the case \( h(u) = 1 - |2u - 1|, u \in [0, 1] \), and \( \lambda = 2 \), \( W \) is called the Takagi function. These are traditional examples of continuous nowhere differentiable functions (Fig. 1).

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It is easy to see that the Hölder exponent of \( W^\gamma_h \) agrees with \( \gamma \). A well-known fact is also that the so-called box counting dimension of the graph of \( W^\gamma_h \) equals \( 2 - \gamma \) (see e.g. [1]). Hu and Lau [3] proved that this number coincides with the fractal dimension of an appropriate measure. The problem whether this is also true for the Hausdorff measure is still open. The best lower estimate for the Hausdorff dimension known from the literature is \( 2 - \gamma - c/\ln \lambda \) for some constant \( c \) (Mauldin and Williams [4]).

In the present paper we will continue the study of fractional derivatives of these functions which is closely related to dimension problems. In [6] the first author proved that the fractional degree of differentiability agrees with \( \gamma \) if \( M \neq 0 \) and is not less than \( \beta \) if \( M = 0 \). In particular, there exist all lower-order Weyl–Marchaud derivatives of \( W \) and \( M \). For \( M \neq 0 \) the corresponding gradual fractional derivatives in the mean are shown to be constant at Lebesgue almost all points.

The aim of the present paper is to calculate the fractional derivatives explicitly and to develop a corresponding computation procedure. The latter results in numerical integration of rapidly oscillating functions. It turns out that for such functions integration by Monte–Carlo methods works well, whereas classical numerical procedures are not practicable here.\(^1\)

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2. Basic notions and related results

In [6] relationships between the Weyl-Marchaud derivatives used in the theory of function spaces and some notions of fractal geometry are worked out. Recall that a version of the Weyl-Marchaud derivative of order $0 < \alpha < 1$ of a real or complex-valued measurable function $f$ on $\mathbb{R}$ at $x$ is given by

$$D^\alpha f(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \frac{f(x) - f(x - y)}{y^{1+\alpha}} \, dy \quad \text{(left-sided)},$$

$$D^\alpha f(x) = \frac{(-1)^\alpha}{\Gamma(1-\alpha)} \int_0^\infty \frac{f(x) - f(x + y)}{y^{1+\alpha}} \, dy \quad \text{(right-sided)}$$

provided that these integrals exist in the sense of absolute convergence. We additionally introduced (upper) (absolute) fractional derivatives of order $\alpha$ in the mean with respect to the logarithmic measure: If $0 < \alpha < 1$,

$$|D^\alpha| f(x) = \limsup_{\delta \to 0} \frac{1}{\ln \delta} \left( \int_{-\delta}^\infty \frac{|f(x) - f(x - y)|}{y^{1+\alpha}} \, dy \right),$$

$$|D^\alpha| f(x) = \limsup_{\delta \to 0} \frac{1}{\ln \delta} \left( \int_{-\delta}^\infty \frac{|f(x + y) - f(x)|}{y^{1+\alpha}} \, dy \right),$$

$$|D^\alpha| f(x) = \lim_{\delta \to 0} \frac{1}{\ln \delta} \left( \int_{-\delta}^\infty \frac{|f(x) - f(x - y)|}{y^{1+\alpha}} \, dy \right),$$

$$|D^\alpha| f(x) = \lim_{\delta \to 0} \frac{1}{\ln \delta} \left( \int_{-\delta}^\infty \frac{|f(x + y) - f(x)|}{y^{1+\alpha}} \, dy \right),$$

$$d^\alpha f(x) = \lim_{\delta \to 0} \frac{1}{\ln \delta} \left( \int_{-\delta}^\infty \frac{f(x) - f(x - y)}{y^{1+\alpha}} \, dy \right),$$

$$d^\alpha f(x) = \lim_{\delta \to 0} \frac{1}{\ln \delta} \left( \int_{-\delta}^\infty \frac{f(x + y) - f(x)}{y^{1+\alpha}} \, dy \right),$$

where (8)–(11) are determined if the limits exist.

For $\alpha \geq 1$ one applies these definitions replacing $f$ by its ordinary derivative of order $[\alpha]$, provided that it exists, and then $\alpha$ by $\alpha - [\alpha]$.

The degree of differentiability of $f$ at $x$ is defined by

$$\gamma = \gamma(x) = \sup \{ \alpha : |D^\alpha| f(x) = 0 \} = \inf \{ \alpha : |D^\alpha| f(x) = \infty \}$$

in the left-sided or right-sided version (cf. [6, Proposition 2]).

In [6, Theorem 2], we proved that the Weyl-Marchaud derivatives exist for all $\alpha < \gamma$, provided that $\int_{-\infty}^\infty |f(x + y) - f(x)| y^{-1-\gamma} \, dy < \infty$, and do not exist if $\alpha > \gamma$.

Finally, the (absolute) gradual derivative in the mean of $f$ at $x$ is given by $(|D^\alpha| f(x)) d^\alpha f(x)$ (left- or right-sided version) if the corresponding expression is determined.
Our aim is to calculate these derivatives for the Weierstrass-type functions $W_h^\gamma$ introduced above. From now on we will assume $h(0) = 0$. First note that in the case $M_h^\gamma \neq 0$ the exponent $\gamma$ agrees with the left-sided and the right-sided degree of differentiability of $W_h^\gamma$ and of $M_h^\gamma$ at all points (cf. [6, Theorem 5(i), Proposition 8]). If $M_h^\gamma = 0$ then the degree of differentiability of $W_h^\gamma$ is not less than the Hölder exponent $\beta$ of $h$ at all points (cf. [6, Proposition 8]). Moreover, in the first case the gradual derivatives $d_{x}^\gamma W_h^\gamma(x)$, $d_{y}^\gamma W_h^\gamma(x)$, $|d_{x}^\gamma|W_h^\gamma(x)$, and $|d_{y}^\gamma|W_h^\gamma(x)$ are constants approximated by

$$d_{x}^\gamma W(x) = \frac{1}{\ln \lambda} \int_0^1 \int_{z-\lambda n}^{z-\lambda n+1} \frac{W(z) - W(z - y)}{y^{1+\gamma}} \, dy \, dz$$

for all $1, 2, \ldots$, at Lebesgue a.a. $x$ (cf. [6, Theorem 5(ii)]). (For the right-sided and absolute versions replace $W(z) - W(z - y)$ by the corresponding expressions.)

3. Calculation of the fractional derivatives

We first consider the Weyl–Marchaud derivatives. Note that $D^\alpha h$ is a Hölder continuous function of order $\beta - \alpha$ with period 1. From this we infer immediately

$$D^\alpha W_h^\gamma(x) = W_h^{\beta - \alpha}(x)$$

(left- and right-sided versions) for all $\alpha < \gamma$ and $x \in \mathbb{R}$ (with the same parameter $\lambda$).

Note that in the classical case, $h(u) = \sin(2\pi u)$, one obtains

$$D^\alpha_h h(u) = (2\pi)^\alpha \sin(2\pi u \pm \frac{1}{2} \pi \alpha).$$

A table of the Weyl–Marchaud derivatives for some basic functions may be found in [5, Ch. II, Section 9].

We now turn to the gradual derivatives. The following formulas will be used as an auxiliary tool.

**Proposition 1.** At Lebesgue a.a. $x$ we have

(i) $d_{x,y}^\gamma W_h^\gamma(x) = \lim_{n \to \infty} \frac{1}{\ln \lambda} \int_0^1 \int_{z-\lambda n}^{z-\lambda n+1} \frac{\pm (W_h^\gamma(z) - W_h^\gamma(z \mp y))}{y^{1+\gamma}} \, dy \, dz$

(ii) $|d_{x,y}^\gamma|W_h^\gamma(x) = \lim_{n \to \infty} \frac{1}{\ln \lambda} \int_0^1 \int_{z-\lambda n}^{z-\lambda n+1} \frac{|W_h^\gamma(z) - W_h^\gamma(z \mp y)|}{y^{1+\gamma}} \, dy \, dz$.

(iii) On the right- or left-hand sides of (i) and (ii) $W$ may be replaced by $M$.

**Proof.** In the notation

$$g_k(x) := \frac{1}{\ln \lambda} \int_{z-(k+1)}^{z-k} \frac{|W_h^\gamma(x) - W_h^\gamma(x \mp y)|}{y^{1+\gamma}} \, dy$$

the derivative $|d_{x,y}^\gamma|W_h^\gamma(x)$ may be rewritten as

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} g_k(x).$$
Furthermore, for any \( n \) and \( m \)

\[
\frac{1}{n+m} \sum_{k=0}^{n+m-1} g_k(x) = \frac{1}{n+m} \sum_{k=0}^{n-1} g_k(x) + \frac{1}{n+m} \sum_{k=0}^{m-1} g_{n+k}(x).
\]

In [6, Theorem 5], we have proved that \( g_{n+k}(x) = g_n(\lambda^k x) + \Theta_{n,k}(x) \), where \( |\Theta_{n,k}(x)| \leq \Theta_n = (1/\ln \lambda) H(h)e^{-(\beta-\gamma)n} \). Therefore,

\[
\left| |d_{\gamma}^x| W_h^\gamma(x) - \lim_{m \to \infty} \frac{1}{m} \sum_{k=0}^{m-1} g_n(\lambda^k x) \right| \leq \Theta_n.
\]

The limit on the left-hand side exists at almost all \( x \) by the following arguments: Because of the periodicity of \( W_h^\gamma \) we have \( g_n(\lambda^k x) = g_n(A^k x) \) for \( A : [0, 1] \mapsto [0, 1] \) mapping \( x \) onto the fractional part of \( \lambda x \). The Lebesgue measure is \( A \)-invariant so that Birkhoff’s ergodic theorem yields

\[
\lim_{m \to \infty} \frac{1}{m} \sum_{k=0}^{m-1} g_n(\lambda^k x) = \int_0^1 g_n(z) \, dz
\]

at Lebesgue a.a. \( x \) (cf. [2]). Consequently,

\[
|d_{\gamma}^x| W_h^\gamma(x) = \lim_{n \to \infty} \int_0^1 g_n(z) \, dz = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \int_0^1 g_k(z) \, dz
\]

which proves (ii).

Similar arguments (without the absolute value signs) lead to (i). The replacement of the continuous limits as \( \delta \to 0 \) in the definition of \( d_{\gamma}^x W_h^\gamma \) by the discrete limits as \( n \to \infty \) is justified by the existence of the absolute derivatives.

For (iii) we estimate the difference on the right-hand sides by

\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \int_0^1 \int_{\lambda^{-l}(1+\gamma)} \frac{y^{-(1+\gamma)}}{\ln \lambda} |M_h^\gamma(z) - M_h^\gamma(z \mp y) - W_h^\gamma(z) + W_h^\gamma(z \mp y)| \, dy \, dz
\]

\[
\leq \limsup_{n \to \infty} \frac{1}{n} \int_0^1 \int_{\lambda^{-l}(1+\gamma)} \frac{1}{\ln \lambda} \sum_{l=-\infty}^{0} \lambda^{-l\beta} y^{-(1+\gamma)} |h(\lambda^l z) - h(\lambda^l (z \mp y))| \, dy \, dz
\]

\[
\leq \frac{1}{\ln \lambda} \limsup_{n \to \infty} \frac{1}{n} \int_0^1 \sum_{l=-\infty}^{0} \lambda^{-l\beta} H(h) \lambda^l \beta y^{-(1+\gamma)} \, dy
\]

\[
\leq \frac{1}{\ln \lambda} \frac{H(h)}{\beta - \gamma} \sum_{l=-\infty}^{0} \lambda^{l(\beta-\gamma)} \limsup_{n \to \infty} \frac{1}{n} (1 - \lambda^{-n(\beta-\gamma)}) = 0.
\]

Similar arguments (see also [6, Proposition 8]) lead to

\[
d_{\gamma}^x M^\gamma(x) = d_{\gamma}^x W^\gamma(x), \quad d_{\gamma}^\gamma M^\gamma(x) = d_{\gamma}^\gamma W^\gamma(x)
\]

\[
|d_{\gamma}^x| W^\gamma(x) = |d_{\gamma}^\gamma| M^\gamma(x), \quad |d_{\gamma}^\gamma| W^\gamma(x) = |d_{\gamma}^\gamma| M^\gamma(x)
\]

proving (iii). \( \square \)
Remark. A by-product of the last proof is the estimation

$$\left| \frac{d_{Y_n}^y W_n^y(x)}{d_{Y_n}^y W_n^y(x)} - \frac{1}{\ln \lambda} \int_0^1 \int_{\lambda^{-n+1}}^{\lambda^{-n}} \frac{|W_n^y(z) - W_n^y(z + y)|}{y^{1+\gamma}} \, dy \, dz \right| \leq \Theta_n.$$ 

However, for large $n$ it is numerically difficult to compute the inner integral. Therefore we suggest another approximation of the absolute derivatives combining the periodicity of $W_n^y$ with the scale invariance of $M_n^y$. (For brevity, we will omit from now on the lower index $h$ in the notation.)

Theorem 2. At Lebesgue a. a. $x$ the following holds:

(a) $d_{y_n}^y W_n^y(x) = d_{y_n}^y M_n^y(x) = d_{y_n}^y M_n^y(x) = 0$;

(b) $|d_{y_n}^y W_n^y(x)| = |d_{y_n}^y M_n^y(x)| = |d_{y_n}^y M_n^y(x)|$;

(c) $\left| \frac{d_{y_n}^y W_n^y(x)}{d_{y_n}^y W_n^y(x)} - \frac{1}{\ln \lambda} \int_0^1 \int_{\lambda^{-n+1}}^{\lambda^{-n}} \frac{|M_{n_1, n_2}^y (\lambda^{n_1} z + y) - M_{n_1, n_2}^y (\lambda^{n_1} z)|}{y^{1+\gamma}} \, dy \, dz \right| \leq \varepsilon_{1, n_1}^y + \varepsilon_{2, n_2}^y,$

where $M_{n_1, n_2}^y(x) = \sum_{k=-n_1}^{n_2} \lambda^{-k} h(\lambda^k x)$ is the truncated Weierstrass–Mandelbrot function and

$$\varepsilon_{1, n_1}^y = \frac{H(h)\lambda^{-(n_1+1)(\beta-\gamma)}}{(\beta - \gamma) \ln \lambda}, \quad \varepsilon_{2, n_2}^y = \frac{2\|h\| \lambda^{-n_2}}{\gamma \ln \lambda},$$

$n_1, n_2 = 1, 2, \ldots$

Proof. Since $W_n^y$ has period 1 we have

$$\int_0^1 \int_{\lambda^{-n-1}}^{\lambda^{-n}} \frac{|W_n^y(z) - W_n^y(z + y)|}{y^{1+\gamma}} \, dy \, dz = \int_0^1 \frac{1}{y^{1+\gamma}} \left( \int_0^1 W_n^y(z) \, dz - \int_{\lambda^{-n}}^{\lambda^{-n} + y} W_n^y(z) \, dz \right) \, dy = 0.$$ 

Therefore Proposition 1(i) implies (a).

The equality of the derivatives of the Weierstrass- and the Weierstrass–Mandelbrot functions is given in Proposition 1(iii).

Furthermore, in view of the periodicity of $W_n^y$,

$$\int_0^1 \int_{\lambda^{-n-1}}^{\lambda^{-n}} \frac{|W_n^y(z) - W_n^y(z + y)|}{y^{1+\gamma}} \, dy \, dz = \int_0^1 \frac{1}{y^{1+\gamma}} \int_{\lambda^{-n}}^{\lambda^{-n} + y} |W_n^y(z) - W_n^y(z)| \, dz \, dy$$

$$= \int_0^1 \int_{\lambda^{-n}}^{\lambda^{-n} + y} |W_n^y(z) - W_n^y(z)| \, dz \, dy,$$

so that Proposition 1(ii) yields

$$|d_{y_n}^y W_n^y(x)| = |d_{y_n}^y W_n^y(x)|,$$

i.e., (b) is true.
Proposition 1(iii) and the scaling property (3) of the Weierstrass–Mandelbrot function provide for a.a. $x$ and any $n$

$$|d^n W^\gamma(x)| = \lim_{m \to \infty} \frac{1}{m \ln \lambda} \int_0^1 \sum_{k=0}^{m-1} \int_{\lambda^{-k+1}}^{\lambda^{-k}} \frac{|M^\gamma(z + y) - M^\gamma(z)|}{y^{1+\gamma}} \, dy \, dz$$

$$= \lim_{m \to \infty} \frac{1}{m \ln \lambda} \int_0^1 \sum_{k=n}^{m-1} \int_{\lambda^{-k+1}}^{\lambda^{-k}} \frac{\lambda^{-k} |M^\gamma(\lambda^k(z + y)) - M^\gamma(\lambda^k z)|}{y^{1+\gamma}} \, dy \, dz$$

$$= \lim_{m \to \infty} \frac{1}{m \ln \lambda} \int_0^1 \sum_{k=n}^{m-1} \int_{\lambda^{-k+1}}^{\lambda^{-k}} \frac{|M^\gamma(\lambda^k z + y) - M^\gamma(\lambda^k z)|}{y^{1+\gamma}} \, dy \, dz$$

$$= \lim_{m \to \infty} \frac{1}{m \ln \lambda} \sum_{k=n}^{m-1} \int_{\lambda^{-k+1}}^{\lambda^{-k}} \int_0^{\lambda^k} |M^\gamma(z + y) - M^\gamma(z)| \, dz \, dy.$$
the estimation

\[ |R_{n_1,n_2}^\beta| \leq \frac{H(h) \lambda^{-(n_1+1)(\beta-\gamma)}}{\ln \lambda} \int_{-1}^{1} y^{-(1+\gamma)} \frac{\lambda^{-n_2}}{\lambda^{\gamma} - 1} \int_{-1}^{1} y^{-(1+\gamma)} \, dy + \frac{2\|h\|}{\ln \lambda} \frac{\lambda^{-n_2}}{\lambda^{\gamma} - 1} \int_{-1}^{1} y^{-(1+\gamma)} \, dy \]

\[ = \frac{H(h) \lambda^{-(n_1+1)(\beta-\gamma)}}{(\beta - \gamma) \ln \lambda} + \frac{2\|h\|}{\gamma \ln \lambda} \lambda^{-n_2} \epsilon_1 + \epsilon_2. \]

This proves (c). \( \square \)

As a by-product of the proof the case \( n_1 = -1 \) and \( n_2 = 0 \) provides the result

\[ |d^\beta W^\gamma(x) \leq \frac{H(h)}{(\beta - \gamma) \ln \lambda} + \frac{2\|h\|}{\gamma \ln \lambda}. \]  \( (15) \)

4. Numerical computations

Theorem 2(c) enables us to compute the values of the absolute fractional derivatives in the mean with arbitrary exactness.

In order to approximate \( |d^\beta W^\gamma(x) \) for a. a. \( x \) up to \( 2\epsilon \), we solve the inequalities

\[ \epsilon_{1,n_1}^\gamma \leq \epsilon \quad \text{for} \quad n_1 \quad \text{and} \quad \epsilon_{2,n_2}^\gamma \leq \epsilon \quad \text{for} \quad n_2. \]

This yields

\[ n_1 \geq \frac{\ln H(h) - \ln(\epsilon(\beta - \gamma) \ln \lambda)}{(\beta - \gamma) \ln \lambda} - 1 \]

and

\[ n_2 \geq \frac{\ln(2\|h\|) - \ln(\epsilon \gamma \ln \lambda)}{\gamma \ln \lambda}. \]

In the classical case, \( h(x) = \sin 2\pi x \), we get, e. g., for \( \lambda = 2, \gamma = 0.5, \epsilon = 0.001 \) as minimal values for \( n_1 \) and \( n_2 \): \( n_1 = 39, n_2 = 31 \).

Because of the large values of \( \lambda^{n_2} \) the integrand of

\[ \int_{-1}^{1} \int_{-1}^{1} \frac{|M_{n_1,n_2}(\lambda z + y) - M_{n_1,n_2}(\lambda z)|}{y^{1+\gamma}} \, dy \, dz \]

is rapidly oscillating in the interval \([0,1]\). So classical numerical integration by smoothing the integrand does not give reliable results to say nothing of the estimation of the error. Therefore, we use a Monte–Carlo method.

For this purpose we substitute in the above integral \( y = \lambda^{-t} \) and obtain

\[ \int_{-1}^{1} \int_{0}^{\lambda} |M_{n_1,n_2}(\lambda^{n_2} z + \lambda^{-t}) - M_{n_1,n_2}(\lambda^{n_2} z)| \, dt \, dz. \]

Denote the integrand by \( A_{n_1,n_2}^\gamma(t,z) \).
Let \((t, z)\) be chosen at random from the uniform distribution on the unit square \([0, 1]^2\). Then

\[ I_{n_1 n_2}^\gamma = E(\Delta_{n_1 n_2}^\gamma), \]

where \(E(\cdot)\) denotes expectations. From \(N\) independent realisations of \(\Delta_{n_1 n_2}^\gamma\) we get the mean \(\bar{x}_N\) and the empirical standard deviation \(s_N\) as estimators for the expectation and the standard deviation of \(\Delta_{n_1 n_2}^\gamma\). Thus, the mean is an estimator for \(I_{n_1 n_2}^\gamma\) and \(s_N/\sqrt{N}\) measures the error in the confidence interval sense; e.g., with probability 0.99 we get \(\bar{x}_N\) and \(s_N\) such that

\[ \bar{x}_N - 2.6 \frac{s_N}{\sqrt{N}} \leq \bar{x}_N \leq \bar{x}_N + 2.6 \frac{s_N}{\sqrt{N}}, \]

and therefore

\[ \bar{x}_N - 2.6 \frac{s_N}{\sqrt{N}} - 2\varepsilon \leq |d^\gamma |W^\gamma(x) - \bar{x}_N \leq \bar{x}_N + 2.6 \frac{s_N}{\sqrt{N}} + 2\varepsilon. \]

For our values in the Figs. 2–4 we have chosen \(\varepsilon \leq 0.001\) and \(N\) such that \(s_N/\sqrt{N} \leq 0.005\) giving with probability 0.99

\[ |d^\gamma |W^\gamma - \bar{x}_N| \leq 0.015 \quad \text{a.e.} \]

Furthermore, to get the relevant parts of \(\lambda^{k+n_1} z\) and \(\lambda^{k-t}\) for the evaluations of the periodic function \(h\), i.e., the parts modulo 1, we have to use enough digits of these expressions. In order that the number of decimal digits of these relevant parts is at least 10 the number of used decimal digits has to be at least

\[ 10 + \lg(\lambda^{n_1+n_2}) = 10 + (n_1 + n_2) \lg \lambda \geq 10 + \left( \frac{\ln H(h) - \ln(\varepsilon(\beta-\gamma) \ln \lambda)}{(\beta-\gamma) \ln \lambda} + \frac{\ln(2 \| h \|) - \ln(\varepsilon \gamma \ln \lambda)}{\gamma \ln \lambda} + 1 \right) \frac{\ln \lambda}{\ln 10} \]

\[ = 10 + \frac{1}{\beta - \gamma} \lg \frac{H(h)}{\varepsilon(\beta-\gamma)} + \frac{1}{\gamma} \lg \frac{2 \| h \|}{\varepsilon \gamma} + \lg \lambda - \left( \frac{1}{\beta - \gamma} + \frac{1}{\gamma} \right) \lg(\ln \lambda). \]

For fixed \(\gamma\) we may thus use

\[ 10 + \frac{1}{\beta - \gamma} \lg \frac{H(h)}{\varepsilon(\beta-\gamma)} + \frac{1}{\gamma} \lg \frac{2 \| h \|}{\varepsilon \gamma} + \lg \lambda \quad \text{digits.} \]

In Fig. 2 the (approximated) absolute fractional derivatives in the mean for the Weierstrass functions \(W^\gamma_h\) with \(h(u) = \sin(2\pi u)\) for varying \(\lambda\) and \(\gamma = 0.3, 0.5, 0.7\) as well as the upper bound (15) for \(\gamma = 0.7\) are presented. The \(\lambda\)-values between \(10^k\) and \(10^{k+1}\) are \(2 \cdot 10^k\) and \(4 \cdot 10^k\), \(k = 1, 2, \ldots, 11\). The
Fig. 2. Absolute fractional derivatives in the mean for the Weierstrass functions $W_h^\alpha$ with $h(u) = \sin(2\pi u)$ for $\gamma = 0.3$ (lower point row), $\gamma = 0.5$ (middle point row), $\gamma = 0.7$ (upper point row) and the upper bound (15) for $\gamma = 0.7$.

Fig. 3. Absolute fractional derivatives in the mean for the Weierstrass functions for $\lambda = 2$ and varying $\gamma$. 
asymptotics as $\lambda \to \infty$ may be determined in the general case:

By using $(\varepsilon_{1,0}^\gamma + \varepsilon_{2,1}^\gamma) \ln \lambda \leq \text{const} \lambda^{-(\beta - \gamma)}$ we get from (15):

$$\tilde{a} := \frac{H(h)}{(\beta - \gamma)} + \frac{2\|h\|}{\gamma} \geq \liminf_{\lambda \to \infty} \left| d^\gamma W^\gamma(x) \ln \lambda \right|$$

$$= \liminf_{\lambda \to \infty} \left( \int_0^1 \int_{\lambda^{-1}}^1 y^{1-\gamma} |h(z + y) - h(z) + \lambda^{-\gamma} h(\lambda z + \lambda y) - \lambda^{-\gamma} h(\lambda z)| \, dy \, dz + R^\gamma_{\delta,1}(x) \ln \lambda \right)$$

$$\geq \int_0^1 \int_0^1 y^{1-\gamma} |h(z + y) - h(z)| \, dy \, dz =: a > 0.$$  

Thus, the absolute fractional derivatives in the mean of the Weierstrass-type functions converge as $a_{\lambda} / \ln \lambda$ to 0 with $a \leq \liminf_{\lambda \to \infty} a_{\lambda} \leq \limsup_{\lambda \to \infty} a_{\lambda} \leq \tilde{a}$.

The (approximated) absolute fractional derivatives in the mean for classical Weierstrass functions with varying parameter $\gamma$ are shown in Fig. 3 for $\lambda = 2$ and in Fig. 4 for $\lambda = 10^{12}$.

References