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Explicit Error Bounds for Spline Interpolation on a Uniform Partition*

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This paper considers the optimality and the evaluation of the constants that appear in the expressions of error bounds for interpolating spline functions over a uniform mesh of the real line when the nodes are uniformly shifted.

1. INTRODUCTION

A n-degree spline s defined over the uniform partition $\pi_h^a = a + \mathbb{Z}h$ of mesh size h of the real line $\mathbb{R}$ is a function $s \in C^{n-1}(\mathbb{R})$ such that s restricted to $[a + lh, a + (l+1)h]$ is an algebraic polynomial of degree at most n for any $l \in \mathbb{Z}$.

To simplify we use the notation $x_{l+t} = a + (l+t)h$. For a function f defined on $\mathbb{R}$ and $t \in \mathbb{R}$, $f_{l+t} = f(x_{l+t})$ and the shift operator is $Ef_t = f_{t+1}$.

Using the polynomials

$$p_n(t, z) = \sum_{i=0}^{n} Q_n(n + t - i)\, z^i \tag{1.1}$$

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for $t \in [0, 1]$ and $z \in \mathbb{R}$, where Q_n is the B-spline of degree n defined on π_1^0, ter Morsche [12], [13] obtained the following linear dependance relationships for a n-degree spline s defined on π_h^a

$$h^k p_n(v, E)\, s_{l+u}^{(k)} - p_{n-k}(u, E)(E-I)^k\, s_{l+v} = 0 \tag{1.2}$$

for any $l \in \mathbb{Z}$, u, $v \in [0, 1]$ and $k = 0, ..., n$.

A spline s is said to be the interpolating spline of f if, for a given $v \in [0, 1]$, we have $s_{l+v} = f_{l+v}$ for all $l \in \mathbb{Z}$.

A function f is said to be of polynomial growth on $\mathbb{R}$ if there exists an integer $\nu \geqslant 0$ such that $f(x) = O(|x|^\nu)$ for $|x| \to +\infty$.

A consequence of (1.2) and the properties of $p_n(t, z)$ is the existence and uniqueness of the n-degree interpolating spline s for any function f of polynomial growth when $p_n(t, -1) \neq 0$ (see [7], [8], [11], [12], and [6]).

Let us consider the following function spaces

$$L^1_{\text{loc}}(\mathbb{R}) = \left\{ f\colon \mathbb{R} \to \mathbb{R} \;\middle|\; \int_a^b |f(x)|\, dx < \infty, \text{ for all interval } [a, b] \subset \mathbb{R} \right\}$$

and

$$AC^{n+1}_{\text{loc}}(\mathbb{R}) = \left\{ f \in C^n(\mathbb{R}) \;\middle|\; \begin{array}{ll} \text{(i)} & f^{(n+1)} \in L^1_{\text{loc}}(\mathbb{R}) \\ \text{(ii)} & \text{For all } [a, b] \subset \mathbb{R}, f^{(n)}(x)|_b^a = \int_a^b f^{(n+1)}(x)\, dx \end{array} \right\}.$$

For any $f \in AC^{n+1}_{\text{loc}}(\mathbb{R})$, using its Taylor expansion and the fact that (1.2) is satisfied for any polynomials of degree at most n, we obtain

$$h^k p_n(v, E)\, f_{l+u}^{(k)} - p_{n-k}(u, E)(E-I)^k\, f_{l+v} = \frac{h^{n+1}}{n!} \int_0^{n+1} K_n^k(u, v, \theta)\, f_{l+\theta}^{(n+1)}\, d\theta \tag{1.3}$$

for any $l \in \mathbb{Z}$, $u \in [0, 1]$, and $k = 0, ..., n$. This is nothing but a consequence of the Peano Kernel Theorem (see also [4] and [6]).

From (1.2) and (1.3) we obtain

$$h^k p_n(v, E)\, e_{l+u}^{(k)} = \frac{h^{n+1}}{n!} \int_0^{n+1} K_n^k(u, v, \theta)\, f_{l+\theta}^{(n+1)}\, d\theta \tag{1.4}$$

where $e = f - s$. Moreover, if $f^{(n+1)}$ is of polynomial growth, we have

$$e_{l+u}^{(k)} = \frac{h^{n+1-k}}{n!}\, p_n(v, E)^{-1} \int_0^{n+1} K_n^k(u, v, \theta)\, f_{l+\theta}^{(n+1)}\, d\theta. \tag{1.5}$$

Finally, since the norm of the operator $p_n(v, E)^{-1}$ on bounded sequences is upper bounded by $1/|p_n(v, -1)|$ (see [3], [5], and [12]), if $f^{(n+1)} \in L^\infty(\mathbb{R})$ we have

$$|e^{(k)}_{l+u}| \leqslant C^k_n(u, v)\, h^{n+1-k}\, \|f^{(n+1)}\|_\infty \tag{1.6}$$

and

$$\|e^{(k)}\|_\infty \leqslant C^k_n(v)\, h^{n+1-k}\, \|f^{(n+1)}\|_\infty \tag{1.7}$$

for $k = 0, ..., n$, where

$$C^k_n(u, v) = \frac{1}{|p_n(v, -1)|} \int_0^{n+1} \frac{|K^k_n(u, v, \theta)|}{n!}\, d\theta, \tag{1.8}$$

$$C^k_n(v) = \sup_{u \in [0, 1]} C^k_n(u, v). \tag{1.9}$$

In this paper we show that $C^0_n(u, v)$ and $C^0_n(v)$ are the best constants in (1.6) and (1.7). We also present explicit expressions for those constants. In the case $k > 0$ we present explicit expressions to bound the constants $C^k_n(u, v)$ and $C^k_n(v)$. These results are presented in sections 4 and 5. In section 2 we present some preliminaries and in section 3 we establish some useful properties of the kernels $K^k_n(u, v, \theta)$.

2. Preliminaries

The B-spline Q_n of degree n with knots 0, 1, ..., $n+1$ can be defined by

$$Q_n(x) = \frac{\nabla^{n+1}(x)^n_+}{n!}$$

where ∇ is the backward difference operator, $(x)^n_+ = x^n \chi_{(0, +\infty)}(x)$ and χ_E is the characteristic function of the set E. It is also equivalent to the formula

$$Q_n(x) = \underbrace{\chi_{(0,1]} * \cdots * \chi_{(0,1]}}_{n+1}(x)$$

where $*$ denotes the convolution operator. Let us remark that

$$Q_n(x) = Q_{n-k} * Q_{k-1}(x)$$

for any $k = 1, ..., n$. Moreover, for any $f \in AC^k(\mathbb{R})$ we have

$$\nabla^k f(v) = Q_{k-1} * f^{(k)}(v). \tag{2.1}$$

The polynomials $p_n(t, z)$ defined by (1.1) for $t \in [0, 1]$ have the following properties (see [12] or [3]): $p_0(t, z) = \chi_{(0, 1]}(t)$ and for $n > 0$:

$p_n(t, z)$ is a polynomial in z of degree n for each $t \in (0, 1]$ and $p_n(0, z)$ is of degree $n-1$;

$$p_n(t, -1) = 0 \quad \text{iff} \quad t = \tau_n = \begin{cases} \frac{1}{2} & \text{for } n \text{ odd}, \\ 0 (\text{or } 1) & \text{for } n \text{ even}; \end{cases} \tag{2.2}$$

$$\frac{\partial^k}{\partial t^k} p_n(t, z) = (z-1)^k p_{n-k}(t, z).$$

We can extend the definition of $p_n(t, z)$ for all $t \in \mathbb{R}$ by the formula

$$p_n(t, z) = \sum_{i=-\infty}^{\infty} Q_n(n + t - i) z^i.$$

It follows that

$$p_n(t+1, z) = z p_n(t, z). \tag{2.3}$$

Moreover $p_n(t, z)$ is a spline of degree n with respect to t and $p_n(t, 1) = 1$.

Finally we have the following useful property

$$p_n(t, z) = Q_{k-1}(t) * p_{n-k}(t, z) z^k \tag{2.4}$$

for $k = 1, ..., n$.

We will also use the Euler splines as defined in [10, p. 152]. The n-degree Euler spline defined on π_1^0, denoted by E_{n+1}, is such that

$$E_1(t) = (-1)^i \quad \text{for} \quad t \in (i, i+1] \text{ and } i \in \mathbb{Z},$$

and for $n > 1$

$$\frac{d}{dt} E_n(t) = 2E_{n-1}(t),$$

$$E_n(t+1) = -E_n(t),$$

$$E_n(1-t) = (-1)^{n+1} E_n(t),$$

$$\operatorname{sign}(E_{n+2}(t)) = -\operatorname{sign}(E_n(t)).$$

From the definition, it follows that

$$\max_{0 \leq t \leq 1} |E_{n+1}(t)| = |E_{n+1}(\tau_n^*)|$$

where

$$\tau_n^* = \begin{cases} \frac{1}{2} & \text{for } n \text{ even}, \\ 0 (\text{or } 1) & \text{for } n \text{ odd}. \end{cases}$$

We also have $|\sin(\pi t)| \leqslant |E_1(t)| \leqslant 1$, and if we set for $n \geqslant 2$

$$g_n(t) = \begin{cases} \cos(\pi t) & \text{for } n \text{ even}, \\ \sin(\pi t) & \text{for } n \text{ odd}, \end{cases}$$

then

$$|g_n(t)| \left(\frac{2}{\pi}\right)^{n-1} \leqslant |E_n(t)| \leqslant |g_n(t)| \left(\frac{2}{\pi}\right)^{n-2}. \tag{2.5}$$

Finally, $p_n(t, -1) = (-1)^n E_{n+1}(t)$ and it follows that

$$\max_{0 \leqslant t \leqslant 1} |p_n(t, -1)| = |p_n(\tau_n^*, -1)| \geqslant \left(\frac{2}{\pi}\right)^n. \tag{2.6}$$

3. Analysis of the Peano Kernels

The Peano kernels in (1.3) are defined by

$$\frac{K_n^k(u, v, \theta)}{n!} = p_n(v, E) \frac{(u-\theta)_+^{n-k}}{(n-k)!} - p_{n-k}(u, E)(E-I)^k \frac{(v-\theta)_+^n}{n!} \tag{3.1}$$

for $k = 0, ..., n$. In the next five lemmas we present some useful properties of these kernels.

The first two lemmas are consequences of the consistency relations (1.2).

Lemma 1. *$K_n^k(u, v, \theta) = 0$ whenever $\theta \notin [\min\{u, v\}, n + \max\{u, v\}]$.*

Proof. Let us define the polynomial $g_\theta(x) = (x-\theta)^n/n!$. If $\theta < \min\{u, v\}$ then

$$\begin{aligned} K_n^k(u, v, \theta) &= p_n(v, E) \frac{(u-\theta)_+^{n-k}}{(n-k)!} - p_{n-k}(u, E)(E-I)^k \frac{(v-\theta)_+^n}{n!} \\ &= p_n(v, E) \frac{(u-\theta)^{n-k}}{(n-k)!} - p_{n-k}(u, E)(E-I)^k \frac{(v-\theta)^n}{n!} \\ &= p_n(v, E)\, g_\theta^{(k)}(u) - p_{n-k}(u, E)(E-I)^k g_\theta(v) = 0. \end{aligned}$$

For $\theta > n + \max\{u, v\}$, we obtain directly that $K_n^k(u, v, \theta) = 0$. ∎

LEMMA 2. *$K_n^k(u, v, i) = 0$ for any integer i.*

Proof. Let us define the function $h_\theta(x) = (x-\theta)_+^n/n!$. Whenever θ is an integer, the function $h_\theta(x)$ is a spline on π_1^0. Thus, by (1.2), we obtain

$$0 = p_n(v, E)\, h_\theta^{(k)}(u) - p_{n-k}(u, E)(E-I)^k\, h_\theta(v)$$
$$= \frac{K_n^k(u, v, \theta)}{n!}. \quad ∎$$

LEMMA 3. *The kernel $K_n^0(u, v, \theta)$ has the following properties.*

(a) *$K_n^0(v, v, \theta) = 0$;*

(b) *$K_n^0(u, v, \theta)$ has no sign change for $\theta \in (i, i+1)$, and has simple zeros at $\theta = i$ for $i = 1, ..., n$;*

(c) *if $\theta \in [n, n + \max\{u, v\}]$ then sign $(K_n^0(u, v, \theta)) = sign(u - v)$.*

Proof. (a) Obvious.

(b) Let $u < v$. The kernel $K_n^0(u, v, \theta)$ is a n-degree spline with respect to θ defined on the partition $\{i+u, i+v \mid i = 0, ..., n\}$ and support $[u, n+v]$. For $\theta \in (u, v)$ we have $K_n^0(u, v, \theta) = -Q_n(n+v)(u-\theta)^n \neq 0$ and for $\theta \in (n+u, n+v)$ we have $K_n^0(u, v, \theta) = -Q_n(u)(n+v-\theta)^n \neq 0$. From [9, p. 155] the intervals $(-\infty, u)$ and $(n+v, \infty)$ are two zeros of multiplicity $n+1$. It follows from lemma 2 that the number of zeros of $K_n^0(u, v, \theta)$ is at least $3n+2$. But from [9, p. 160–161] the number of zeros of $K_n^0(u, v, \theta)$ is at most $3n+2$. Hence the result follows. A similar proof holds for $u > v$.

(c) If $u < v$ and $\theta \in (n+u, n+v)$ we have $K_n^0(u, v, \theta) = -Q_n(u)(n+v-\theta) < 0$, and if $u > v$ and $\theta \in (n+v, n+u)$ we have $K_n^0(u, v, \theta) = Q_n(v)(n+u-\theta) > 0$. The result follows. ∎

The next lemma is a direct consequence of (2.3).

LEMMA 4. *For any $i \in \mathbb{Z}$ we have*

$$K_n^k(u+i, v, \theta) = K_n^k(u, v+i, \theta) = K_n^k(u, v, \theta - i). \quad ∎$$

The next lemma relates $K_n^k(u, v, \theta)$ to $K_{n-k}^0(u, v, \theta)$ using the convolution operator.

LEMMA 5. *For $k = 1, ..., n$ we have*

$$\frac{K_n^k(u, v, \theta)}{n!} = Q_{k-1}(v) * \frac{K_{n-k}^0(u, v, \theta - k)}{(n-k)!}.$$

Proof. From (2.4) we have $p_n(v, E) = Q_{k-1}(v) * p_{n-k}(v, E)\, E^k$. Also $(E-I)^k = \nabla^k E^k = E^k \nabla^k$, and from (2.1), we have

$$\nabla^k \frac{(v-\theta)_+^n}{n!} = Q_{k-1}(v) * \frac{(v-\theta)_+^{n-k}}{(n-k)!}.$$

Finally, using the lemma 4 we obtain the result. ∎

4. The Case $k=0$

This section contains the proof of the optimality of the constants $C_n^0(u, v)$ and $C_n^0(v)$ defined by (1.8) and (1.9).

Let us consider the following class of functions

$$\mathscr{C} = \{ f \in AC_{\text{loc}}^{n+1}(\mathbb{R}) \mid f^{(n+1)} \in L^\infty(\mathbb{R}) \}.$$

The results of this section are based on the following lemma.

Lemma 6. *Let $v \in [0, 1]$ such that $p_n(v, -1) \neq 0$. Then*

$$\int_0^{n+1} \frac{|K_n^0(u, v, \theta)|}{n!}\, d\theta$$
$$= \frac{\operatorname{sign}(u-v)}{2^{n+1}} (E_{n+1}(v)\, E_{n+2}(u) - E_{n+1}(u)\, E_{n+2}(v)) \tag{4.1}$$

for any $u \in [0, 1]$.

Proof. Let $f(x) = (1/2^{n+1})\, E_{n+2}(x/h)$. Then $f^{(n+1)}(x) = (1/h^{n+1})\, E_1(x/h)$, $\|f^{(n+1)}\|_\infty = 1/h^{n+1}$, and $f \in \mathscr{C}$. Its n-degree interpolating spline s such that $f_{l+v} = s_{l+v}$ is $s(x) = (E_{n+2}(v)/2^{n+1} E_{n+1}(v))\, E_{n+1}(x/h)$. From (1.4) we have

$$p_n(v, E)\, e_{l+u} = \int_0^{n+1} \frac{K_n^0(u, v, \theta)}{n!} E_1(l+\theta)\, d\theta.$$

Using the definition of E_1 and the properties of the zeros of $K_n^0(u, v, \theta)$ given by the lemma 3, we have

$$\int_0^{n+1} \frac{K_n^0(u, v, \theta)}{n!} E_1(l+\theta)\, d\theta = (-1)^{l+n} \operatorname{sign}(u-v) \int_0^{n+1} \frac{|K_n^0(u, v, \theta)|}{n!}\, d\theta.$$

Moreover, we also have $p_n(v, E)\, e_{l+u} = p_n(v, -1)\, e_{l+u}$ because $e_{l+i+u} = (-1)^i e_{l+u}$. But

$$e_u = \frac{1}{2^{n+1}} \left(E_{n+2}(u) - E_{n+1}(u) \frac{E_{n+2}(v)}{E_{n+1}(v)} \right)$$

and $p_n(v, -1) = (-1)^n E_{n+1}(v)$. Hence the result follows. ▮

THEOREM 1. *Let $v \in [0, 1]$ be such that $p_n(v, -1) \neq 0$. If $f \in \mathscr{C}$, s is the n-degree interpolating spline of f such that $f_{l+v} = s_{l+v}$ $(l \in \mathbb{Z})$ and $e = f - s$, then*

$$C_n^0(u, v) = \max_{f \in \mathscr{C}} \frac{\sup_{l \in \mathbb{Z}} |e_{l+u}|}{h^{n+1} \|f^{(n+1)}\|_\infty}. \tag{4.2}$$

and

$$C_n^0(u, v) = \frac{1}{2^{n+1}} \left| E_{n+2}(u) - E_{n+1}(u) \frac{E_{n+2}(v)}{E_{n+1}(v)} \right|. \tag{4.3}$$

Moreover

$$C_n^0(v) = \max_{f \in \mathscr{C}} \frac{\|e\|_\infty}{h^{n+1} \|f^{(n+1)}\|_\infty} \tag{4.4}$$

and

$$C_n^0(v) = \frac{1}{2^{n+1}} \max_{u \in [0, 1]} \left(|E_{n+2}(u)| + |E_{n+1}(u)| \frac{|E_{n+2}(v)|}{|E_{n+1}(v)|} \right). \tag{4.5}$$

Proof. Equations (4.2) and (4.3) are direct consequences of the proof of the lemma 6. To obtain (4.5) we first observe that the righthand side of (4.3) is a continuous function of $u \in [0, 1]$. It remains to observe that the maximum is at a value u such that

$$\text{sign}(E_{n+2}(u)) = -\text{sign}\left(E_{n+1}(u) \frac{E_{n+2}(v)}{E_{n+1}(v)} \right).$$

This fact comes from the properties of the Euler splines. Finally (4.4) is a consequence of (4.2) and (4.5). ▮

The next result indicates the best choice of v for the interpolating problem with equispaced data on a uniform partition.

THEOREM 2. *We have*

$$\min_{0 \leqslant v \leqslant 1} C_n^0(v) = C_n^0(\tau_n^*)$$

and

$$C_n^0(\tau_n^*) = \frac{|E_{n+2}(\tau_{n+1}^*)|}{2^{n+1}} = \frac{|p_{n+1}(\tau_{n+1}^*, -1)|}{2^{n+1}}. \tag{4.6}$$

Proof. From (4.5) we have

$$C_n^0(v) \geqslant \frac{1}{2^{n+1}} \max_{u \in [0,1]} |E_{n+2}(u)| = \frac{|E_{n+2}(\tau_{n+1}^*)|}{2^{n+1}} \tag{4.7}$$

for all $v \in [0, 1]$ such that $p_n(v, -1) \neq 0$. But the righthand side of (4.7) is exactly the value of $C_n^0(\tau_n^*)$ because $\tau_n^* = \tau_{n+1}$ and $E_{n+2}(\tau_{n+1}) = 0$. ∎

Results similar to (4.6) appear elsewhere, for example [9, Theorem 5, p. 291], [2, theorem 3, p. 47], and [14].

EXAMPLE 1. If $\bar{E}_n$ is the Euler polynomial of degree n, we have

$$p_n(t, -1) = \frac{(-2)^n}{n!} \bar{E}_n(t)$$

(see [1, pp. 804–805]). Also the nth Euler number is given by $\bar{E}_n = 2^n \bar{E}_n(\frac{1}{2})$ and $\bar{E}_n(0) = -2((2^{n+1} - 1)/(n+1)) \bar{B}_{n+1}$ where $\bar{B}_n$ is the nth Bernoulli number. It follows that

(a) for n odd, $\tau_n^* = 0$ (or 1), $\tau_{n+1}^* = \frac{1}{2}$ and

$$C_n^0(0) = \frac{|\bar{E}_{n+1}|}{2^{n+1}(n+1)!},$$

(b) for n even, $\tau_n^* = \frac{1}{2}$, $\tau_{n+1}^* = 0$ (or 1) and

$$C_n^0\left(\frac{1}{2}\right) = 2\frac{2^{n+2} - 1}{(n+2)!} |\bar{B}_{n+2}|.$$

Remark 1. The results of this section can be applied directly to periodic functions on an interval $[a, b]$ when $b - a = Nh$ and N is even. For N odd we can prove the following asymptotic result based on the absolute convergence of the Laurent series $p_n(v, z)^{-1} = \sum_{j=-\infty}^{\infty} \alpha_j(v) z^j$.

THEOREM 3. *Let $v \in [0, 1]$ be such that $p_n(v, -1) \neq 0$. For any $\varepsilon > 0$ there exists $N(\varepsilon) > 0$ such that for each N odd $\geqslant N(\varepsilon)$ there exists a periodic function $f \in \mathscr{C}$ of period $b - a$ such that*

$$C_n^0(u, v) - \varepsilon \leqslant \frac{\sup_{l \in \mathbb{Z}} |e_{l+u}|}{h^{n+1} \|f^{(n+1)}\|_\infty} \leqslant C_n^0(u, v). \quad \blacksquare$$

5. THE CASE $k > 0$

In this section we obtain bounds for the constants $C_n^k(u, v)$ and $C_n^k(v)$. These results are based on the following lemma.

LEMMA 7. *For $k = 1, ..., n$ we have*

$$\int_0^{n+1} \frac{|K_n^k(u, v, \theta)|}{n!} d\theta \leqslant \int_{\mathbb{R}} \int_0^1 \frac{|K_{n-k}^0(u, \mu, \theta)|}{(n-k)!} d\mu\, d\theta.$$

Proof. From lemma 5 and lemma 4 we have

$$\begin{aligned}
&\int_0^{n+1} \frac{|K_n^k(u, v, \theta)|}{n!} d\theta \\
&\leqslant \int_{\mathbb{R}} \int_{\mathbb{R}} Q_{k-1}(v-\mu) \frac{|K_{n-k}^0(u, \mu, \theta - k)|}{(n-k)!} d\mu\, d\theta \\
&= \sum_{i=-\infty}^{\infty} \int_{\mathbb{R}} \int_0^1 Q_{k-1}(v - i - \mu) \frac{|K_{n-k}^0(u, \mu, \theta - k - i)|}{(n-k)!} d\mu\, d\theta \\
&= \sum_{i=-\infty}^{\infty} \int_0^1 \int_{\mathbb{R}} Q_{k-1}(v - i - \mu) \frac{|K_{n-k}^0(u, \mu, \theta)|}{(n-k)!} d\theta\, d\mu \\
&= \int_{\mathbb{R}} \int_0^1 \sum_{i=-\infty}^{\infty} Q_{k-1}(v - i - \mu) \frac{|K_{n-k}^0(u, \mu, \theta)|}{(n-k)!} d\theta\, d\mu
\end{aligned}$$

and the result follows because $\sum_{i=-\infty}^{\infty} Q_{k-1}(v - i - \mu) = 1$. $\blacksquare$

By a similar method we can prove the following result.

LEMMA 8. *Let $v \in [0, 1]$ be such that $p_n(v, -1) \neq 0$. If $f \in AC_{\text{loc}}^{n+1}(\mathbb{R})$ and f is of polynomial growth, then*

$$p_n(v, E)\, e_{l+u}^{(k)} = h^{n-k+1} \int_{\mathbb{R}} \int_0^1 \frac{K_{n-k}^0(u, \mu, \theta)}{(n-k)!} E p_{k-1}(v - \mu, E)\, f_{l+\theta}^{(n+1)}\, d\mu\, d\theta$$

for $k = 1, ..., n$.

A direct consequence of lemma 6 and the properties of the Euler splines is the following result.

LEMMA 9.

$$\int_{\mathbb{R}}\int_0^1 \frac{|K_n^0(u,\mu,\theta)|}{n!}\,d\mu\,d\theta=\frac{E_{n+2}^2(u)-E_{n+1}(u)\,E_{n+3}(u)}{2^{n+1}}. \quad \blacksquare$$

THEOREM 4. *Let $v\in[0,1]$ be such that $p_n(v,-1)\neq 0$. If we set*

$$D_n^k(u,v)=\frac{1}{|p_n(v,-1)|}\int_{\mathbb{R}}\int_0^1 \frac{|K_{n-k}^0(u,\mu,\theta)|}{(n-k)!}\,d\mu\,d\theta$$

then

$$C_n^k(u,v)\leqslant D_n^k(u,v) \tag{5.1}$$

and

$$D_n^k(u,v)=\frac{1}{2^{n-k+1}\,|E_{n+1}(v)|}\left(E_{n-k+2}^2(u)-E_{n-k+1}(u)\,E_{n-k+3}(u)\right) \tag{5.2}$$

for $k=1,\ldots,n$.

Proof. Equation (5.1) is a direct consequence of lemma 7 and (5.2) comes from lemma 9. $\blacksquare$

LEMMA 10. *For any $u\in[0,1]$ and $n\geqslant 0$ we have*

$$\left(\frac{2}{\pi}\right)^{2n+2}\leqslant E_{n+2}^2(u)-E_{n+1}(u)\,E_{n+3}(u)\leqslant\left(\frac{2}{\pi}\right)^{2n}. \tag{5.3}$$

Proof. We consider two cases. For $n=0$, $E_1(u)=1$, $E_2(u)=2u-1$, and $E_3(u)=2u(u-1)$. Then

$$E_2^2(u)-E_1(u)\,E_3(u)=1+2u(u-1)$$

and the result follows for $n=0$. For $n>0$, we first observe that

$$E_{n+2}^2(u)-E_{n+1}(u)\,E_{n+3}(u)=E_{n+2}^2(u)+|E_{n+1}(u)|\,|E_{n+3}(u)|.$$

We obtain the result using (2.5). $\blacksquare$

If we set

$$D_n^k(v)=\sup_{0\leqslant u\leqslant 1} D_n^k(u,v)$$

then, from (5.2) and (5.3)

$$C_n^k(u, v) \leqslant D_n^k(u, v) \leqslant \frac{(\pi^2/2)^{k-n}}{2\,|p_n(v, -1)|}.$$

Finally

$$D_n^k(\tau_n^*) \leqslant \frac{(\pi^2/2)^{k-n}}{2\,|p_n(\tau_n^*, -1)|}. \tag{5.4}$$

Remark 2. We can show that

$$\max_{u \in [0,1]} (E_{n+2}^2(u) - E_{n+1}(u)\, E_{n+3}(u)) = E_{n+2}^2(1) - E_{n+1}(1)\, E_{n+3}(1),$$

for $0 \leqslant n \leqslant 3$, but it is an open problem for $n > 3$. From this result we can obtain Table I.

EXAMPLE 2. As in section 4, we have

(a) for n odd, $\tau_n^* = 0$ (or 1), $p_n(\tau_n^*, -1) = (-2)^{n+1}((2^{n+1}-1)/(n+1)!)\,\bar{B}_{n+1}$, and

$$C_n^k(0) \leqslant D_n^k(0) \leqslant \frac{(\pi^2/2)^{k-n}\,(n+1)!}{2^{n+2}(2^{n+1}-1)\,|\bar{B}_{n+1}|},$$

(b) for n even, $\tau_n^* = \frac{1}{2}$, $p_n(\tau_n^*, -1) = \bar{E}_n/n!$, and

$$C_n^k\left(\frac{1}{2}\right) \leqslant D_n^k\left(\frac{1}{2}\right) \leqslant \frac{(\pi^2/2)^{k-n}\,n!}{2\,|\bar{E}_n|}.$$

TABLE I

Some Values of $D_n^k(\tau_n^*)$

$k \mid n$	2	3	4	5
1	1/6	1/24	1/75	...
2	1	1/4	3/45	1/48
3		3/2	2/5	15/144
4			12/5	15/24
5				15/4

6. Conclusion

We have obtained expressions for the optimal bounds when $k=0$. For $k>0$, we have obtained closed expressions to bound the constants $C_n^k(u, v)$ and $C_n^k(v)$. If we put together (4.6) and (5.4), and use (2.5) and (2.6) we obtain the following result.

THEOREM 5. *Let* $v=\tau_n^*$. *If* $f \in \mathscr{C}$, *s is the n-degree interpolating spline of f such that* $f_{l+v}=s_{l+v}$ *for all* $l \in \mathbb{Z}$, *and* $e=f-s$, *then*

$$\|e^{(k)}\|_\infty \leqslant \frac{(\pi^2/2)^k}{2\pi^n} h^{n+1-k} \|f^{(n+1)}\|_\infty$$

for $k=0, ..., n$.

A similar result has already been presented in [9, Theorem 6, p. 293] for $k=0$ and n odd.

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