On Vinogradov's Constant in Goldbach's Ternary Problem

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This paper shows that under the assumption of the Generalized Riemann Hypothesis, every odd integer greater than $10^{20}$ can be written as a sum of three primes. Using the computational results it may be possible to check the Ternary Conjecture for all odd numbers.

0. INTRODUCTION

Goldbach's Ternary Conjecture (GTC) asserts that any odd number ($\geq 9$) can be expressed as a sum of three prime numbers. Hardy and Littlewood showed that the Generalized Riemann Hypothesis (GRH) implies the existence of a constant $N_0 > 0$ such that GTC holds for any odd number $N \geq N_0$. Applying his method of trigonometric sums, Vinogradov proved—without assuming the GRH—the existence of such a constant $N_0$. The constant $N_0$ is called Vinogradov's constant. There were several results on the magnitude of this constant. For example, it was shown in 1989 by Wang and Chen that the magnitude of this constant is about $10^{43,000}$. However, no computations would be able to fill the gap. It is natural to ask what can be done assuming the GRH. In 1926 Lucke [5] showed in his unpublished doctoral thesis that by assuming the GRH any odd number greater than $10^{32}$ can be written as a sum of three primes. We show that this constant can be reduced to $10^{20}$. We hope that with additional effort it is possible to cover all odd numbers and claim the complete solution of Goldbach's Ternary Problem under the assumption of GRH.

Fix $N \geq 9$. We are interested in the number of triples $(p_1, p_2, p_3)$ of prime numbers which satisfy the equation

$$N = p_1 + p_2 + p_3.$$  \hspace{1cm} (0.1)

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Following [2] we introduce the function
\[
J(N) = \sum_{p_1 + p_2 + p_3 = N} \ln(p_1) \ln(p_2) \ln(p_3),
\]
where the sum ranges over all triples of primes \((\geq 2)\), and \(\ln(x)\) denotes the natural logarithm with base \(e\). If \(J(N) > 0\) then there is at least one solution of (0.1). Here by \(A(n)\) (\(n\) is always a natural number) we denote Mangold’s function: \(A(n) = \ln(p)\), if \(n = p^k\) (\(p\) is prime), and \(A(n) = 0\) otherwise. For any real number \(x\) set
\[
S(x) = \sum_{n \geq 1} A(n) e^{-2\pi m e^{-x/N}}.
\]
Then we have
\[
S(x) = \sum_{p \geq 1} \ln(p) e^{-2\pi m e^{-x/N}} + 0 N^{0.5} \ln^3(N),
\]
where \(|\theta| \leq 1\). Clearly, for any integer \(m\)
\[
\int_0^1 e^{2\pi m x} dx = \begin{cases} 1, & \text{if } m = 0, \\ 0, & \text{if } m \neq 0. \end{cases}
\]
Changing the order of summation and integration (see [2]), for some new real \(\theta (|\theta| \leq 1)\) we obtain
\[
J(N) = e \int_0^1 S^1(x) e^{2\pi m N} dx + 0 N^{1.5} \ln^3(N)
= e \int_{-w}^{1-w} S^1(x) e^{2\pi m N} dx + 0 N^{1.5} \ln^3(N),
\]
where \(w = w(N)\) is a small number defined later. We will express \(J(N)\) as a sum of the leading term and the remainder. Estimating the remainder from above, we will show that it is less than the leading term when \(N \geq 10^{20}\). We then conclude that \(J(N) > 0\).

Following Linnik and Vinogradov, we subdivide the interval \([-w, 1-w]\) into the disjoint union of subsets \(E_1, E_2\). We split the integral \(J(N)\) into two integrals, over \(E_1\) and \(E_2\), denoted respectively by \(J_1(N)\) and \(J_2(N)\). Our strategy is to obtain the leading term from \(J_1(N)\) and to estimate the other two from above. They will contribute to the remainder term. We consider the integral \(J_1(N)\) in Lemmas 3.1, 3.2, and 3.3 (Section 3). We follow [2] and use the circular method of Hardy and Littlewood. We use Lemma 2.1 of Section 2 to bound \(J_2(N)\) from above. This lemma estimates \(S(x)\) when \(x \in E_1 \cup E_2\). We prove this lemma on using the
Riemann–Hadamard method which involves summation over the zeroes of $L$-functions. Our computations are similar but simpler than those of [1] and [2]. This is due to our usage of GRH, and our choice of constants and the subsets $E_1^1, E_1^2, E_2$. In particular, we estimate all constants which appear in the remainder terms (in all lemmas and theorems).

The first part (Section 1) of this paper has some auxiliary computations used in Sections 2 and 3. In particular, the key result of that section is Proposition 1.5, which gives a formula (under the assumption of GRH) for $\pi(x; k, l)$, the number of primes less than $x$ congruent to $l$ modulo $k$.

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1. SOME LEMMAS

Recall that $L(s, \chi) = \sum_{n=1}^{\infty} (\chi(n)/n^s)$. For $\Re(s) > 1$, we know (see [2, Chap. VIII, Section 2]) that

$$\frac{L(s, \chi)}{L(s, \chi)} = \sum_{n=1}^{\infty} A(n) \chi(n)n^{-s}.$$ 

The proof of the following two lemmas follows [1, Chap. VIII, Section 5], but here we estimate all constants explicitly from above. Similar results can be found in [3].

**Lemma 1.1.** Under the assumption of GRH, the number of zeroes $p = \beta + \gamma$ of $L(s, \chi)$ ($\chi$ is a primitive character modulo $q$) is less than

$$2.17(1 + 0.5 \ln(qT)), \quad \text{if} \quad T - 1 \leq \gamma < T + 1, \quad \text{and} \quad T \geq 2,$$

$$1.09(1 + 0.5 \ln(q)), \quad \text{if} \quad 0 \leq \gamma < 1.$$

**Proof.** This lemma follows from the following:

**Lemma 1.2.** Let $\rho_n = \beta_n + \gamma_n$ be all the nontrivial zeroes of $L(s, \chi)$, where $\chi$ is a primitive character mod $q$, $T > 2$. Assuming GRH,

$$\sum_{n=1}^{\infty} \frac{1.5}{2 \gamma + (T - \gamma)_n} < 1 + 0.5 \ln(qT).$$
Proof. It is known (see [1, Chap. VIII, Section 3]) that

\[ \zeta(s, \chi) = (\pi/q)^{-s+1/2} \Gamma \left( \frac{s+d}{2} \right) L(s, \chi), \quad \text{where } d = 0 \text{ or } d = -1, \]

(1.1)

\[ \zeta(s, \chi) = e^{s+R_\gamma} \prod_{n=1}^{\infty} (1-s/n) e^{\pi i \rho_n}, \quad \text{where } A = A(\chi), B = B(\chi), \]

(1.2)

\[ \zeta(1-s, \chi) = \frac{\sqrt{\gamma}}{\pi(\gamma)} \zeta(s, \chi). \]

(1.3)

Taking the logarithmic derivative of (1.1) and (1.2), and equating their real parts, we obtain

\[ \sum_{n=1}^{\infty} \Re \frac{1}{s - \rho_n} = \Re \frac{L'(s, \chi)}{L(s, \chi)} + \frac{1}{2} \ln \left( \frac{\pi}{\gamma} \right) - \frac{\gamma}{2} - \Re \frac{1}{s+d} \]

\[ - \sum_{n=1}^{\infty} \Re \left( \frac{1}{s+d+2n} - \frac{1}{2n} \right) - \Re B(\chi) + \sum_{n=1}^{\infty} \Re \frac{1}{\rho_n}. \]

(1.4)

Lemma 12 of [1, Chap. VIII, Section 4] asserts that

\[ \Re B(\chi) + \sum_{n=1}^{\infty} \Re \frac{1}{\rho_n} = 0. \]

So the last two terms on the right-hand side of (1.4) cancel each other.

Next, for \( s = 2 + iT \) (\( T \geq 2 \)) consider

\[ \Re \sum_{n=1}^{\infty} \left( \frac{1}{2n} - \frac{1}{s+2n} \right). \]

We have

\[ \Re \sum_{n=1}^{\infty} \left( \frac{1}{2n} - \frac{1}{s+2n} \right) = \frac{2(2+2n)+T^2}{2n(2+2n)^2+T^2} \]

\[ \leq \begin{cases} \frac{1}{2n}, & \text{if } n \leq T, \\ \frac{T}{2n}, & \text{if } n > T. \end{cases} \]
So, we obtain (the same estimate is true for any \( s \) with \( \Re s > 0 \)) that
\[
\Re \sum_{n=1}^{\infty} \left( \frac{1}{2n} - \frac{1}{s+2n} \right) < \sum_{n=1}^{T} \frac{1}{2n} + \sum_{n>T} \frac{T}{2n^2} < 1 + \frac{1}{2} \ln(T). \tag{1.5}
\]

Note that when \( T = \Im(s) = 0 \), we have
\[
\Re \sum_{n=1}^{\infty} \left( \frac{1}{2n} - \frac{1}{s+2n} \right) < 1. \tag{1.6}
\]

Finally, we have
\[
\Re \frac{L(s, \chi)}{L(s, \overline{\chi})} \leq \sum_{n=1}^{\infty} \frac{A(n) n^{-2}}{2} < 0.61, \tag{1.7}
\]
\[
\Re \frac{1}{s-\rho_n} = \frac{2-\beta_n}{(2-\beta_n) - (T-\gamma_n)^2} = \frac{1.5}{2.25 + (T-\gamma_n)^2}. \tag{1.8}
\]

Applying (1.5), (1.7), and (1.8) to (1.4), we obtain
\[
0 < \sum_{n=1}^{\infty} \frac{1.5}{2.25 + (T-\gamma_n)^2}
< 0.61 + \frac{1}{2} \ln(q) - \frac{1}{2} \ln(\pi) - \frac{\gamma}{2} + 1 + \frac{1}{2} \ln(T) < 1 + 0.5 \ln(qT).
\]

If \( T = 0 \), then we use (1.6) instead of (1.5),
\[
\sum_{n=1}^{\infty} \frac{1.5}{2.25 + \gamma_n^2} < 1 + 0.5 \ln(q),
\]
which proves the lemma.

Now we return to the proof of Lemma 1.1. Let \( N \) be the number of zeroes of \( L(s, \chi) \) with \( T-1 \leq \gamma < T+1 \). Then \( (T-\gamma)^2 \leq 1 \) and
\[
\frac{1.5N}{2.25 + 1} \leq \sum_{n=1}^{\infty} \frac{1.5}{2.25 + (T-\gamma_n)^2} < 1 + 0.5 \ln(kT).
\]
The second estimate is obtained on setting \( T = 0 \).

**Lemma 1.3.** Let \( \chi \) be a primitive character \( \bmod k \), where \( k \leq x^{0.37} \). Assuming GRH, the following holds \( (x \geq 10^9) \):
\[
|\psi(x, \chi)| \leq 0.83 \sqrt{x \ln^2(x)}.
\]
Proof. Lemma 1 of [3] asserts that for any $x > 10^{7.8}$ and any primitive character $\chi \mod q$, $q < x$, we have

$$\psi(x, \chi) = \sum_{n \leq x} A(n) \chi(n) = - \sum_{|\text{Im}(\rho)| < T} \frac{x^\rho}{\rho} + 40x^{0.25} \ln^2(x),$$

where $|\theta| \leq 1$, $|T - x^{0.75}| \leq 1$, and $\rho$ is an arbitrary nontrivial zero of $L(s, \chi)$. Applying Lemma 1.1, we have ($\gamma = \text{Im}(\rho)$)

$$\sum_{|\text{Im}(\rho)| < T} \frac{1}{|\rho|} < 2.17(1 + 0.5 \ln(q)) + 2 \sum_{1 < \gamma < T} \frac{1}{\gamma}$$

$$< 2.17(1 + 0.5 \ln(q)) + 2 \sum_{t=1}^{T/2} \frac{2.17(1 + 0.5 \ln(2qt))}{2t - 1}$$

$$< 2.17(1 + 0.5 \ln(q)) + 2.17(1 + 0.5 \ln(2q)) \sum_{t=1}^{T/2} \frac{1}{t - 0.5}$$

$$+ 1.09 \sum_{t=1}^{T/2} \ln(t) \frac{1}{t - 0.5}.$$  (1.9)

Estimating the sums by the appropriate integrals, we find

$$\sum_{t=1}^{T} \frac{1}{t - 0.5} < 2.3 + \ln(T) \quad \text{and}$$

$$\sum_{t=1}^{T} \frac{\ln(t)}{t - 0.5} < 0.5 \ln^2(T) + 0.16 \ln(T) + 0.6.$$  

Hence (1.9) is less than

$$2.17(1 + 0.5 \ln(q)) + 2.17(1 + 0.5 \ln(2q))(1.7 + \ln(T))$$

$$+ 1.09(0.5 \ln^2(T) - 0.53 \ln(T) + 0.6).$$

Using that $x > 10^9$, $T = x^{0.75}$, and $q < x^{0.37}$, this is less than 0.83 $\sqrt{x \ln^2(x)}$. 

**Proposition 1.4. Assuming GRH, we have** ($x \geq 10^9$)

$$\pi(x; q, l) = \text{Li}(x) \frac{l}{\phi(q)} + 1.3\theta_1 \sqrt{x \ln(x)} + \theta_2 \cdot 10^3, \quad |\theta_1| \leq 1, \quad |\theta_2| \leq 1.$$
Proof. Note that the number of all \( n \leq x, n \equiv l (\mod q) \), such that \( n = p^k \) where \( k \geq 2 \) is bounded by \( \sqrt{x} \ln(x)/\phi(q) \). Thus we have

\[
\pi(x; q, l) = \sum_{n \equiv l (\mod q)} 1 = \sum_{n \equiv l (\mod q)} \frac{A(n)}{\ln(n)} - \theta_1 \frac{\sqrt{x} \ln(x)}{\phi(q)} , \quad 0 < \theta_1 < 1.
\]

Applying Abel's transformation [1, Chap. I, Section 4], we obtain \((0 < \theta_1 < 1)\)

\[
\pi(x; q, l) = \psi(x; q, l) + \frac{\psi(x; q, l)}{\ln(x)} - \theta_1 \frac{\sqrt{x} \ln(x)}{\phi(q)} ,
\]

(1.10)

where

\[
\psi(x; q, l) = \sum_{n \leq x, \quad n = l (\mod q)} A(n) .
\]

Let \( \chi \) be an arbitrary character \( \mod q \). Using the orthogonality property of characters we have

\[
\psi(x; q, l) = \frac{1}{\phi(q)} \sum_{\chi \mod q} \psi(x; l) \overline{\chi}(l) = \frac{1}{\phi(q)} \sum_{n \leq x, \quad (n, q) = 1} A(n) + \frac{1}{\phi(q)} \sum_{\chi \neq \chi_0} \psi(x; l) \overline{\chi}(l) .
\]

Note that

\[
\left| \psi(x) - \sum_{(n, q) = 1} A(n) \right| \leq \left| \sum_{(n, q) > 1} A(n) \right| \leq \ln^2(x) .
\]

Moreover, if \( \chi_1 \) is a primitive character \( \mod q, q_1 \mid q \) generated by character \( \chi \), then

\[
|\psi(x; \chi) - \psi(x, \chi_1)| \leq \sum_{(n, q) > 1} A(n) \leq \ln^2(x) .
\]

Lemma 1.3 asserts that (for primitive character \( \chi \)) \(|\psi(x, \chi)| < 0.83 \sqrt{x} \ln^2(x)\). So we obtain (for \( x \geq 10^6\))

\[
\psi(x; q, l) = \frac{\psi(x)}{\phi(q)} + c \sqrt{x} \ln^2(x), \quad |c| \leq 0.84 .
\]
For any $x > 75$, it is shown in [4] that (under the assumption of the Riemann Hypothesis)

$$\psi(x) = \sum_{n \leq x} A(n) = x + \frac{\theta}{8\pi} \sqrt{x \ln^2(x)},$$

where $|\theta| \leq 1$. Finally,

$$\psi(x; q, l) = \frac{x}{\phi(q)} + c \sqrt{x \ln^2(x)}, \quad |c| \leq 0.85.$$

We substitute this into (1.10) and use that

$$\text{Li}(x) = \int_2^x \frac{du}{\ln(u)} = \frac{x}{\ln(2)} - \frac{2}{\ln(2)} + \int_2^x \frac{du}{\ln^2(u)}.$$

Thus (1.10) becomes

$$\pi(x; q, l) = \frac{\text{Li}(x)}{\phi(k)} + c \sqrt{x \ln(x)} + R,$$

where $|c| \leq 0.85 + 1/q$. Set $x_0 = 10^9$. Using the trivial estimate $\psi(t; q, l) \leq 2t$ when $t < x_0$, we obtain

$$|R| \leq 2 \int_2^{x_0} \frac{du}{\ln^2(u)} + \frac{1}{\phi(q)} \int_2^{x_0} \frac{du}{\ln^2(u)} + c \int_2^{x_0} \frac{du}{\sqrt{u}} + \frac{3}{\phi(q)}$$

$$\leq 0.1 \sqrt{x \ln(x)} + 10^8.$$

Note that if $q \geq 3$ then we obtain the remainder term claimed in the proposition. If $q = 1, 2$ then $\pi(x; q, l)$ is $\pi(x)$. The formula for $\pi(x)$ follows from [4] and is covered by the formula for $\pi(x; q, l)$ of this proposition.

**Lemma 1.5.** The following holds:

$$\sum_{p \geq N} \ln^2(p) e^{-2p/N} < 0.96N \ln(N).$$

**Proof.** Split the sum into two terms

$$\sum_{p < N} \ln^2(p) e^{-2p/N} + \sum_{p \geq N} \ln^2(p) e^{-2p/N}. \quad (1.11)$$
The first sum can be written as
\[
\sum_{p < N/2} \ln^2(p) e^{-2p^2/N} + \sum_{N/2 < p < N} \ln^2(p) e^{-2p^2/N} < 0.55N \ln(N) + 0.55e^{-1}N \ln(N) < 0.76N \ln(N). \tag{1.12}
\]

The second sum of (1.11) is equal to
\[
\sum_{k=1}^{\infty} \sum_{kN \leq p < (k+1)N} \ln^2(p) e^{-2p^2/N}. \tag{1.13}
\]

Trivially, for \(k \geq 1\), we have
\[
\ln((k+1)N) = \ln(kN) + \ln(1 + k^{-1}) < \ln(kN) + 0.7.
\]

So (1.13) is less than
\[
\sum_{k=1}^{\infty} (\ln(kN) + 0.7)^2 e^{-2k} \sum_{kN \leq p < (k+1)N} 1. \tag{1.14}
\]

The results of [4] imply that
\[
\sum_{kN \leq p < (k+1)N} 1 < 1.1 \frac{N}{\ln(kN)}.
\]

Substituting this into (1.14) we obtain that (1.13) can be bounded by
\[
1.1N \sum_{k=1}^{\infty} (\ln(kN) + 0.7)^2 e^{-2k} < 1.2N \sum_{k=1}^{\infty} \ln(kN) e^{-2k}. \tag{1.15}
\]

By elementary computations
\[
\sum_{k=1}^{\infty} \ln(k) e^{-2k} < 0.05 \quad \text{and} \quad \sum_{k=1}^{\infty} e^{-2k} = \frac{e^{-2}}{1 - e^{-2}} < 0.16. \tag{1.16}
\]

Substituting (1.16) into (1.15) and using that \(\ln(N) > 46\), we bound the sum (1.13) with
\[
1.2N(0.05 + 0.16 \ln(N)) < 0.2N \ln(N). \tag{1.17}
\]

Combining (1.12) with (1.17), we find
\[
\sum_{p > 2} \ln^2(p) e^{-2p^2/N} < 0.96N \ln(N).
\]

The lemma is proved. \(\blacksquare\)
2. THE REMAINDER TERM

Let \( Q = [1.1 \ln^2(N)] \), \( \tau = 4900 \ln^4(N) \), and \( w = 1/\tau \). Any \( x \) from \([-w, 1-w] \) can be written (see [1, Chap. X, Sect. 1]) as

\[
x = \frac{a}{q} + z, \quad \text{where} \quad 0 < q \leq \tau, \quad (a, q) = 1, \quad |z| \leq \frac{1}{q^\tau}.
\]

Here \( 1 \leq a \leq q - 1 \) if \( q > 1 \) and \( a = 0 \) if \( q = 1 \).

**Definition.** Denote by \( E(a, q) \) \( ((a, q) = 1, \ 0 \leq a \leq q - 1) \) the interval centered at \( a/q \) with radius \( 1/(q\tau) \), i.e.,

\[
E(a, q) = \left[ \frac{a - 1}{q} \frac{a + 1}{q} \right].
\]

Let \( E_1 \) be the set of \( E(a, q) \) with \( q \leq Q \), and \( E_2 \) its complement in \([-w, 1-w]\).

Note that if \( E(a, q) \) and \( E(a_1, q_1) \) are two different intervals then they are noninterlacing. Indeed, the distance between their centers: \(|(a/q) - (a_1/q_1)| \geq 1/qq_1 \), while the sum of their radii is \((1/q\tau) + (1/q_1 \tau) < 1/qq_1 \) (because \( q + q_1 < \tau \)). Thus, \( E_1 \) is a finite set of noninterlacing intervals

\[
\left[ \frac{a - 1}{q} \frac{a + 1}{q} \right], \quad 0 \leq a < q, \quad (a, q) = 1, \quad q = 1, 2, ..., Q.
\]

**Definition.** Denote by \( E'_1 \) the subset of \( E_1 \) consisting of intervals centered at \( a/q \), \( a < q \), \( (a, q) = 1 \), \( q = 1, 2, ..., Q \) of length \( 2\delta_N(q) \), where \( \delta_N(q) = 2 \ln(N)/\phi(q)N \). By \( E''_1 \) we denote the complement of \( E'_1 \) in \( E_1 \).

Throughout this paper we denote by \( \gamma \) a multiplicative character of integers, see [1, Chap. VIII, Sect. 1], and by \( \theta (\theta_1, \theta_2, \text{etc.}) \) we denote an arbitrary constant (possibly different every time we use it), s.t. \( |\theta| \leq 1 \).

The following lemma is used to estimate the remainder term, i.e., the integral \( J_2 \).

**Lemma 2.1.** Take any \( x \in E''_1 \cup E_2 \), and for any \( N > 10^{20} \) (not necessarily odd), GRH implies that

\[
|S(x)| < 0.18 \frac{N}{\ln(N)}.
\]
Proof. Set $x = (1/N) + 2\pi iz$. Using the orthogonality relation for characters [1, Chap. VIII, Sect. 1], we have

$$S(x) = \sum_{n \geq 1} A(n)e^{-nx}$$

$$= \sum_{n \geq 1} A(n) \frac{1}{\phi(q)} \sum_{m=1 \atop (m, q)=1}^q \sum_{n \geq 1 \atop (n, q)>1} \tau(n)\bar{\tau}(m)e^{-nx} + \sum_{n \geq 1 \atop (n, q)>1} A(n)e^{-n/N}.$$  

Here $\phi(q)$ ($q$ is a positive integer) is the Euler function: $\phi(q)$ is the number of integers $a$ ($a > 2$), s.t. $(a, q) = 1$. Since the sum on the right-hand side converges absolutely, we can change the order of summation. Thus it is equal to

$$\frac{1}{\phi(q)} \sum_{n \geq 1} A(n) \sum_{\tau \in \tau_{qr}} \tau(n) \bar{\tau}(m) e^{2\pi im/\phi(q)} + \theta_1 \ln^3(N)$$

$$= \frac{1}{\phi(q)} \sum_{n \geq 1 \atop (n, q)>1} A(n) \tau(n) \bar{\tau}(a) S(N, \tau, x, z) + \theta_1 \ln^3(N),$$

where $\tau(\chi)$ is the Gauss sum [2, Chap. VIII, Sect. 1] and

$$S(N, \tau, x, z) = \sum_{n \geq 1} A(n) \tau(n) x^{-n}.$$  

It is known [1, Chap. VIII, Sect. 1] that $|\tau(\chi_0)| = 1$, $\chi_0$ is the trivial character, $|\tau(a)| = 1$, and $|\tau(\chi)| \leq q^{0.5} \chi$ is any character). Littlewood's formula (see [2]) asserts that for a primitive character $\chi$ (see [1, Chap. VIII, Sect. 1]), we have

$$S(N, \chi, x, z) = E(\chi) x^{-1} - \sum_{\rho} \Gamma(\rho) x^{-\rho} - \frac{L'(0, \chi)}{L(0, \chi)}$$

$$+ \frac{1}{2\pi i} \int_{-0.5+it}^{-0.5-it} x^{-w} \left( -\frac{L'(w, \chi)}{L(w, \chi)} \right) \Gamma(w) dw,$$  

where $E(\chi) = 1$ if $\chi = \chi_0$ and $E(\chi) = 0$ otherwise, $\rho$ are all the nontrivial zeroes of $L(s, \chi)$. If $\chi$ is not primitive, then it is induced by some primitive character $\chi_1$ and

$$S(N, \chi, x, z) = S(N, \chi_1, x, z) + \theta \ln^3(N).$$

Our strategy is to estimate all terms on the right-hand side of (2.1). Its crucial term is $\sum_{\rho} \Gamma(\rho) x^{-\rho}$. To estimate this term we use GRH.
Lemma 1.2 and Theorem 4 of \cite[Chap. VIII, Sect. 5]{1} imply that
\[
\left| \frac{L'(w, \chi)}{L(w, \chi)} \right| < 10(3 + \ln(q |t|)).
\]

Again, with \( w = 0 \), we obtain
\[
\left| \frac{L'(0, \chi)}{L(0, \chi)} \right| < 2(2 + \ln(q)) < 2 \ln(N).
\]

Sterling’s formula \cite[Chap. III, Sect. 2]{1} asserts that
\[
I(\sigma + it) = \sqrt{2\pi} t^{-\frac{1}{2}} e^{-\frac{1}{2}t} \left(\frac{1}{\sqrt{\pi}} + \frac{\ln(t)}{2t} + \frac{\ln(q)}{2t} + o\left(\frac{1}{t}\right)\right),
\]
where \( |\theta| \leq 1 \).

If \( z = re^{i\phi} \) is a complex number, \( r > 0 \), then \( \phi (0 \leq \phi < 2\pi) \) is denoted by \( \arg(z) \). We will need the following two facts which can be obtained easily:
\[
\frac{\pi}{2} - \arg(x) = \arctan\left(\frac{1}{2\pi N |x|}\right),
\]
\[
\int_{1}^{\infty} \frac{e^{-t/A}}{t} \ln(qt) \, dt < 2 \ln(Aq) \ln(3A), \quad \text{for any } A > 3.
\]

To estimate the integral in (2.1) we first use Lemma 1.3 and then (2.3) and (2.4). We obtain
\[
\left| \frac{1}{2\pi i} \int_{-0.5 - i\infty}^{-0.5 + i\infty} x^{-w} \left(\frac{L'(w, \chi)}{L(w, \chi)}\right) I(w) \, dw \right|
\leq \frac{3}{\pi} \int_{1}^{\infty} 10(3 + \ln(qt)) e^{\left|\arg(x) - 0.5\right|t} \, dt + 20 \ln(q) < 30 \ln^3(N).
\]

Consider the term \( \sum I(\rho) x^{-\rho} \) of (2.1). Assuming GRH, we have \( \text{Re}(\rho) = 0.5 \). Using (2.2),
\[
\left| \sum_{\rho} I(\rho) x^{-\rho} \right| < 2\sqrt{2\pi} |x|^{-0.5} \sum_{\gamma > 0} e^{\left|\arg(x) - 0.5\right|\gamma}.
\]

We have the following three cases.

\textit{Case 1.} Let \( 2\pi N |x| < 1 \). Then \( \arg(x) - 0.5\pi \leq 0.25\pi \) and \( |x|^{-1} \leq N \). The sum (2.6) is less than
\[
2\sqrt{2\pi N} \sum_{\gamma > 0} e^{-\gamma/4}.
\]
Applying Lemma 1.1, this is

$$\leq 2 \sqrt{2\pi N} \left( 1.09(1 + 0.5 \ln(q)) + 2.17 \sum_{t=2, 4, \ldots} (1 + 0.5 \ln(qt))e^{-\pi t/4} \right).$$

Changing the variable of summation, this is

$$\leq 2 \sqrt{2\pi N} \left( 1.09(1 + 0.5 \ln(q)) + 2.17 \sum_{k=1}^{\infty} (1 + 0.5 \ln(2qk))e^{-nk/2} \right).$$

By elementary computations

$$\sum_{k=1}^{\infty} \ln(k)e^{-nk/2} < 0.05 \quad \text{and} \quad \sum_{k=1}^{\infty} e^{-nk/2} < 0.27.$$  

So, the expression above is

$$\leq 2 \sqrt{2\pi N} \left( 1.83 + 0.84 \ln(q) \right) < \sqrt{N} \left( 9.18 + 4.22 \ln(q) \right).$$

Since $N \geq 10^{50}$ and $q \leq \tau = 4900 \ln(N)$, this is less than $2.75 \sqrt{N \ln(N)}$.

**Case 2.** Let $2\pi N |z| > 10$. Then

$$\arg(x) - \frac{\pi}{2} < \frac{0.996}{2\pi N |z|}.$$  

Using (2.7) and Lemma 1.1, the sum (2.6) is

$$\leq 2 \sqrt{|z|} \cdot \frac{1.09(1 + 0.5 \ln(q)) + 2.17}{\sqrt{|z|}} \sum_{t=2, 4, \ldots} (1 + 0.5 \ln(qt))e^{-0.996/2\pi N |z|}$$

$$\leq 2.17 \sqrt{|z|} \frac{1 + 0.5 \ln(q)}{\sqrt{|z|}} \left( 2.17(1 + 0.5 \ln(2q)) \sum_{k=1}^{\infty} e^{-0.996k/\pi N |z|} \right)$$

$$+ 1.09 \sum_{k=1}^{\infty} \ln(k)e^{-0.996k/\pi N |z|}.$$  

We need the following inequality:

$$\sum_{t=2}^{\infty} \ln(t)e^{-t/4} < A \ln(10.4), \quad A > 9.$$  

(2.8)
Indeed,
\[
\int_{1}^{\infty} \ln(t) e^{-t/A} dt = A \int_{1}^{\infty} \ln(t/A) e^{-t/A} d(t/A) + \int_{1}^{\infty} \ln(A) e^{-t/A} dt
\]
\[
< A \ln(A) + A \int_{1/A}^{\infty} \ln(x) e^{-x} dx
\]
\[
< A \ln(A) + A \int_{1}^{\infty} \ln(x) e^{-x} dx
\]
\[
< A \ln(A) + A < A \ln(3A).
\]
Using (2.8), our sum can be bounded by
\[
\frac{2}{\sqrt{|z|}} (1.09(1+0.5 \ln(q)) + 6.85 N |z|(1+0.5 \ln(q)) + 3.44N |z| \ln(10N |z|)).
\]
Since $|z| \leq 1/q$, this is
\[
\leq 7(q)^{-0.5} N \ln(N).
\]

**Case 3.** The last possibility is when $1 \leq 2\pi N |z| \leq 10$. We do the same computations as in the first two cases. The required estimate follows from the facts that
\[
\arctan(1/2\pi |z| N) > 1/3\pi |z| N, \quad \ln(3\pi |z| N) < 2.8, \quad 3\pi |z| N \leq 15.
\]
Finally, we have the following estimate for $S(x)$:
\[
|S(x)| \leq \frac{1}{\phi(q) |x|} + \left( \frac{2.75 \sqrt{Nq} \ln(N)}{7 \sqrt{\pi}} \right)
\]
(2.9)

We distinguish between the cases:

**Case 1.** Let $x \in E_1$. We have $|x| > 2\pi |z|$ where $|z| > 2 \ln(x)/(\phi(q) N)$. Thus
\[
\frac{1}{\phi(q) |x|} \leq \frac{1}{2\pi |z|} \leq \frac{N}{4\pi \ln(N)} < 0.08 \frac{N}{\ln(N)}.
\]
The upper term of (2.9) can be bounded by
\[
4 \sqrt{N \ln^2(N)} \leq 0.01 \frac{N}{\ln(N)}.
\]
Since $\sqrt{\tau} = 70 \ln^2(N)$, the lower term of (2.9) is equal to $0.1N/\ln(N)$. So, when $x \in E_1^*$, we have

$$|S(x)| \leq 0.18 \frac{N}{\ln(N)}.$$

**Case 2.** Let $x \in E_2$. Since $N \geq 10^{29}$, for such $x$, we have $q > Q = [1.1 \ln^2(N)] > 2330$. By elementary computations (for $q > 2330$), the following estimate holds:

$$\phi(q) > 14.1 \sqrt{q} > 14.78 \ln(N).$$

Using that $|x| > 1/N$, we obtain

$$\frac{1}{\phi(q)} |x| \leq \frac{N}{\phi(q)} < 0.068 \frac{N}{\ln(N)}.$$ (2.10)

So, $S(x)$ with the lower term of (2.9) is

$$|S(x)| < \frac{1}{\phi(q)} |x| + 7 \frac{N}{\sqrt{\tau}} \ln(N)$$

$$< 0.068 \frac{N}{\ln(N)} + 0.1 \frac{N}{\ln(N)} < 0.18 \frac{N}{\ln(N)}.$$

Using (2.10) and that $\tau = 4900 \ln^4(N)$, for $S(x)$ with the upper term of (2.9), we have

$$|S(x)| < \frac{1}{\phi(q)} |x| + 2.75 \sqrt{Nq} \ln(N) < \left( 0.68 + \frac{192.5}{\ln(N)} \right) \frac{N}{\ln(N)}.$$

Since $N \geq 10^{29}$, we have $N > \ln^{12}(N)$. The above expression is $\leq 0.16N/\ln(N)$. The lemma follows.

### 3. ESTIMATION OF THE VINOGRADOV CONSTANT

This section computes the leading term in the expression for $J(N)$. This is done in Lemmas 3.1, 3.2, and 3.3. We are following the techniques of [1, Chap. X, Sect. 2].

Recall that $x = a/q + z$, where $q \leq \tau$ and $|z| \leq 1/q\tau$. 
Lemma 3.1. For $S(x)$, where $x \in E_1$, we have

$$S(x) = \frac{\mu(q)}{\phi(q)} M(z) + 40(\phi(q) + 4\pi \ln(N)) \sqrt{N} \ln^2(N), \quad |\theta| \leq 1,$$

where

$$M(z) = \sum_{n=\infty}^{\infty} e^{-nx} = \int_{\gamma}^{\infty} e^{-xu} \, du + 4\theta_1, \quad |\theta_1| \leq 1, \quad x = \frac{1}{N} + 2\pi iz.$$

Proof. When $n > 10^9$, Theorem 2.4 asserts that

$$\pi(n; q, l) = \frac{\text{Li}(n)}{\phi(q)} + 1.3\theta_1 \sqrt{n} \ln(n) + \theta_2 \cdot 10^3,$$

where $|\theta_1| \leq 1$, $|\theta_2| \leq 1$. We have ($p$ is a prime number)

$$S \left( \frac{a}{q} + z \right) = \sum_{p > \sqrt{N}} e^{-2\pi ip/q} e^{-pN} \ln(p) + 0N^{0.5}$$

$$= \sum_{l=1}^{q} e^{2\pi il/q} T(l) + 0N^{0.5} \quad (3.1)$$

where

$$T(l) = \sum_{p = \left( l \mod q \right)} e^{-\pi l/p} \ln(p) = \sum_{\sqrt{N} < n} \left( \pi(n; q, l) - \pi(n-1; q, l) \right) e^{-\pi l/n}(n).$$

Fix a big positive number $M$ and set

$$T(l)_M = \sum_{\sqrt{N} < n < M} \left( \pi(n; q, l) - \pi(n-1; q, l) \right) e^{-\pi l/n}(n). \quad (3.2)$$

Since this sum converges absolutely, $T(l)$ is the limit of $T(l)_M$ when $M \to \infty$. We use Abel's transformation [1, Chap. I, Sect. 4] with

$$c_n = \pi(n; q, l) - \pi(n-1; q, l), \quad f(u) = e^{-\pi l/u} \ln(u),$$

$$C(u) = \sum_{\sqrt{N} < n < u} c_n = \frac{\text{Li}(u)}{\phi(q)} + 2.6\theta_1 \sqrt{u} \ln(u) + 2\theta_2 \cdot 10^3.$$
Applying this transformation to (3.2) gives

\[ T(l)_M = -\int_M^\infty C(u) \, d(e^{-uM} \ln(u)) + C(M) \, e^{-uM} \ln(M) \]

\[ = -\frac{1}{\phi(q)} \int_M^\infty \text{Li}(u) \, d(e^{-uM} \ln(u)) + \frac{\text{Li}(M)}{\phi(q)} \, e^{-uM} \ln(M) \]

\[ + R_1(M) + R_2(M), \]

where

\[ |R_1(M)| \leq 2.6 \left| \int_3^M \sqrt{u} \ln(u) \, d(e^{-uM} \ln(u)) \right| + 2 \sqrt{M} \ln^2(M) \, |e^{-uM}| \]

\[ |R_2(M)| \leq 2 \cdot 10^8 \left| \int_3^M |f'(u)| \, du \right|.

Note that \( \sqrt{M} \ln^2(M) \, |e^{-uM}| \to 0 \) as \( M \to \infty \). Thus, by integrating by parts the leading terms and taking the limit of \( T(l)_M \) as \( M \) goes to \( \infty \), we obtain

\[ T(l) = \frac{1}{\phi(q)} \int_3^\infty e^{-uM} \, du + 2.6 \theta_1 \left| \int_3^\infty \sqrt{u} \ln(u) \, |f'(u)| \, du \right| 

\[ + 2 \theta_2 \cdot 10^8 \left| \int_3^\infty |f'(u)| \, du \right|. \tag{3.3} \]

We have

\[ |f'(u)| \leq \left[ \frac{1}{u} + \frac{1}{N} \left( 1 + \frac{4\pi \ln(N)}{\phi(q)} \right) \ln(u) \right] \, e^{-w/N}. \]

First, consider the second term of (3.3). It is bounded by

\[ \frac{2.6}{N} \left( 1 + \frac{4\pi \ln(N)}{\phi(q)} \right) \left| \int_3^\infty \sqrt{u} \ln^2(u) e^{-w/N} \, du \right| + 2.6 \left| \int_3^\infty \frac{\ln(u)}{\sqrt{u}} e^{-w/N} \, du \right|. \tag{3.4} \]

The first integral of (3.4) is equal to

\[ \int_3^N \sqrt{u} \ln^2(u) \, e^{-w/N} \, du + \int_N^\infty \sqrt{u} \ln^2(u) \, e^{-w/N} \, du. \]
We estimate the first term trivially and make a change of variable $u \mapsto Nu$ in the second one. This is equal to

$$\ln^2(N) \int_3^N \sqrt{u} e^{-u/N} du + N \int_1^\infty \sqrt{u} \ln^2(uN) e^{-u} du.\]

By elementary computations, this is less than

$$0.54N \sqrt{N} \ln^2(N) + 0.61N \sqrt{N} \ln^2(N) = 1.15N \sqrt{N} \ln^2(N).$$

To estimate the second integral of (3.4), we first split it into two integrals and then estimate each of them. We have

$$\int_3^N \frac{\ln(u)}{u} e^{-u/N} du = \int_3^N \frac{\ln(u)}{u} e^{-u/N} du + \int_1^\infty \frac{\ln(u)}{u} e^{-u} du$$

$$< \int_3^N \frac{\ln(u)}{u} du + \ln(N) \int_1^\infty e^{-u} du$$

$$< 2 \sqrt{N} (\ln(N) + 2) + e^{-1} \sqrt{N} \ln(N) < 0.05 \sqrt{N} \ln^2(N).$$

Thus (3.4) is less than

$$\left[2.99 \left(1 + \frac{4\pi \ln(N)}{\phi(q)}\right) + 0.14\right] \sqrt{N} \ln^2(N).$$

Consider the third term of (3.3). It is bounded by

$$2 \cdot 10^8 \int_3^\infty \left[\frac{1}{u} + \frac{1}{N} \left(1 + \frac{4\pi \ln(N)}{\phi(q)}\right) \ln(u)\right] e^{-u/N} du.$$ 

Since $N \geq 10^{20}$, we have $2 \cdot 10^8 \leq 0.02 \sqrt{N}$. Applying similar estimates, the above integral is bounded by $0.001 \sqrt{N} \ln^2(N)$. By a crude estimate, the remainder terms of (3.3) are less than

$$3.9 \left(1 + \frac{4\pi \ln(N)}{\phi(q)}\right) \sqrt{N} \ln^2(N).$$

By the well known formula

$$\mu(q) = \sum_{\substack{l=1, (l, q) = 1}}^q e^{2\pi i q l} , \quad (a, q) = 1.$$
Substituting the expressions for $T(l)$ and $\mu(q)$ into (3.1), we obtain

$$S(x) = \frac{\mu(q)}{\phi(q)} \int_{-\infty}^{\infty} e^{-ux} du + 3.9\theta(\phi(q) + 4\pi \ln(N)) \sqrt{N} \ln^2(N),$$

where $|\theta| \leq 1$.

The following facts are obvious:

$$\int_{-\infty}^{\infty} e^{-ux} du = -\frac{1}{x} e^{-ux} \bigg|_{-\infty}^{\infty} = \frac{1}{x} e^{-3x},$$

$$\sum_{n=3}^{\infty} e^{-nx} = e^{-3x}(1 - e^{-x})^{-1},$$

$$e^{-x} = 1 - \frac{x}{1} + \frac{x^2}{2} - \frac{x^3}{6} + \cdots = 1 - x + 2\theta x^2, \quad |\theta| \leq 1.$$

Finally, we have

$$\left| \int_{-\infty}^{\infty} e^{-ux} du - \sum_{n=3}^{\infty} e^{-un} \right| < 2 \left| x^{-1} - (1 - e^{-x})^{-1} \right| < \left| \frac{2}{x} \left( 1 - \frac{x}{1 - e^{-x}} \right) \right| < 4. \quad \Box$$

**Lemma 3.2.** For the integral $J_1(N)$ (when $N > 10^{20}$), the following formula holds:

$$J_1(N) = \sigma a + 0.03\theta N^2, \quad \text{where} \quad \sigma = \sum_{q=1}^{Q} \frac{\mu(q)}{\phi(q)} \sum_{a=1}^{A} e^{2\pi q a / N},$$

and

$$a = \int_{-1}^{+0.5} M(z)e^{2\pi izN} dz, \quad M(z) = \sum_{n=3}^{\infty} e^{-nx}, \quad x = \frac{1}{N} + 2\pi iz.$$

**Proof.** Write $J_1(N)$ as a finite sum of integrals over the intervals of the set $E_1$:

$$J_1(N) = \int_{E_1} S(x)e^{2\pi inxN} dx = \sum_{q=Q}^{q} \sum_{a=1}^{A} \sum_{(a, q) = 1}^{A} I(a, q), \quad (3.5)$$
where
\[ I(a, q) = \int_{-\delta_{a,q}}^{\delta_{a,q}} S^3 \left( \frac{a}{q} + z \right) e^{2\pi i (a/q + z) N} dz. \]

Using Lemma 3.1, the third power of \( S(x) \) is
\[
S^3(x) = \frac{\mu(q)}{\phi(q)} M^3(z) + 12\theta_1 \frac{M^2(z)}{\phi(q)} (\phi(q) + 4\pi \ln(N)) \sqrt{N} \ln^2(N) \\
+ 48\theta_2 \frac{M(z)}{\phi(q)} (\phi(q) + 4\pi \ln(N))^2 N \ln^4(N) \\
+ 64\theta_2 (\phi(q) + 4\pi \ln(N))^3 N \sqrt{N} \ln^5(N), \quad (3.6)
\]
where \( |\theta_i| \leq 1, i = 1, 2, 3 \). We will substitute the expression for \( S^3(x) \) from (3.6) into the sum (3.5). Then we will estimate the contribution from the nonleading terms (there are three of them) in this expression.

**Case 1.** Consider the contribution to the remainder term of (3.5) from the first non-leading term of (3.6). It is
\[
\leq 12 \sum_{q < Q} \left| \sum_{u = 1}^{\varphi(q)} \int_{-\delta_{u/q}}^{\delta_{u/q}} \frac{M^2(z)}{\phi(q)} (\phi(q) + 4\pi \ln(N)) \sqrt{N} \ln^2(N) dz \right| \\
\leq 152 \sqrt{N} \ln^3(N) \int_{E_5} |M(z)|^2 dz < 152 \sqrt{N} \ln^3(N) \int_0^1 |M(z)|^2 dz. \quad (3.7)
\]

We have
\[
\int_0^1 |M(z)|^2 dz = \int_0^1 \sum_{n_1 = 3}^{\infty} \sum_{n_2 = 3}^{\infty} e^{-(n_1 + n_2)/N} e^{-2\pi i (n_1 - n_2) z} dz \\
= \sum_{n_1 = 3}^{\infty} \sum_{n_2 = 3}^{\infty} e^{-(n_1 + n_2)/N} \int_{-w}^{1-w} e^{-2\pi i (n_1 - n_2) z} dz = \sum_{n_1 = 3}^{\infty} e^{-2\pi i /N} \\
\leq \int_2^{\infty} e^{-2\pi x/N} dx = -\frac{N}{2} e^{-2\pi x/N}_{x=\infty} = 0.5 Ne^{-4/N} < 0.5 N.
\]
Hence, using that \( N > \ln^2(N) \), (3.7) is
\[
< 76N \sqrt{N} \ln^3(N) < 76 \frac{N^2}{\ln(N)} < 0.0008N^2. \quad (3.8)
\]
Case 2. Consider the contribution to the remainder term of (3.5) from the second nonleading term of (3.6). It is
\[
\begin{align*}
\leq 48 \sum_{q \leq Q} \sum_{(a, q) = 1}^{\mathcal{S}_2(q)} \int_{\delta_2(q)}^{\mathcal{S}_1(q)} \frac{M(z)}{\phi(q)} (\phi(q) + 4\pi \ln(N))^2 N \ln^4(N) \, dz.
\end{align*}
\tag{3.9}
\]

The following estimate is trivial:
\[
|M(z)| = \left| \sum_{a = 3}^{\infty} e^{-a} \right| < N.
\]

The sum (3.9) is
\[
\begin{align*}
\leq 48N^2 \ln^4(N) \sum_{q \leq Q} \frac{4 \ln(N)}{\phi(q) N} (\phi(q) + 4\pi \ln(N))^2 \\
< 192N \ln^5(N) \sum_{q \leq Q} (q + 8\pi \ln(N) + 16\pi^2 \ln^2(N)/\phi(q)).
\end{align*}
\]

Using that \(\phi(q) \geq \sqrt{q}\), \(Q = 1.1 \ln^2(N) < N^{0.2}\) and \(\ln(N) > 46\), this is
\[
\begin{align*}
< 192N \ln^5(N) \left[ \frac{Q}{2} + 8\pi Q \ln(N) + 16\pi^2 \ln^3(N) \right] 2 \sqrt{Q} \\
< 192N \ln^5(N) \cdot 5.04 \ln^4(N) < 968 \frac{N}{\ln^3(N)} < 0.012N^2. 
\end{align*}
\tag{3.10}
\]

Case 3. Consider the contribution from the third nonleading term of (3.6). It is
\[
\begin{align*}
\leq 64 \sum_{q \leq Q} \sum_{(a, q) = 1}^{\mathcal{S}_2(q)} \int_{\delta_2(q)}^{\mathcal{S}_1(q)} (\phi(q) + 4\pi \ln(N))^3 N^{1.5} \ln^4(N) \, dz \\
< 64N \sqrt{N} \ln^7(N) \sum_{q \leq Q} 4 \frac{\ln(N)}{N} (\phi(q) + 4\pi \ln(N))^3 \\
< 256 \sqrt{N} \ln^7(N) \sum_{q \leq Q} (q^3 + 3q^2 \cdot 4\pi \ln(N) \\
+ 3q \cdot 16\pi^2 \ln^2(N) + 64\pi^3 \ln^3(N)) \\
< 256 \sqrt{N} \ln^7(N) (0.25Q + 4\pi Q^2 \ln(N) + 24\pi^2 Q^2 \ln^2(N) + 64\pi^3 Q \ln^3(N)).
\end{align*}
\]

Using that \(Q = [1.1 \ln^2(N)]\) and \(\ln(N) > 46\), this is
\[
< 256 \sqrt{N} \ln^7(N) \cdot 0.91 \ln^8(N) < 233 \frac{N^2}{\ln^2(N)} < 0.003N^2. 
\tag{3.11}
\]
The contribution of (3.8), (3.10), and (3.11) to the remainder is less than 0.016$N^2$. Thus

$$J_1(N) = \sum_{q \leq Q} \sum_{\alpha=1}^{q} \frac{\mu(q)}{\phi(q)} e^{2\pi i \alpha q/N} \int_{-\delta(q)}^{\delta(q)} M^*(z) e^{2\pi i z} + 0.0160 N^2, \tag{3.12}$$

where $|\theta| \leq 1$. Changing the range of integration, we have

$$\int_{-\delta(q)}^{\delta(q)} M^*(z) e^{2\pi i z} \, dz = \int_{-0.5}^{+0.5} M^*(z) e^{2\pi i z} \, dz + R = I(N) + R. \tag{3.12}$$

Note that

$$|M(z)| < \left| \int_{0}^{\infty} e^{-t^2} \, dt \right| + 4 = \frac{1}{\sqrt{\pi}} \left| e^{-u^2} \right|_{0}^{\infty} + 4 < \frac{1}{\sqrt{\pi}} |z| + 4 < 0.16 \frac{1}{|z|}. \tag{3.12}$$

Thus, for $R$ we have

$$|R| < 2 \left| \int_{-\delta(q)}^{\delta(q)} |M(z)|^3 \, dz \right| < 0.01 \left| \int_{-\delta(q)}^{\delta(q)} \frac{dz}{z} \right| < 0.01 \frac{N^2 \phi(q)^2}{4 \ln^2(N)}. \tag{3.12}$$

Once integrated over the set $E_1^*$, it contributes to the remainder

$$\left| \sum_{q \leq Q} \sum_{\alpha=1}^{q} \frac{e^{2\pi i \alpha q/N}}{\phi(q)^3} R \right| < 0.0025 \frac{N^2}{\ln^2(N)} \sum_{q \leq Q} 1 \sum_{\alpha=1}^{q} \frac{1}{\phi(q)^3} \left| \sum_{q \leq Q} \frac{\mu(q)}{\phi(q)} e^{2\pi i \alpha q/N} R \frac{N^2}{\ln^2(N)} \cdot 1.1 \ln^2(N) \right| < 0.003N^2. \tag{3.13}$$

Finally, we need to estimate

$$\left| \sum_{q \leq Q} \frac{\mu(q)}{\phi(q)} \sum_{\alpha=1}^{q} \frac{e^{2\pi i \alpha q/N} I(N)}{\phi(q)^3} \right|. \tag{3.14}$$
By elementary computations \(\sum_{q>0} (1/\phi(q)^2) < 0.03\), and
\[
\left| I(N) \right| < N \int_{-0.5}^{0.5} |M(z)|^2 \, dz
\]
\[
< N \int_{-0.5}^{0.5} \sum_{n=3}^{\infty} \sum_{m=3}^{\infty} e^{-2\pi i (n-m)e^{-m/m}} d\zeta < 0.5 N^2.
\]
Combining this with (3.14) gives
\[
\left| \sum_{q>0} \mu(q) \phi(q)^{-1} \sum_{(a,q)=1}^{q} e^{2\pi i N/a} I(N) \right| < |I(N)| \sum_{q>0} \frac{1}{\phi(q)^2} < 0.015 N^2. \quad (3.15)
\]
Substituting (3.13), (3.15), (3.8), (3.10), and (3.11) into (3.5), we obtain
\[
J_1(N) = \sum_{q>0} \mu(q) \phi(q)^{-1} \sum_{(a,q)=1}^{q} e^{2\pi i N/a} M^l(z) e^{2\pi i N} d\zeta + c,
\]
where \(|c| < 0.016 N^2 + 0.003 N^2 + 0.015 N^2 < 0.04 N^2. \]

**Lemma 3.3.** The following holds:
\[
I(N) = \frac{N^2}{2} + 8\theta N.
\]

**Proof.** We have
\[
I(N) = \int_{-0.5}^{0.5} M^l(z) e^{2\pi i N} d\zeta
\]
\[
= \int_{-0.5}^{0.5} \sum_{n=3}^{\infty} \sum_{m=3}^{\infty} \sum_{k=3}^{\infty} e^{-2\pi i (n-m+k-N)e^{-(n+m+k)}} d\zeta
\]
\[
= \sum_{n=3}^{\infty} \sum_{m=3}^{\infty} \sum_{k=3}^{\infty} e^{-(n+m+k)} \int_{-0.5}^{0.5} e^{2\pi i (N-n-m-k)} d\zeta = S(N)/e,
\]
where \(S(N)\) is the number of solutions to the equation
\[
n_1 + n_2 + n_3 = N, \quad 2 < n_1, n_2, n_3 < N - 5.
\]
Once \(n_3\) is fixed \((2 < n_3 < N - 5)\), the equation
\[
n_1 + n_2 = N - n_3, \quad 2 < n_1, n_2 < N - 5
\]
has \( N - n_3 - 5 \) solutions. Hence

\[
S(N) = \sum_{n_3=3}^{N-6} (N - n - 5) = \sum_{k=1}^{N-8} k = \frac{(N-8)(N-7)}{2} = \frac{N^2}{2} + 8\theta N.
\]

**Theorem 3.4.** Assuming GRH, every odd number greater than \( 10^{20} \) is a sum of three prime numbers.

**Proof.** Using the expression of Lemma 3.2 for \( J_1 = J_1(N) \), and the value of \( I(N) \) of Lemma 3.3, we have

\[
J_1 = \frac{\sigma}{e^{\left(\frac{N^2}{2} + 8\theta N\right)}} + 0.04\theta N^2.
\]

When \( N \) is odd, it is known that \( \sigma > 12/\pi^2 > 1.21 \) (see [2, Chap. X, Sect. 2]). Hence

\[
J_1 > 0.444(N^2/2 - 8N) - 0.04N^2 > 0.18N^2.
\]

For the integral \( J_2 \), we have

\[
|J_2| \leq \int_{E_1 + E_2} |S(x)e^{-2mxN}| \, dx < \max_{x \in E_1 + E_2} |S(x)| \int_0^1 |S(x)|^2 \, dx. \tag{3.16}
\]

We have

\[
\int_0^1 |S(x)|^2 \, dx = \int_0^1 \sum_{p_1 > 2} \sum_{p_2 > 2} e^{-(p_1 + p_2/N)} e^{2\pi(p_1 - p_2)^2} \ln(p_1) \ln(p_2) \, dx
\]

\[
= \sum_{p_1 > 2} \sum_{p_2 > 2} e^{-(p_1 + p_2/N)} \ln(p_1) \ln(p_2) \int_0^1 e^{2\pi(p_1 - p_2)^2} \, dx
\]

\[
< \sum_{p > 2} \ln^2(p) e^{-2\pi p/N}. \tag{3.17}
\]

Using Lemmas 1.5 and 2.1, the integral (3.16) is less than

\[
0.18 \frac{N}{\ln(N)} \times 0.96 \ln(N) < 0.179N^2.
\]

Finally, \( J > |J_1| - |J_2| > 0.18N^2 - 0.179N^2 > 0.001N^2 > 0 \). The theorem follows.
REFERENCES