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Two New Mappings Associated with Hadamard's Inequalities for Convex Functions

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Abstract—In this paper, we shall introduce two new mappings closely connected with Hadamard's inequality for convex mappings and study their main properties. Some applications are also included.

Keywords—Convex functions, Hadamard's inequality.

1. INTRODUCTION

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function on the real interval I and $a,\ b \in I^o$ (I^o is the interior of I) with $a < b$. The following inequality due to Hermite [1] and Hadamard [2] is well known:

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2}. \tag{1}$$

In [3] (see also [4]), the first author has introduced the following mappings $H,\ F : [0,1] \to \mathbb{R}$,

$$H(t) = \frac{1}{b-a}\int_a^b f\left(tx + (1-t)\frac{a+b}{2}\right)\,dx,$$

and

$$(t) = \frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(tx + (1-t)y\right)\,dx\,dy$$

associated with Hermite-Hadamard's inequality which give two "continuous scales" of refinements of Hermite-Hadamard inequality.

(i) H is a convex nondecreasing function on $[0,1]$ and

$$f\left(\frac{a+b}{2}\right) = H(0) \le H(t) \le H(1) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

(ii) F is nonincreasing on $[0,1/2]$, nondecreasing on $[1/2,1]$, and it is convex on $[0,1]$. Moreover,

$$\sup_{t\in[0,1]} F(t) = \frac{1}{b-a}\int_a^b f(x)\,dx, \quad \inf_{t\in[0,1]} F(t) = \frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{x+y}{2}\right)\,dx\,dy$$

and one has the inequality

$$H(t) \le F(t), \qquad \text{for all } t \in [0,1].$$

For some other properties of H and F, see [3,4], where some applications are also given.

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The aim of this paper is to study Hermite-Hadamard's inequality from a different point of view. Namely, we shall consider the "difference" mappings L and P defined by

$$L : [a, b] \to \mathbb{R}, \qquad L(t) = \frac{f(t) + f(a)}{2}(t - a) - \int_a^t f(s)\, ds$$

and

$$P : [a, b] \to \mathbb{R}, \qquad P(t) = \int_a^t f(s)\, ds - (t - a) f\left(\frac{t + a}{2}\right),$$

and will discuss the main properties of these, and then obtain some refinements of (1). Finally, some applications in connection with well-known elementary inequalities are also given. The motivation for the present work stems from many recent refinements of (1) in [1,3–15].

2. MAIN RESULTS

The main properties of the mapping L are embodied in the following theorem.

THEOREM 1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping on the interval I and let $a < b$ be fixed in I^o. Then, we have the following.*

(i) *The mapping L defined above is nonnegative, monotonically nondecreasing, and convex on $[a, b]$.*

(ii) *The following refinement of Hadamard's inequality holds:*

$$\frac{1}{b - a} \int_a^b f(s)\, ds \le \frac{1}{b - a} \int_y^b f(s)\, ds + \left(\frac{y - a}{b - a}\right) \frac{f(a) + f(y)}{2} \le \frac{f(a) + f(b)}{2}, \tag{2}$$

for each $y \in [a, b]$.

(iii) *The following inequality holds:*

$$\begin{aligned} \alpha \frac{f(t) + f(a)}{2}(t - a) &+ (1 - \alpha) \frac{f(s) + f(a)}{2}(s - a) \\ &- \frac{f(\alpha t + (1 - \alpha)s) + f(a)}{2}[\alpha t + (1 - \alpha)s - a] \\ &\ge \alpha \int_a^t f(u)\, du + (1 - \alpha) \int_a^s f(u)\, du - \int_a^{\alpha t + (1 - \alpha)s} f(u)\, du, \end{aligned} \tag{3}$$

for every $t, s \in [a, b]$ and each $\alpha \in [0, 1]$.

PROOF.

(i) The fact that L is nonnegative follows from Hadamard's inequality.

In order to prove the monotonicity and the convexity of L, we shall show the following inequality:

$$L(x) - L(y) \ge (x - y) L'_+(y), \qquad \text{for all } x, y \in [a, b]. \tag{4}$$

For this, suppose that $x > y$. Then, we have

$$L(x) - L(y) = \frac{f(x) + f(a)}{2}(x - a) - \frac{f(y) + f(a)}{2}(y - a) - \int_y^x f(s)\, ds. \tag{5}$$

By the inequality (1), we deduce

$$\frac{L(x) - L(y)}{x - y} \ge \frac{(f(x) + f(a))(x - a)}{2(x - y)} - \frac{(f(y) + f(a))(y - a)}{2(x - y)} - \frac{f(x) + f(y)}{2}.$$

On the other hand, since f is convex, $f'_+(y)$ exists for all $y \in [a, b)$, and thus, a simple calculation yields

$$L'_+(y) = \frac{f'_+(y)(y-a)}{2} - \frac{f(y)-f(a)}{2}, \qquad y \in [a, b). \tag{6}$$

Therefore, the inequality (4) holds provided

$$A = \frac{(f(x)+f(a))(x-a)}{x-y} - \frac{(f(y)+f(a))(y-a)}{x-y} - (f(x)+f(a)) \geq f'_+(y)(y-a). \tag{7}$$

But, a simple calculation shows that

$$A = \frac{(y-a)(f(x)-f(y))}{x-y},$$

and hence, the relation (7) is equivalent to

$$\frac{f(x)-f(y)}{x-y} \geq f'_+(y),$$

which holds by the convexity of f.

The proof of (4) for the case $y > x$ is similar, and we omit the details. Consequently, the mapping L is convex on $[a, b]$.

Now let $x > y$, $x, y \in [a, b]$. Since L is convex on $[a, b]$, we find

$$\frac{L(x)-L(y)}{x-y} \geq L'_+(y) = \frac{f'_+(y)(y-a) - (f(y)-f(a))}{2} \geq 0,$$

as, by the convexity of f, we have $f(a) - f(y) \geq (a-y)f'_+(y)$, for all $y \in [a, b]$. Thus, L is nondecreasing on $[a, b]$.

(ii) By (i), we have $0 \leq L(y) \leq L(b)$, for all $y \in [a, b]$, and hence,

$$\frac{f(y)+f(a)}{2}(y-a) - \int_a^y f(s)\,ds \leq \frac{f(b)+f(a)}{2}(b-a) - \int_a^b f(s)\,ds,$$

which gives

$$\int_a^b f(s)\,ds - \int_a^y f(s)\,ds \leq \frac{f(b)+f(a)}{2}(b-a) - \frac{f(y)+f(a)}{2}(y-a).$$

Therefore,

$$\frac{1}{b-a}\int_y^b f(s)\,ds \leq \frac{f(b)+f(a)}{2} - \frac{f(y)+f(a)}{2}\left(\frac{y-a}{b-a}\right),$$

which is the right inequality in (2).

By Hadamard's inequality, we also have

$$\begin{aligned}
\frac{1}{b-a}\int_y^b f(s)\,ds + \left(\frac{y-a}{b-a}\right)\frac{f(a)+f(y)}{2} &\geq \frac{1}{b-a}\int_y^b f(s)\,ds + \frac{y-a}{b-a}\cdot\frac{1}{y-a}\int_a^y f(s)\,ds \\
&= \frac{1}{b-a}\left(\int_y^b f(s)\,ds + \int_a^y f(s)\,ds\right) \\
&= \frac{1}{b-a}\int_a^b f(s)\,ds,
\end{aligned}$$

for all $y \in [a, b]$. This completes the proof of the left inequality in (2).

(iii) The inequality (3) follows by the convexity of L, i.e.,

$$L(\alpha t + (1-\alpha)s) \le \alpha L(t) + (1-\alpha)L(s),$$

for all $s,t \in [a,b]$ and $\alpha \in [0,1]$.

REMARK 1. Since L is nondecreasing, we have the following:

$$\inf_{t\in[a,b]} L(t) = L(a) = 0$$

and

$$\sup_{t\in[a,b]} L(t) = L(b) = \frac{f(b)+f(a)}{2}(b-a) - \int_a^b f(s)\,ds \ge 0.$$

REMARK 2. If f is a monotonically nondecreasing function on $[a,b]$, then the mapping $\Phi(t) = \int_a^t f(u)\,du$ is convex on $[a,b]$. Consider the new mapping $\Psi : [a,b] \to \mathbb{R}$ given by $\Psi(t) = 1/2(f(t)+f(a))(t-a)$. If f is assumed to be convex and nondecreasing, then Ψ is also convex on $[a,b]$ and, by the inequality (3), the following holds:

$$\alpha\Psi(t) + (1-\alpha)\Psi(s) - \Psi(\alpha t + (1-\alpha)s) \ge \alpha\Phi(t) + (1-\alpha)\Phi(s) - \Phi(\alpha t + (1-\alpha)s) \ge 0, \quad (8)$$

for all $s,t \in [a,b]$ and $\alpha \in [0,1]$.

The main properties of the mapping P are given in the following theorem.

THEOREM 2. *Let f be as in Theorem 1. Then,*

(i) *The mapping P is nonnegative and monotonically nondecreasing on $[a,b]$.*
(ii) *The following inequality holds:*

$$0 \le P(t) \le L(t), \qquad \textit{for all } t \in [a,b]. \quad (9)$$

(iii) *The following refinement of Hadamard's inequality holds:*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\le \frac{1}{b-a}\left[(b-a)f\left(\frac{a+b}{2}\right) - (y-a)f\left(\frac{a+y}{2}\right)\right] \\ &\quad + \frac{1}{b-a}\int_a^y f(s)\,ds \le \frac{1}{b-a}\int_a^b f(s)\,ds, \end{aligned} \quad (10)$$

for all $y \in [a,b]$.

PROOF.

(i) Clearly, by (1) the mapping P is nonnegative. Let $a \le x < y \le b$. Then, we have

$$L(y) - L(x) = \int_x^y f(s)\,ds - (y-a)f\left(\frac{y+a}{2}\right) + (x-a)f\left(\frac{x+a}{2}\right).$$

By Hermite-Hadamard's inequality, we have

$$\int_x^y f(s)\,ds \ge (y-x)f\left(\frac{x+y}{2}\right),$$

and hence,

$$L(y) - L(x) \ge (y-x)f\left(\frac{x+y}{2}\right) - (y-a)f\left(\frac{y+a}{2}\right) + (x-a)f\left(\frac{x+a}{2}\right).$$

Now, using the convexity of f, we get

$$\frac{y-x}{y-a}f\left(\frac{x+y}{2}\right)+\frac{x-a}{y-a}f\left(\frac{x+a}{2}\right)\geq f\left(\frac{(y-x)(x+y)}{2(y-a)}+\frac{(x-a)(x+a)}{2(y-a)}\right)=f\left(\frac{y+a}{2}\right),$$

and thus, $L(y)-L(x)\geq 0$, which shows that L is nondecreasing on $[a,b]$.

(ii) By Hermite-Hadamard's inequality, we have

$$\frac{2}{t-a}\int_a^{(a+t)/2} f(s)\,ds\leq\frac{f((t+a)/2)+f(a)}{2}$$

and

$$\frac{2}{t-a}\int_{(a+b)/2}^{b} f(s)\,ds\leq\frac{f((a+t)/2)+f(b)}{2},$$

for all $a<t<b$. On summing these inequalities, we obtain

$$\frac{2}{t-a}\int_a^b f(s)\,ds\leq f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2},\qquad t\in[a,b],$$

which implies the inequality (9).

(iii) The left inequality in (10) follows from the fact that

$$\int_a^y f(s)\,ds\geq(y-a)f\left(\frac{a+y}{2}\right),\qquad\text{for all } y\in[a,b].$$

For the right inequality of (10), we use the fact that, by (i), $0\leq P(y)\leq P(b)$, for all $y\in[a,b]$, i.e.,

$$\int_a^y f(s)\,ds-(y-a)f\left(\frac{y+a}{2}\right)\leq\int_a^b f(s)\,ds-(b-a)f\left(\frac{a+b}{2}\right),$$

which is clearly equivalent with the right inequality of (10).

REMARK 3. From the above assumptions, we have

$$\inf_{t\in[a,b]}P(t)=P(a)=0$$

and

$$\sup_{t\in[a,b]}P(t)=P(b)=\int_a^b f(s)\,ds-(b-a)f\left(\frac{a+b}{2}\right)\geq 0.$$

REMARK 4. The condition "f is convex on $[a,b]$" does not imply the convexity of P on $[a,b]$. Indeed, if $f(t)=1/t$, $t\in[1,6]$, then f is convex on $[1,6]$ and

$$P'(t)=\ln t-\frac{2(t-1)}{t+1},\qquad P''(t)=\frac{8t^2-(t+1)^3}{t^2(t+1)^3},$$

and $P''(5)<0$, which shows that P is not convex on $[1,6]$.

REMARK 5. Let f be twice differentiable on I^o and suppose that f and f' are convex on I^o. Then, P is also convex. Indeed, we have

$$P'(t)=f(t)-f\left(\frac{t+a}{2}\right)-\left(\frac{t-a}{2}\right)f'\left(\frac{t+a}{2}\right)$$

and

$$P''(t)=f'(t)-f'\left(\frac{t+a}{2}\right)-\left(\frac{t-a}{4}\right)f''\left(\frac{t+a}{2}\right),$$

for all $t\in[a,b]$, then from the convexity of f', we have

$$f'(t)-f'\left(\frac{t+a}{2}\right)\geq\left(\frac{t-a}{2}\right)f''\left(\frac{t+a}{2}\right),\qquad\text{for all } t\in[a,b],$$

which in view of the convexity of f implies that

$$P''(t)\geq\left(\frac{t-a}{4}\right)f''\left(\frac{t+a}{2}\right),\qquad\text{for all } t\in[a,b].$$

Consequently, P is also convex on $[a,b]$.

3. APPLICATIONS

1. Suppose that $0 \le a < b$ and $y \in [a,b]$. Then, for all $p \ge 1$, we have

$$\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)} \le \frac{1}{b-a}\left[\frac{b^{p+1}-y^{p+1}}{p+1}+\frac{(y-a)(a^p+y^p)}{2}\right] \le \frac{a^p+b^p}{2}.$$

Indeed, it follows from (2) applied to the convex mapping $f:[0,\infty)\to[0,\infty)$, $f(x)=x^p$.

2. Let $0 < a < b$ and $y \in [a,b]$. Then,

$$\left(\frac{b}{a}\right)^{1/(b-a)} \le \left(\frac{b}{y}\right)^{1/(b-a)} \exp\left[\frac{y^2-a^2}{2ay(b-a)}\right] \le \exp\left(\frac{a+b}{2ab}\right).$$

This follows from (2) applied to the convex mapping $f:(0,\infty)\to(0,\infty)$, $f(x)=1/x$.

3. Let $0 \le a < b$ and $t,s \in [a,b]$, $\alpha \in [0,1]$. Then, for all $p \ge 1$, the following inequality holds:

$$\frac{1}{2}\left[\alpha t^p(t-a)+(1-\alpha)s^p(s-a)-(\alpha t+(1-\alpha)s)^p(\alpha t+(1-\alpha)s-a)\right]$$
$$\ge \frac{1}{p+1}\left[\alpha t^{p+1}+(1-\alpha)s^{p+1}-(\alpha t+(1-\alpha)s)^{p+1}\right] \ge 0.$$

This is the inequality (3) applied to the convex function $f(x)=x^p$ defined on $[0,\infty)$.

4. Suppose that $0 < a$ and $t,s \ge a$. Then, for all $\alpha \in [0,1]$, we have the following refinement of the arithmetic-geometric means inequality:

$$\alpha t+(1-\alpha)s \ge t^\alpha s^{1-\alpha}\exp\left[a\left(\frac{\alpha}{t}+\frac{1-\alpha}{s}-\frac{1}{\alpha t+(1-\alpha)s}\right)\right] \ge t^\alpha s^{1-\alpha},$$

which follows from the inequality (3) on applying for the convex function $f:(0,\infty)\to(0,\infty)$, $f(x)=1/x$.

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