ELSEVIER Journal of Computational and Applied Mathematics 58 (1995) 117–133

JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

# The construction of a null basis for a discrete divergence operator

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Received 16 June 1992; revised 2 March 1993

**Abstract**

The divergence free finite element method (DFFEM) is a method to find an approximate solution of the Navier–Stokes equations in a divergence free space. That is, the continuity equation is satisfied a priori. DFFEM eliminates the pressure from the calculations and significantly reduces the dimension of the system to be solved at each time step. For the standard 9-node velocity and 4-node pressure DFFEM, a basis for the divergence-free subspace is constructed such that each basis function has nonzero support on at most 4 contiguous elements. Given this basis, discretely divergence free macro elements can be constructed and used in the implementation of the DFFEM.

*Keywords:* Divergence free; Finite elements; Navier–Stokes; Continuity equation; Dual variables

## 1. Introduction

In this paper, we study one aspect of the divergence free finite element method (DFFEM) (the construction of an appropriate basis) for the efficient numerical solution of the two dimensional incompressible Navier–Stokes problem:

$$N(\boldsymbol{u}) \equiv \frac{\partial \boldsymbol{u}}{\partial t} - \frac{1}{R}\Delta \boldsymbol{u} + (\boldsymbol{u}\nabla)\boldsymbol{u} = -\nabla p + f(\boldsymbol{u}), \quad \boldsymbol{x} \in \Omega \tag{1}$$

$$\nabla \cdot \boldsymbol{u} = 0, \ \boldsymbol{x} \in \Omega \tag{2}$$

$$\boldsymbol{u} = \boldsymbol{u}_{\mathrm{b}}, \ \boldsymbol{x} \in \partial\Omega \tag{3}$$

where $\boldsymbol{u} = (u_1, u_2)$ is the velocity vector, $p$ is the pressure and $\boldsymbol{f}(\boldsymbol{u})$ is a source term. We assume that the domain $\Omega$ is a bounded open connected set with a piecewise smooth boundary $\partial\Omega$. The Reynolds number is defined to be $Re = u_0 d\rho/\mu$ where $\mu$ is the fluid viscosity, $\rho$ is the fluid density, $d$ is a characteristic length and $u_0$ is a characteristic velocity. We assume that the velocity $\boldsymbol{u}_{\mathrm{b}}$ has

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been specified on the boundary $\partial\Omega$, though more general boundary conditions can be handled in similar ways.

In the finite element analysis, two approaches are used to solve Eqs. (1)–(2). One is the standard finite element approach which treats the momentum and continuity equations equally and a large primitive system with both velocity and pressure variables is solved. Another, more efficient approach is the DFFEM approach. The DFFEM treats the continuity equation as a constraint, thus the velocity is approximated not from the standard finite element vector space but from a discretely divergence free finite element subspace. In this approach, the pressures are eliminated from the calculations and the dimension of the system to be solved at each time step is significantly reduced. However, the main obstacle to implementing the divergence free finite element method (DFFEM) is the construction of a basis for the appropriate discretely divergence free subspace, or equivalently, a basis for the associated discrete divergence operator. In Section 3, the previous efforts to obtain such bases are reviewed. We show in Section 4 that, for 9-node isoparametric velocity and 4-node pressure elements, a complete set of basis functions can be chosen with the support of at most 4 contiguous element. Macro elements can then be constructed which are discretely divergence free, and the DFFEM can be applied using these macro elements.

## 2. Two finite element approaches

The finite element method applied to the incompressible Navier–Stokes equations as given in [19, 5] is as follows. Let $S^h$ and $\Lambda^h$ be finite-dimensional subspaces of $H_0^1(\Omega)$ and $L_2(\Omega)/\mathscr{R}$, respectively,

$$S^h = \operatorname{span}\{\Phi_1, \dots, \Phi_L\}, \tag{4}$$

$$\Lambda^h = \operatorname{span}\{\lambda_1, \dots, \lambda_N\}. \tag{5}$$

Then the standard finite element formulation of (1)–(3) is the following.

**Approach 1.** *Find* $(\boldsymbol{u}, p) \in S^h \times \Lambda^h$ *such that*

$$(N\boldsymbol{u}, \boldsymbol{v}) - (\operatorname{div} \boldsymbol{v}, p) = (f, \boldsymbol{v}), \quad \textit{for all } \boldsymbol{v} \in S^h \tag{6}$$

*and*

$$(\operatorname{div} \boldsymbol{u}, \lambda) = 0 \quad \textit{for all } \lambda \in \Lambda^h \tag{7}$$

*where* $(p, q) = \int_\Omega pq \, \mathrm{d}\Omega$.

In order to eliminate the pressure from the computation and hence reduce the number of unknowns, we define the discrete divergence free subspace:

$$D^h = \{\boldsymbol{u} \in S^h \mid (\operatorname{div} \boldsymbol{u}, \lambda) = 0 \text{ for all } \lambda \in \Lambda^h\}. \tag{8}$$

Under this definition, Approach 1 is equivalent to the following which has been studied, for example, in [7, 8, 10, 11].

**Approach 2.** *Find $\boldsymbol{u} \in D^h$ such that*

$$(N\boldsymbol{u}, \boldsymbol{v}) = (f, \boldsymbol{v}) \quad \text{for all } \boldsymbol{v} \in D^h, \tag{9}$$

*and then find $p \in \Lambda^h$ such that*

$$(p, \operatorname{div} \boldsymbol{v}) = (N\boldsymbol{u}, \boldsymbol{v}) - (f, \boldsymbol{v}) \quad \text{for all } \boldsymbol{v} \in I^h, \tag{10}$$

*where $I^h$ is a subspace of $S^h$ and satisfies*

$$S^h = D^h \oplus I^h.$$

The method described in Approach 2 is the *divergence free finite element method* (DFFEM). Efficient use of this method depends on whether the divergence free subspace $D^h$ can be derived at a reasonable cost. If $L = \dim(S^h)$ and $N = \dim(\Lambda^h)$, then the system resulting from (6)–(7) of Approach 1 is of dimension $L + N$, while the system resulting from (9) of Approach 2 is of dimension $L - N$.

As $\lambda$ in (7) varies over a basis for $\Lambda^h$, a system of $N$ equations in $L$ unknowns is generated. The coefficient matrix of this system is called the *discrete divergence operator*. The problem of finding a basis of vector functions for $D^h$ is equivalent to that of finding a null basis for this $N \times L$ matrix.

## 3. The literature

There have been many constructions of explicit bases of the divergence-free subspace for various finite element and finite difference schemes. Griffiths [7–9] obtained an element level divergence free basis for several finite element schemes on triangular and quadrilateral elements. Approximate values of the stream function at corner nodes are used to eliminate the unknown velocity components at midside nodes so that a typical divergence free function on each element is derived. In [7, 8] three types of finite element schemes were investigated on triangular elements which were given in [2]. A divergence free basis was given for a nonconforming velocity field where the components of velocity are represented by piecewise linear functions defined in terms of their values at the midside nodes of the triangles. A divergence free basis also was given for a velocity field where the components of the velocity are piecewise quadratic functions defined in terms of values at the vertices and midside nodes of each triangle. The (discontinuous) piecewise constant pressure space was used for both of the above velocity spaces. Another divergence free subspace derived in [7] involved a velocity field which comes from adding a cubic term to the quadratic representation. The pressure space used was a piecewise linear function with a single element support. Griffiths [9] derived a basis for the divergence free subspace of the 9-node biquadratic element velocity field on quadrilateral elements. The following corresponding pressure spaces were investigated: constant, linear and bilinear elements. The basis functions for these pressure spaces have support on a single element. This allows the incompressible constraint to be analyzed one element at time. But unlike our approach here, the basis functions for the pressure space are discontinuous at element boundaries.

Gustafson and Hartman [10, 11] combine group theory and principles of fluid mechanics to obtain a basis for the divergence free subspace associated with the choice of quadratic velocity and constant pressure triangular elements in two dimensions. Similar results have been obtained in three dimensions for the scheme referred to as APX 3 in [19]. The later work in [11] can be viewed as augmenting and extending their previous work.

Stephens et al. [18] and Goodrich and Soh [6] applied the Galerkin finite difference method (GFDM) to Eqs. (1)–(3). This approach is similar to the Galerkin finite element method. For the GFDM, the discrete finite difference equations approximating (1)–(3) are considered using various subspaces of mesh vector functions and mesh scalar functions (i.e., vector and scalar functions defined only at the nodes of specified finite difference meshes). The subspaces of discrete divergence-free mesh vectors are constructed for several finite difference schemes. It is required that the discrete divergence and discrete gradient operators are formally adjoint. Stephens et al. [18] gave a more general form of GFDM which did not require the adjointness of the discrete divergence and gradient operators. In [4] a subspace of $S^h$ is constructed in which a function satisfies (7) for a subspace of the pressure space. This subspace is the orthogonal complement of the piecewise constant pressure space. This reduces the 5-node velocity and linear discontinuous pressure element to a 4-node velocity and discontinuous constant pressure element.

All of the above constructions of divergence free basis velocity vectors require that the approximations to pressure be discontinuous. Ye [20] and Hall and Ye [15] constructed a divergence free basis for 8-node velocity and 4-node pressure elements where the finite element approximation to pressure is continuous. It was proved that this is the optimal basis in the sense of minimal nonzero support. There must be basis functions with nonzero support of 9 elements. In contrast, the present paper establishes that if $S^h$ is chosen as 9-node velocity elements then there is a basis for $D^h$ with maximal nonzero support of 4 elements. Further, it was shown in [15, 20] that DFFEM is equivalent to applying the dual variable method (DVM) to the standard finite element system (6) and (7). The DVM [1, 3, 12] applied in the context of finite element methods (see [13]) also involves construction of a basis for the discrete divergence operator in (7) and uses this to eliminate the pressure from (6) through a matrix transformation.

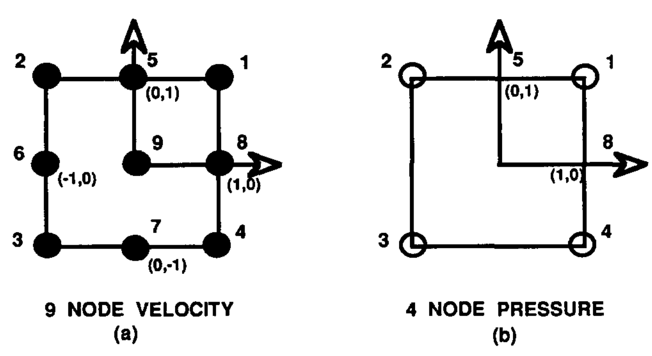



Fig. 1. 9-node velocity and 4-node pressure element.

## 4. Divergence free basis: 9-node velocity and 4-node pressure

In this section we will determine the dimension of and obtain a basis for the null space of the discrete divergence operator or equivalently the divergence free space $D^h$ for the popular choice of 9-node isoparametric velocity elements to generate $S^h$ and 4-node isoparametric pressures to generate $\Lambda^h$. We first consider only uniform meshes on rectangular domains, however extensions to curved domains follow in a fashion similar to [15].

The typical element is shown in Fig. 1. The two velocity components are associated with each of the 9 nodes shown in Fig. 1(a) and the pressures are associated with the 4 corner nodes shown in Fig. 1(b). The nodes shown for velocity and pressure are labeled independently. The 9-node element construction on a master element ($[-1,1]\times[-1,1]$) involves a biquadratic polynomial $\Psi_i$ associated with node $i$ which is one at node $i$ and is zero at the other nodes, and they are defined as

$$\begin{aligned}
&\psi_1 = \tfrac{1}{4}x(x+1)y(y+1), \qquad \psi_2 = \tfrac{1}{4}x(x-1)y(y+1),\\
&\psi_3 = \tfrac{1}{4}x(x-1)y(y-1), \qquad \psi_4 = \tfrac{1}{4}x(x+1)y(y-1),\\
&\psi_5 = -\tfrac{1}{2}(x-1)(x+1)y(y+1), \qquad \psi_6 = -\tfrac{1}{2}x(x-1)(y-1)(y+1),\\
&\psi_7 = -\tfrac{1}{2}(x-1)(x+1)y(y-1), \qquad \psi_8 = -\tfrac{1}{2}x(x+1)(y-1)(y+1),\\
&\psi_9 = (x-1)(x+1)(y-1)(y+1).
\end{aligned} \tag{11}$$

At each corner node $j$, $j = 1,2,3,4$ shown in Fig. 1(b), there is a bilinear polynomial $\beta_j$ defined as

$$\begin{aligned}
&\beta_1 = \tfrac{1}{4}(x+1)(y+1), \qquad \beta_2 = -\tfrac{1}{4}(x-1)(y+1),\\
&\beta_3 = \tfrac{1}{4}(x-1)(y-1), \qquad \beta_4 = -\tfrac{1}{4}(x+1)(y-1).
\end{aligned} \tag{12}$$

We assume, for simplicity, that the velocities on the boundary of the domain are specified and hence we need not consider velocity nodes on the boundary. For the mesh shown in Fig. 2(a), a piecewise biquadratic basis function $\phi_i$ is associated with an interior node $i$ and can be constructed by using the local functions $\psi_j$ defined in (11). Associated with each corner node of an element (including boundary nodes) shown in Fig. 2(b), a piecewise bilinear tent function $\lambda_i$ can be derived by using the local functions $\beta_j$ in (12). The number of $\phi_i$ is the same as the number of interior nodes which is $l \equiv (2m-1)(2n-1)$. The number of $\lambda_j$ is same as the number of corner nodes which is $(m+1)(n+1)$.

We choose the following basis for the finite element velocity space $S^h$:

$$\Phi_1 = (\phi_1, 0)^{\mathrm{T}},\ \Phi_2 = (0, \phi_1)^{\mathrm{T}},\ \ldots,\ \Phi_{L-1} = (\phi_l, 0)^{\mathrm{T}},\ \Phi_L = (0, \phi_l)^{\mathrm{T}} \tag{13}$$

and the following basis for the finite element pressure space $\Lambda^h$:

$$\lambda_1, \lambda_2, \ldots, \lambda_N,$$

where $L = 2l$ is the dimension of $S^h$ and $N = (m+1)(n+1)$ is the dimension of $\Lambda^h$.

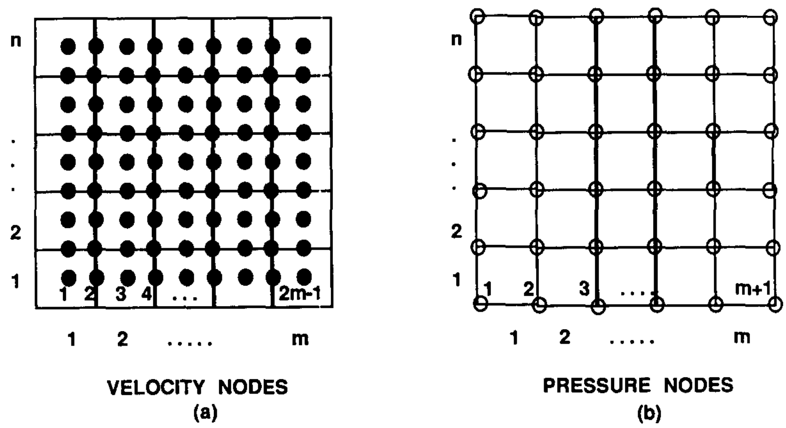



Fig. 2. $m \times n$ mesh.

Define the discrete divergence free subspace $D^h$ of $S^h$:

$$D^h = \{\boldsymbol{v} \in S^h\colon (\operatorname{div} \boldsymbol{v}, \lambda_i) = 0, \quad i = 1, \dots, N\}. \tag{14}$$

We will find the dimension of, and a basis for, $D^h$.

The simple fact is that any element in the vector space $S^h$ (in particular, those in $D^h$) can be expressed as a linear combination of the basis functions $\Phi_i, i = 1, \dots, L$. Therefore, the basis functions of the discrete divergence free subspace $D^h$ can be found if the appropriate coefficients of this combination can be found.

Suppose $D^h = \operatorname{span}\{\Psi_1, \Psi_2, \dots, \Psi_t\}$. We have

$$\Psi_i = \sum_{j=1}^{L} c_{ij}\Phi_j, \quad i = 1, \dots, t.$$

Since $\Psi_i$ is in $D^h$, $c_{ij}$ must satisfy the following equation (see (8) and (14)):

$$(\operatorname{div} \Psi_i, \lambda_k) = \sum_{j=1}^{L} (\operatorname{div} \Phi_j, \lambda_k) c_{ij} = 0, \quad i = 1, \dots, t, \quad k = 1, 2, \dots, N. \tag{15}$$

This implies that $(c_{i1}, c_{i2}, \dots, c_{iL})$ must be the solution of the equation

$$AX = 0,$$

where

$$A = (a_{ij}) = ((\operatorname{div} \Phi_j, \lambda_i)), \quad i = 1, \dots, N, \quad j = 1, \dots, L. \tag{16}$$

To derive the dimension of, and basis for, the discrete divergence free space $D^h$ for any $m \times n$ mesh, the $2 \times 2$ mesh is considered first. A $2 \times 2$ mesh and the order of the nodes for velocity and pressure are shown in Figs. 3(a) and (b), respectively. For the $2 \times 2$ mesh, there are 9 interior nodes and $S^h = \operatorname{span}\{\Phi_1, \Phi_2, \dots, \Phi_{18}\}$, where $\Phi_1, \Phi_2, \dots, \Phi_{18}$ are defined in (13). There are 9 corner nodes and $\Lambda^h = \operatorname{span}\{\lambda_1, \dots, \lambda_9\}$.

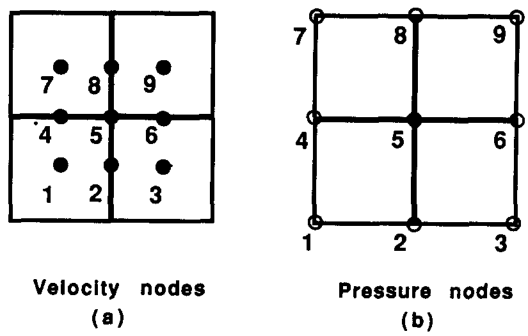



Fig. 3. $2 \times 2$ mesh.

The $9 \times 18$ matrix $A$ in (16) for the $2 \times 2$ mesh is

$$\frac{2h}{9}\begin{vmatrix} 4 & 4 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -4 & 4 & 0 & 4 & 4 & 4 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -4 & 4 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & -4 & 1 & 0 & 0 & 0 & 4 & 0 & 1 & 0 & 0 & 0 & 4 & 4 & 1 & 0 & 0 & 0 \\ -4 & -4 & 0 & -4 & 4 & -4 & -4 & 0 & 0 & 0 & 4 & 0 & -4 & 4 & 0 & 4 & 4 & 4 \\ 0 & 0 & -1 & 0 & -4 & -4 & 0 & 0 & -1 & 0 & -4 & 0 & 0 & 0 & -1 & 0 & -4 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 4 & -4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & -1 & -4 & -4 & 0 & -4 & 4 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & -4 & -4 \end{vmatrix}. \tag{17}$$

Based on Theorem 4 in [15],

$$\dim(D^h) \leqslant (\text{number of the velocity variables}) - (\text{number of the pressure variables}) + 1, \tag{18}$$

and in fact for all cases considered the above inequality was an equality. Hence, the dimension of $D^h \leqslant 18 - 9 + 1 = 10$ where 18 is the number of velocity variables and 9 is the number of pressure variables for the $2 \times 2$ mesh. Using the turnback algorithm [16, 17] the matrix $A$ in (17) is shown to have rank 8 and a basis for the null space is constructed. Manipulation of this basis leads to the following more symmetrical basis $\Psi_1, \Psi_2, \ldots, \Psi_{10}$ for the space $D^h$:

$$\begin{aligned} &\Psi_1 = \Phi_{13} - 4\Phi_{15} + \Phi_{17}, \qquad \Psi_2 = \Phi_1 - 4\Phi_3 + \Phi_5, \\ &\Psi_3 = \Phi_2 - 4\Phi_8 + \Phi_{14}, \qquad \Psi_4 = \Phi_6 - 4\Phi_{12} + \Phi_{18}, \\ &\Psi_5 = -\Phi_1 + \Phi_2 + 8\Phi_9 - 8\Phi_{10} - \Phi_{17} + \Phi_{18}, \\ &\Psi_6 = \Phi_5 + \Phi_6 - 8\Phi_9 - 8\Phi_{10} + \Phi_{13} + \Phi_{14}, \\ &\Psi_7 = 2\Phi_{11} - 2\Phi_{16} - \Phi_{17} + \Phi_{18}, \qquad \Psi_8 = \Phi_1 - 1\Phi_2 + 2\Phi_4 - 2\Phi_7, \\ &\Psi_9 = -2\Phi_4 + \Phi_5 + \Phi_6 - 2\Phi_{11}, \qquad \Psi_{10} = 2\Phi_7 - \Phi_{13} - \Phi_{14} + 2\Phi_{16}. \end{aligned} \tag{19}$$

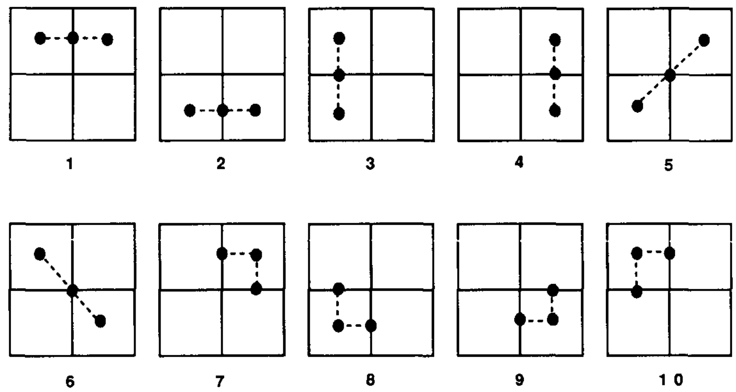



Fig. 4. Support nodes for the divergence free functions of a $2 \times 2$ mesh.

It is easy to verify that $\Psi_1, \Psi_2, \ldots, \Psi_{10}$ belong to the divergence free space $D^h$ for a $2 \times 2$ mesh by checking that the coefficient vector of $\Psi_i$ belongs to the null space of the matrix $A$ in (17).

The linear independence of $\Psi_1, \Psi_2, \ldots, \Psi_{10}$ can be verified by computing the rank of the matrix with the linear combination coefficients of $\Psi_i$ as its columns. In fact, such a matrix has linearly independent columns, therefore $\Psi_1, \Psi_2, \ldots, \Psi_{10}$ are linearly independent. Combining this with (18), we have that $\Psi_1, \Psi_2, \ldots, \Psi_{10}$ form a basis of $D^h$ for a $2 \times 2$ mesh.

The basis vector function $\Psi_i$ has nonzero linear combination coefficients only for certain $\Phi_j$'s. In Fig. 4, for each $\Psi_i$, the nodes are marked if the coefficients are nonzero.

Since any $m \times n$ mesh contains many $2 \times 2$ submeshes, the divergence free functions $\Psi_1, \Psi_2, \ldots, \Psi_{10}$ in (19) for a $2 \times 2$ mesh or macro element can be used as blocks to build a divergence free basis of $D^h$ for an $m \times n$ mesh. We now discuss such a construction. For an $m \times n$ mesh, there are $(m-1)(n-1)$ such macro elements. If 10 functions in (19) are generated for each macro element, then a total of $10(m-1)(n-1)$ functions can be derived. However, by the inequality (18), the dimension of $D^h$ for an $m \times n$ mesh satisfies

$$\dim(D^h) \leqslant 2(2m-1)(2n-1) - (m+1)(n+1) + 1 = 7mn - 5m - 5n + 2, \tag{20}$$

where $(2m-1)(2n-1)$ is the number of interior nodes for velocity and $(m+1)(n+1)$ is the number of the nodes for pressure. Now $10(m-1)(n-1) \geqslant 7mn - 5m - 5n + 2$ for $m, n \geqslant 2$. Hence, the $10(m-1)(n-1)$ functions in $D^h$ generated above will be linearly dependent. In fact, among these $10(m-1)(n-1)$ functions, many of them coincide with each other. For example, if the $2 \times 3$ mesh is considered in Fig. 5, 20 discrete divergence free functions can be derived corresponding to two $2 \times 2$ macro elements. The $\Psi_1$ in (19) for the $2 \times 2$ submesh shaded in Fig. 5(a) is the same as $\Psi_2$ in (19) for the $2 \times 2$ submesh shaded in Fig. 5(b). Similarly, if a $3 \times 2$ mesh is considered, $\Psi_4$ in one $2 \times 2$ submesh will be $\Psi_5$ in another $2 \times 2$ submesh. Based on these simple cases, we have Observation 1.

**Observation 1.** For an $m \times n$ mesh, if 10 functions defined in (19) are derived associated with each $2 \times 2$ submesh, a total of $10(m-1)(n-1)$ such functions can be obtained and $(m-1)(n-2) + (m-2)(n-1) = 2mn - 3m - 3n + 4$ functions of types 1–4 are duplicates.

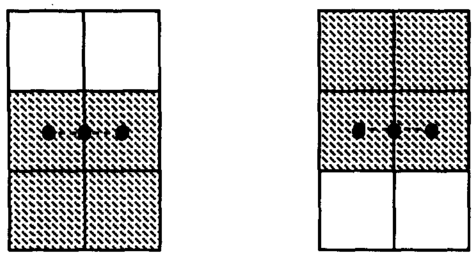



Fig. 5. $2 \times 3$ mesh.

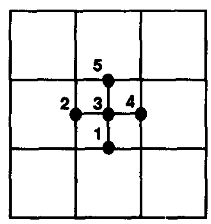



Fig. 6. $3 \times 3$ mesh.

Therefore, the number of the possible independent functions among the $10(m-1)(n-1)$ functions is $10(m-1)(n-1)-(2mn-3m-3n+4)=8mn-7m-7n+6$. By the inequality (20), these $8mn-7m-7n+6$ functions are still linearly dependent for $m, n \geqslant 2$. To obtain a basis for $D^h$, more functions need to be eliminated.

Consider a $3 \times 3$ mesh: there are four $2 \times 2$ macro elements and 10 divergence free functions $\Psi_1, \Psi_2, \ldots, \Psi_{10}$ in (19) can be constructed associated with each of them. Let the five nodes be labeled as in Fig. 6 and $\phi'_1, \phi'_2, \phi'_3, \phi'_4$ and $\phi'_5$ be constructed as piecewise biquadratic basis functions corresponding to these nodes. Then vector functions $\Phi'_1, \Phi'_2, \ldots, \Phi'_{10}$ can be defined as in (13) that is $\Phi'_1=(\phi'_1, 0)^{\mathrm{T}}$, $\Phi'_2=(0, \phi'_1)$, etc. The divergence free vector function $\Psi_7$ in (19) for the $2 \times 2$ macro element shaded in Fig. 7(a) can be written as the linear combination of $\Phi'_1, \ldots, \Phi'_{10}$ as follows:

$$\Psi_7 = 2\Phi'_1 - 2\Phi'_4 - \Phi'_5 + \Phi'_6.$$

Similarly, $\Psi_8$, $\Psi_9$ and $\Psi_{10}$ in (19) for the $2 \times 2$ macro elements shaded in Figs. 7(b), (c) and (d), respectively, can be written as

$$\Psi_8 = \Phi'_5 - \Phi'_6 + 2\Phi'_8 - 2\Phi'_9,$$

$$\Psi_9 = -2\Phi'_4 + \Phi'_5 + \Phi'_6 - 2\Phi'_9,$$

$$\Psi_{10} = 2\Phi'_1 - \Phi'_5 - \Phi'_6 + 2\Phi'_8.$$

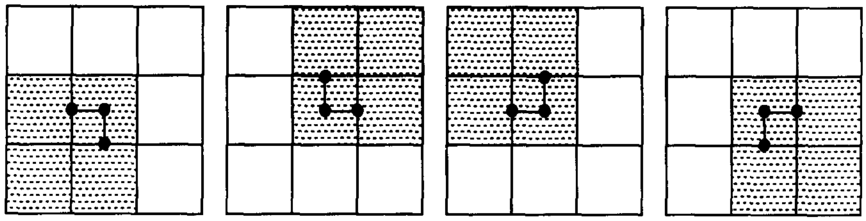



Fig. 7. Four $2 \times 2$ macro elements in a $3 \times 3$ mesh.

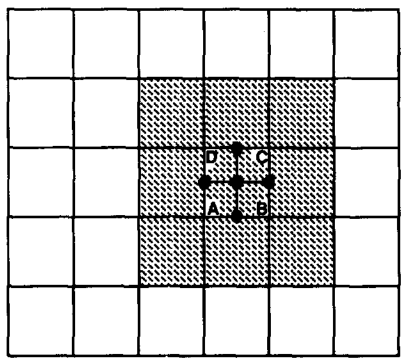



Fig. 8. Typical interior element in an $m \times n$ mesh.

Hence for a $3 \times 3$ mesh, the divergence free vector functions $\Psi_7$, $\Psi_8$, $\Psi_9$ and $\Psi_{10}$ in (19) associated with the four $2 \times 2$ macro elements are linearly dependent since

$$\Psi_7 + \Psi_8 - \Psi_9 - \Psi_{10} = 0.$$

Based on this fact, Observation 2 is stated as follows.

**Observation 2.** For each interior element (ABCD in Fig. 8) in an $m \times n$ mesh, one of the four vector functions of the types $\Psi_7$, $\Psi_8$, $\Psi_9$ and $\Psi_{10}$ has to be eliminated to guarantee their linear independence. For an $m \times n$ mesh, the total number of interior elements is $(m-2)(n-2)$. This implies that another $(m-2)(n-2)$ functions of types 7, 8, 9 or 10 among the $10(m-1)(n-1)$ functions can be eliminated. Now we are ready to construct a discrete divergence free basis of $D^h$ for an $m \times n$ mesh.

**Theorem 1.** *For an $m \times n$ uniform mesh on a rectangular domain shown in Fig.* 2, $7mn - 5m - 5n + 2$ *vector functions in $D^h$ can be formed in the following way.*

*Step* 1. *For each $2 \times 2$ macro element, construct* 10 *functions $\Psi_1, \ldots, \Psi_{10}$ defined as in* (19). *Since there are a total of $(m-1)(n-1)$ different $2 \times 2$ submeshes in an $m \times n$ mesh, a total of* $10(m-1)(n-1)$ *functions can be derived in this way.*

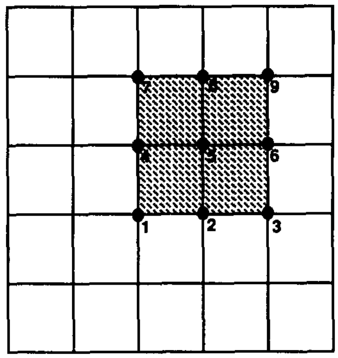



Fig. 9. $2 \times 2$ macro element.

*Step 2. By Observations* 1 *and* 2, *a total of* $(m-2)(n-1)+(m-1)(n-2)+(m-2)(n-2)$ *functions can be eliminated from the* $10(m-1)(n-1)$ *functions obtained in Step* 1. *Therefore, the number of the functions remaining in* $7mn-5m-5n+2$.

*These* $7mn-5m-5n+2$ *functions form a basis for the discrete divergence free space* $D^h$ *such that each basis function has a maximum support of four elements arranged in a* $2 \times 2$ *submesh.*

**Proof.** This proof includes three parts:
(1) prove that these functions are in $D^h$,
(2) prove that they are linearly independent, and
(3) prove that the dimension of $S^h$ is $7mn-5m-5n+2$.
It is obvious that all the functions generated by Steps 1 and 2 are defined on a $2 \times 2$ macro element and vanish outside that element. Let $\Psi$ be one of $7mn-5m-5n+2$ functions and assume it has support on the $2 \times 2$ mesh as shown in Fig. 9. On this $2 \times 2$ submesh, it has been verified

$$(\operatorname{div} \Psi, \lambda_i) = 0, \quad i = 1, 2, \ldots, 9$$

where $\lambda_1, \lambda_2, \ldots, \lambda_9$ are bilinear polynomial basis functions for pressure associated with nodes 1 to 9. For the $\lambda_i \in \Lambda^h$, $i \neq 1, \ldots, 9$, $\Psi$ and $\lambda_i$ have no common nonzero support, hence it is also true that

$$(\operatorname{div} \Psi, \lambda_j) = 0$$

for any other $\lambda_j$ in $\Lambda^h$. Thus $\Psi$ satisfies condition (15), and it is in $D^h$. This finishes the proof of part (1).

Now we prove that the discrete divergence free functions constructed in this theorem are also linearly independent by using mathematical induction on the number of elements in the mesh.

(1) By construction $\Psi_1, \Psi_2, \ldots, \Psi_{10}$ defined in (19) are linearly independent for a $2 \times 2$ mesh.

(2) Assume that the divergence free functions constructed in Steps 1 and 2 are linearly independent for an $m \times n$ mesh and prove that this is also true for an $m \times (n+1)$ mesh.

The $m \times (n+1)$ mesh is constructed by adding one row of elements at the top of the $m \times n$ mesh. At the same time, $2(2m-1)$ interior nodes are added to the mesh marked in Fig. 10 and another $(m-1)$ $2 \times 2$ macro elements are added.

For an $m \times (n+1)$ mesh, $7m(n+1)-5m-5(n+1)+2$ discrete divergence free functions can be obtained by the following two stages.

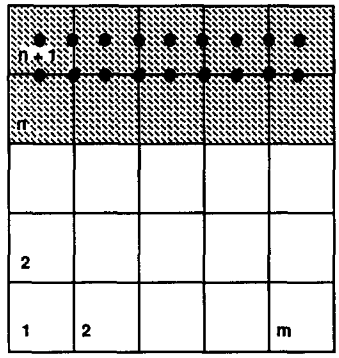



Fig. 10. $2(2m-1)$ new nodes for $m \times (n+1)$ mesh.

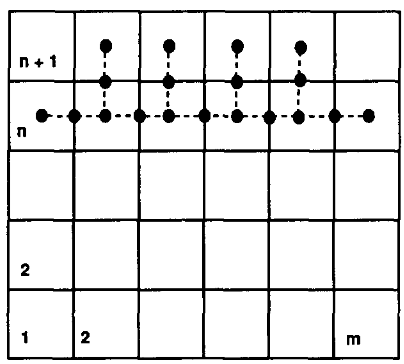



Fig. 11. Supporting nodes of the repeated functions.

*Stage* 1. $7mn - 5m - 5n + 2$ basis functions in $D^h$ for the $m \times n$ submesh are generated by the procedure described in the theorem. Obviously, they also belong to $D^h$ for the $m \times (n+1)$ mesh and are linearly independent.

*Stage* 2. Corresponding to the $m - 1$ $2 \times 2$ macro elements shaded in Fig. 10, $10(m-1)$ discrete divergence free functions can be generated. However, many of them will be duplicates. The $\Psi_i$ of types 1–4 in Fig. 4 with supporting nodes as marked in Fig. 11 are repeated. The total number of them is $2m - 3$. Also from an $m \times n$ mesh to the $m \times (n-1)$ mesh, another $m - 2$ interior elements are added. By Observation 2, $m - 2$ functions should be eliminated from the $10(m-1)$ functions generated in the first stage. Therefore, a total of $10(m-1) - (2m-3) - (m-2)$ discrete divergence free functions can be constructed.

Since $D^h$ is a subspace of $S^h$, these $7m(n+1) - 5m - 5(n+1) + 2$ functions can be expressed as a linear combination of the basis functions for $S^h$. These linear combination coefficients can be used as columns to form a matrix, say $C^{m \times (n+1)}$. $C^{m \times (n+1)}$ is a $2(2m-1)(2n+1) \times 7m(n+1) - 5m - 5(n+1) + 2$ matrix where $2(2m-1)(2n+1)$ is the dimension of $S^h$ for an $m \times (n+1)$ mesh. To prove that these $7m(n+1) - 5m - 5(n+1) + 2$ divergence free functions for the $m \times (n+1)$ are linearly independent is equivalent to proving that the matrix $C^{m\times(n+1)}$ has linearly independent columns. This will follow from the linear independence of the basis functions $\Phi_i$ in $S^h$.

Like the divergence free basis functions, the columns of the matrix $C^{m\times(n+1)}$ can also be divided into two parts as shown in Fig. 12. The first part contains the columns corresponding to the discrete divergence free functions defined on the $m \times n$ submesh generated in Stage 1. Since all of them have support on the $m \times n$ submesh, the linear combination coefficients of these functions corresponding to the basis functions of $S^h$ defined at the $2(2m-1)$ nodes shown in Fig. 10 are zero. This is why the matrix $C^{m\times(n+1)}$ has a zero block matrix in the (2, 1) position. Since the divergence free functions generated in Stage 1 form a basis of $D^h$ associated with the $m \times n$ mesh (by the assumption of mathematical induction), they are linearly independent; consequently the submatrix $C^{m\times n}$ has linearly independent columns. The second block column of the matrix $C^{m\times(n+1)}$ contains the columns corresponding to the discrete divergence free functions constructed in Stage 2.

If the submatrix $C^1$ has linearly independent columns, the matrix $C^{m\times(n+1)}$ has linearly independent columns and the proof is complete.

Now we prove that $C^1$ has linearly independent columns. We first remark that we can consider an $m \times n$ mesh as an extension of an $m \times (n-1)$ mesh. The $7mn - 5m - 5n + 2$ discrete divergence free functions can be constructed also by 2 stages similar to the case for an $m \times (n+1)$ mesh. The corresponding matrix $C^{m\times n}$ has a form in Fig. 13, where $C^{m\times(n-1)}$ has $7m(n-1) - 5m - 5(n-1) + 2$ columns corresponding to the basis functions in $D^h$ for the $m \times (n-1)$ mesh. The matrices $C^1$ in $C^{m\times n}$ and $C^1$ in $C^{m\times(n+1)}$ are identical. But the matrix $C^1$ in Fig. 13 must have linearly independent columns as we now prove by contradiction. Assume this is not the case. Then, there exists a vector $X \neq 0$ such that $C^1 X = 0$ and

$$Y = \begin{pmatrix} C^{n-1} \\ C^1 \end{pmatrix}, \qquad X = \begin{pmatrix} C^{n-1}X \\ 0 \end{pmatrix}.$$

$$C^{m\times(n+1)} = \begin{array}{|c|c|} \hline C^{m\times n} & C^{n} \\ \hline 0 & C^{1} \\ \hline \end{array}$$

Fig. 12. The matrix $C^{m\times(n+1)}$.

$$C^{m\times n} = \begin{array}{|c|c|} \hline C^{m\times(n-1)} & C^{n-1} \\ \hline 0 & C^{1} \\ \hline \end{array}$$

Fig. 13. The matrix $C^{m\times n}$.

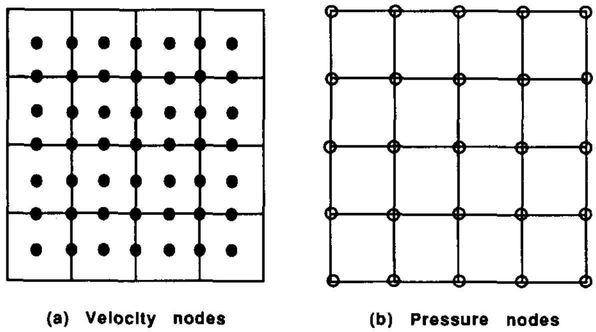



Fig. 14. $4 \times 4$ mesh.

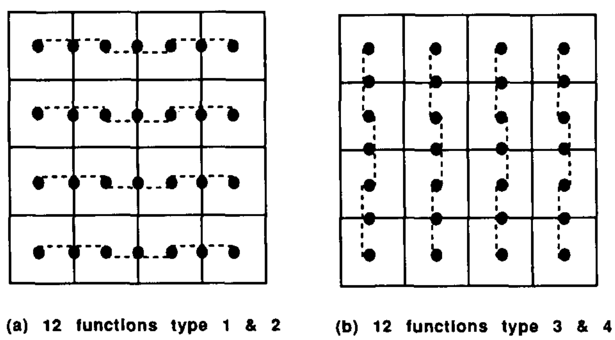



Fig. 15. 24 discrete divergence free functions of types 1–4.

Let $A$ be the matrix in (16) for the $m \times n$ mesh. By the definition of the matrix $A$ and Eq. (15), we have $AC^{m \times n} = 0$. By the definition of the vector $Y$, $AY = 0$. This implies that a function, say $\Psi$ with the components of the vector $Y$ as linear combination coefficients of the basis functions of $S^h$ is in $D^h$ for the $m \times n$ mesh. Now $Y$ has its last $4(2m - 1)$ components zero. These correspond to the basis functions of $S^h$ associated with nodes marked in Fig. 10. Thus, $\Psi$ is defined on the $m \times (n - 1)$ submesh and belongs to $D^h$ for the $m \times (n - 1)$ mesh. Hence, vector $C^{n-1}X$ can be expressed as the linear combination of the columns in $C^{m \times (n-1)}$ because the columns in $C^{m \times (n-1)}$ are corresponding to the basis functions of $D^h$ for $m \times (n - 1)$ mesh. But this implies that the columns in the matrix $C^{m \times n}$ are linearly dependent and contradicts the assumption in mathematical induction that the discrete divergence free functions generated by this theorem for an $m \times n$ mesh are linearly independent. Therefore, the matrix $C^1$ has linearly independent columns. Hence, we have proven that the matrix $C^{m \times (n+1)}$ has linearly independent columns, which is equivalent to the fact that

$7m(n+1) - 5m - 5(n+1) + 2$ discrete divergence free functions generated in the theorem for an $m \times (n+1)$ mesh are linearly independent.

The exact same method can be used to prove that if the discrete divergence free functions derived in this theorem are linearly independent for an $m \times n$ mesh, then they will be linearly independent for an $(m+1) \times n$ mesh. Since the two proofs are very similar, the second proof is omitted here.

Hence, by mathematical inducation the $7mn - 5m - 5n + 2$ discrete divergence free functions derived in this theorem are linearly independent. This finishes the second part of the proof.

By Eq. (18), we have

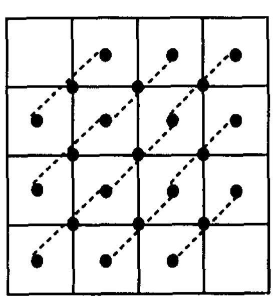
$$\dim(D^h) \leqslant 2(2m-1)(2n-1) - (m+1)(n+1) + 1 = 7mn - 5m - 5n + 2, \tag{21}$$

where $2(2m-1)(2n-1)$ is the number of the velocity variables and $(m+1)(n+1)$ is the number of the pressure variables. Combining Eq. (21) and the linear independence of the $7mn - 5m - 5n + 2$ discrete divergence functions, we have that the dimension of $D^h$ is $7mn - 5m - 5n + 2$. $\square$

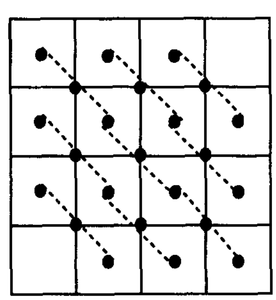
The result of Theorem 1 can be extended to curved domain using appropriate domain transformations as described in [15, 20].

## 5. An example

To illustrate Theorem 1, we consider the $4 \times 4$ mesh in Fig. 14. There are 49 interior nodes for velocity marked in Fig. 14(a) and 25 corner nodes for pressure marked in Fig. 14(b). Hence, the dimensions of $S^h$ and $\Lambda^h$ are 98 and 25, respectively. By Theorem 1, a total of $7 \times 4 \times 4 - 5 \times 4 - 5 \times 4 + 2 = 74$ discrete divergence free function can be constructed. Twelve functions of types 1 and 2 are illustrated in Fig. 15(a) and 12 functions of types 3 and 4 are illustrated in Fig. 15(b). Nine functions of types 5 and 6 are shown in Figs. 16(a) and (b), respectively. Finally, 32 functions of types 7–10 are shown in Fig. 17. Note that in each of these figures, the nodes common to the support of two or more basis functions are physically offset so as to clarify the support of these basis functions.



(a) 9 functions type 5



(b) 9 functions type 6

Fig. 16. 18 discrete divergence free functions of types 5 and 6.

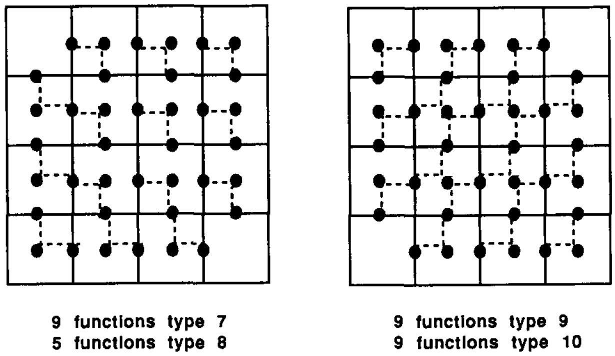



Fig. 17. 32 discrete divergence free functions of types 7–10.

## 6. Conclusions

The divergence free finite element method (DFFEM) for the numerical solution of the incompressible Navier–Stokes equations requires that the velocity be approximated not in the standard finite element space but in the discretely divergence free finite element subspace. There are two inherent advantages to this method. One is that the number of variables is reduced and the other is that the discrete divergence free condition is satisfied a priori, not just approximated as with many other methods.

However the main difficulty to implementing the DFFEM is the lack of an explicit basis for the discretely divergence free subspace. In this paper, a divergence free basis is constructed for the 9-node velocity and 4-node pressure elements. This problem is equivalent to finding a basis for the null space of a discrete divergence operator. The construction of a basis makes use of the turnback algorithm for finding the null space of associated matrices for a $2 \times 2$ macro element. This basis is then modified in such way that translates of the macro element yields a basis for a general $m \times n$ mesh. The basis functions have maximal support of 4 elements arranged in a $2 \times 2$ submesh. This construction also verifies that a basis with smaller maximal support does not exist.

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