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Journal of Functional Analysis 261 (2011) 2741–2796

JOURNAL OF Functional Analysis

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Estimates of operator moduli of continuity ☆

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Received 20 April 2011; accepted 8 July 2011

Communicated by K. Ball

Abstract

In Aleksandrov and Peller (2010) [2] we obtained general estimates of the operator moduli of continuity of functions on the real line. In this paper we improve the estimates obtained in Aleksandrov and Peller (2010) [2] for certain special classes of functions. In particular, we improve estimates of Kato (1973) [18] and show that

$$\big\| |S| - |T| \big\| \leqslant C\|S-T\| \log\left(2 + \log\frac{\|S\|+\|T\|}{\|S-T\|}\right)$$

for all bounded operators S and T on Hilbert space. Here $|S| \stackrel{\text{def}}{=} (S^*S)^{1/2}$. Moreover, we show that this inequality is sharp. We prove in this paper that if f is a nondecreasing continuous function on $\mathbb{R}$ that vanishes on $(-\infty, 0]$ and is concave on $[0, \infty)$, then its operator modulus of continuity Ω_f admits the estimate

$$\Omega_f(\delta) \leqslant \text{const} \int_e^\infty \frac{f(\delta t)\,dt}{t^2 \log t}, \qquad \delta > 0.$$

We also study the problem of sharpness of estimates obtained in Aleksandrov and Peller (2010) [2,3]. We construct a C^∞ function f on $\mathbb{R}$ such that $\|f\|_{L^\infty} \leqslant 1$, $\|f\|_{\mathrm{Lip}} \leqslant 1$, and

☆ The first author is partially supported by RFBR grant 11-01-00526-a; the second author is partially supported by NSF grant DMS 1001844.

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doi:10.1016/j.jfa.2011.07.009

$$\Omega_f(\delta) \geqslant \operatorname{const}\delta\sqrt{\log\frac{2}{\delta}}, \quad \delta \in (0,1].$$

In the last section of the paper we obtain sharp estimates of $\|f(A) - f(B)\|$ in the case when the spectrum of A has n points. Moreover, we obtain a more general result in terms of the ε-entropy of the spectrum that also improves the estimate of the operator moduli of continuity of Lipschitz functions on finite intervals, which was obtained in Aleksandrov and Peller (2010) [2].

Keywords: Operator modulus of continuity; Operator Lipschitz function; Self-adjoint operator; Commutators

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1. Introduction

In this paper we study operator moduli of continuity of functions on subsets of the real line. For a closed subset $\mathfrak{F}$ of the real line $\mathbb{R}$ and for a continuous function f on $\mathfrak{F}$, the *operator modulus of continuity* $\Omega_{f,\mathfrak{F}}$ is defined by

$$\Omega_{f,\mathfrak{F}}(\delta) \stackrel{\text{def}}{=} \sup\bigl\|f(A) - f(B)\bigr\|, \quad \delta > 0,$$

where the supremum is taken over all self-adjoint operators A and B such that

$$\sigma(A) \subset \mathfrak{F}, \qquad \sigma(B) \subset \mathfrak{F}, \quad \text{and} \quad \|A - B\| \leqslant \delta.$$

If $\mathfrak{F} = \mathbb{R}$, we use the notation $\Omega_f \stackrel{\text{def}}{=} \Omega_{f,\mathbb{R}}$. Recall that a continuous function f on $\mathfrak{F}$ is called *operator Lipschitz* if $\Omega_{f,\mathfrak{F}}(\delta) \leqslant \operatorname{const}\delta$, $\delta > 0$.

It turns out that a *Lipschitz function* f on $\mathbb{R}$, i.e., a function f satisfying

$$\bigl|f(x) - f(y)\bigr| \leqslant \operatorname{const}|x - y|, \quad x, y \in \mathbb{R},$$

does not have to be operator Lipschitz. This was established for the first time by Farforovskaya [9]. It was shown later in [18] that the function $x \mapsto |x|$ on $\mathbb{R}$ is not operator Lipschitz.

The paper [18] followed the paper [22], in which it was shown that the function $x \mapsto |x|$ is not commutator Lipschitz. We refer the reader to Section 5 for the definition of commutator Lipschitz functions. Note that nowadays it is well known that operator Lipschitzness is equivalent to commutator Lipschitzness.

We would like to also mention that in [27] necessary conditions for operator Lipschitzness were found that also imply that Lipschitzness is not sufficient for operator Lipschitzness. On the other hand, it was shown in [27] and [28] that if f belongs to the Besov class $B^1_{\infty 1}(\mathbb{R})$, then f is operator Lipschitz (we refer the reader to [25] and [30] for the definition of Besov classes).

In our joint papers [1] and [2] we obtain the following upper estimate for continuous functions f on $\mathbb{R}$:

$$\Omega_f(\delta) \leqslant \text{const}\,\delta \int\limits_\delta^\infty \frac{\omega_f(t)}{t^2}\,dt = \text{const} \int\limits_1^\infty \frac{\omega_f(t\delta)}{t^2}\,dt, \quad \delta > 0, \tag{1.1}$$

where ω_f is the modulus of continuity of f, i.e.,

$$\omega_f(\delta) \stackrel{\text{def}}{=} \sup\bigl\{\bigl|f(x) - f(y)\bigr| \colon x, y \in \mathbb{R},\ |x - y| \leqslant \delta\bigr\}, \quad \delta > 0.$$

We deduced from (1.1) in [2] that for a Lipschitz function f on $[a, b]$, the following estimate for the operator modulus of continuity $\Omega_{f,[a,b]}$ holds:

$$\Omega_{f,[a,b]}(\delta) \leqslant \text{const}\,\delta\biggl(1 + \log\biggl(\frac{b-a}{\delta}\biggr)\biggr)\|f\|_{\text{Lip}},$$

where

$$\|f\|_{\text{Lip}} \stackrel{\text{def}}{=} \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|}.$$

A similar estimate was obtained earlier in [18] in the very special case $f(x) = |x|$. Namely, it was shown in [18] that for bounded self-adjoint operators A and B on Hilbert space, the following inequality holds:

$$\bigl\||A| - |B|\bigr\| \leqslant \frac{2}{\pi}\|A - B\|\biggl(2 + \log\frac{\|A\| + \|B\|}{\|A - B\|}\biggr).$$

It turns out, however, that for the function $x \mapsto |x|$ the operator modulus of continuity admits a much better estimate. Namely, we show in Section 6 that under the same hypotheses

$$\bigl\||A| - |B|\bigr\| \leqslant \text{const}\,\|A - B\| \log\biggl(2 + \log\frac{\|A\| + \|B\|}{\|A - B\|}\biggr).$$

We also prove in this paper that this estimate is sharp.

Note that in [24] an estimate slightly weaker than (1.1) was obtained by a different method.

In Section 8 we show that if f is a continuous nondecreasing function on $\mathbb{R}$ such that $f(x) = 0$ for $x \leqslant 0$ and the restriction of f to $[0, \infty)$ is a concave function, then estimate (1.1) can also be improved considerably:

$$\Omega_f(\delta) \leqslant \operatorname{const} \int_e^\infty \frac{f(\delta t)\,dt}{t^2 \log t}, \quad \delta > 0.$$

We also obtain other estimates of operator moduli of continuity in Section 8.

It is still unknown whether inequality (1.1) is sharp. It follows easily from (1.1) that if f is a function on $\mathbb{R}$ such that $\|f\|_{L^\infty} \leqslant 1$, $\|f\|_{\mathrm{Lip}} \leqslant 1$, then

$$\Omega_f(\delta) \leqslant \operatorname{const} \delta \left(1 + \log \frac{1}{\delta}\right), \quad \delta \in (0, 1].$$

We construct in Section 9 a C^∞ function f on $\mathbb{R}$ such that $\|f\|_{L^\infty} \leqslant 1$, $\|f\|_{\mathrm{Lip}} \leqslant 1$, and

$$\Omega_f(\delta) \geqslant \operatorname{const} \delta \sqrt{\log \frac{2}{\delta}}, \quad \delta \in (0, 1].$$

To construct such a function f, we use necessary conditions for operator Lipschitzness found in [27]. We do not know whether the results of Section 9 are sharp.

In Section 10 we obtain lower estimates in the case of functions on the unit circle and unitary operators.

Finally, we obtain in Section 11 the following sharp estimate for the norms $\|f(A) - f(B)\|$ for Lipschitz functions f and self-adjoint operators A and B on Hilbert space such that the spectrum $\sigma(A)$ of A has n points:

$$\big\|f(A) - f(B)\big\| \leqslant C(1 + \log n)\|f\|_{\mathrm{Lip}}\|A - B\|. \tag{1.2}$$

Moreover, we obtain in Section 11 an upper estimate in the general case (see Theorem 11.5) in terms of the ε-entropy of the spectrum of A, where $\varepsilon = \|A - B\|$. It includes inequalities (1.1) and (1.2) as special cases. Note that (1.2) improves earlier estimates in [9] and [10].

In Section 2 we give a brief introduction to Schur multipliers, in Section 3 we collect auxiliary estimates of certain functions in the space of functions with absolutely converging Fourier integrals. The estimates collected in Section 3 are used in Section 4 to estimate the Schur multiplier norms of certain functions of two variables. To obtain upper estimates for operator moduli of continuity of concave functions, we estimate in Section 7 the operator modulus of continuity of a very special piecewise continuous function on $\mathbb{R}$.

2. Schur multipliers

In this section we define Schur multipliers and discuss their properties. Note that the notion of a Schur multiplier can be defined in the case of two spectral measures (see, e.g., [27]). In this section we define Schur multipliers in the case of two scalar measures. This corresponds to the case of spectral measures of multiplicity 1.

Let $(\mathcal{X},\mu)$ and $(\mathcal{Y},\nu)$ be σ-finite measure spaces. Let $k \in L^2(\mathcal{X}\times\mathcal{Y},\mu\otimes\nu)$. Then k induces the integral operator $\mathcal{I}_k = \mathcal{I}_k^{\mu,\nu}$ from $L^2(\mathcal{Y},\nu)$ to $L^2(\mathcal{X},\mu)$ defined by

$$(\mathcal{I}_k f)(x) = \int\limits_{\mathcal{Y}} k(x,y) f(y)\,d\nu(y), \quad f \in L^2(\mathcal{Y},\nu).$$

Denote by $\|k\|_{\mathcal{B}} = \|k\|_{\mathcal{B}^{\mu,\nu}_{\mathcal{X},\mathcal{Y}}}$ the operator norm of $\mathcal{I}_k$. Let Φ be a $\mu\otimes\nu$-measurable function defined almost everywhere on $\mathcal{X}\times\mathcal{Y}$. We say that Φ is a *Schur multiplier with respect to* μ *and* ν if

$$\|\Phi\|_{\mathfrak{M}^{\mu,\nu}_{\mathcal{X},\mathcal{Y}}} \stackrel{\text{def}}{=} \sup\bigl\{\|\Phi k\|_{\mathcal{B}}\colon k, \Phi k \in L^2(\mathcal{X}\times\mathcal{Y},\mu\otimes\nu),\ \|k\|_{\mathcal{B}} \leqslant 1\bigr\} < \infty.$$

We denote by $\mathfrak{M}^{\mu,\nu}_{\mathcal{X},\mathcal{Y}}$ the space of Schur multipliers with respect to μ and ν. It can be shown easily that $\mathfrak{M}^{\mu,\nu}_{\mathcal{X},\mathcal{Y}} \subset L^\infty(\mathcal{X}\times\mathcal{Y},\mu\otimes\nu)$ and $\|\Phi\|_{L^\infty(\mathcal{X}\times\mathcal{Y},\mu\otimes\nu)} \leqslant \|\Phi\|_{\mathfrak{M}^{\mu,\nu}_{\mathcal{X},\mathcal{Y}}}$. Thus if $\Phi \in \mathfrak{M}^{\mu,\nu}_{\mathcal{X},\mathcal{Y}}$, then

$$\|\Phi\|_{\mathfrak{M}^{\mu,\nu}_{\mathcal{X},\mathcal{Y}}} = \sup\bigl\{\|\Phi k\|_{\mathcal{B}}\colon k \in L^2(\mathcal{X}\times\mathcal{Y},\mu\otimes\nu),\ \|k\|_{\mathcal{B}} \leqslant 1\bigr\}.$$

Note that $\mathfrak{M}^{\mu,\nu}_{\mathcal{X},\mathcal{Y}}$ is a Banach algebra:

$$\|\Phi_1\Phi_2\|_{\mathfrak{M}^{\mu,\nu}_{\mathcal{X},\mathcal{Y}}} \leqslant \|\Phi_1\|_{\mathfrak{M}^{\mu,\nu}_{\mathcal{X},\mathcal{Y}}}\|\Phi_2\|_{\mathfrak{M}^{\mu,\nu}_{\mathcal{X},\mathcal{Y}}}.$$

It is easy to see that $\|\Phi\|_{\mathfrak{M}^{\mu,\nu}_{\mathcal{X},\mathcal{Y}}} = \|\Psi\|_{\mathfrak{M}^{\nu,\mu}_{\mathcal{Y},\mathcal{X}}}$ for $\Psi(y,x) = \Phi(x,y)$.

If $\mathcal{X}_0$ is a μ-measurable subset of $\mathcal{X}$, then we denote by $(\mathcal{X}_0,\mu)$ the corresponding measure space on the σ-algebra of μ-measurable subsets of $\mathcal{X}_0$.

Let $\mathcal{X} = \bigcup_{n=1}^\infty \mathcal{X}_n$ and $\mathcal{Y} = \bigcup_{n=1}^\infty \mathcal{Y}_n$, where the $\mathcal{X}_n$ are μ-measurable subsets of $\mathcal{X}$, and the $\mathcal{Y}_n$ are ν-measurable subsets of $\mathcal{Y}$. It is easy to see that

$$\sup_{m,n\geqslant 1} \|k\|^2_{\mathcal{B}^{\mu,\nu}_{\mathcal{X}_m,\mathcal{Y}_n}} \leqslant \|k\|^2_{\mathcal{B}^{\mu,\nu}_{\mathcal{X},\mathcal{Y}}} \leqslant \sum_{m=1}^\infty\sum_{n=1}^\infty \|k\|^2_{\mathcal{B}^{\mu,\nu}_{\mathcal{X}_m,\mathcal{Y}_n}}$$

for every $k \in L^2(\mathcal{X}\times\mathcal{Y},\mu\otimes\nu)$, and

$$\sup_{m,n\geqslant 1} \|\Phi\|^2_{\mathfrak{M}^{\mu,\nu}_{\mathcal{X}_m,\mathcal{Y}_n}} \leqslant \|\Phi\|^2_{\mathfrak{M}^{\mu,\nu}_{\mathcal{X},\mathcal{Y}}} \leqslant \sum_{m=1}^\infty\sum_{n=1}^\infty \|\Phi\|^2_{\mathfrak{M}^{\mu,\nu}_{\mathcal{X}_m,\mathcal{Y}_n}} \tag{2.1}$$

for every $\Phi \in L^\infty(\mathcal{X}\times\mathcal{Y},\mu\otimes\nu)$.

We state the following elementary theorem:

Theorem 2.1. *Let $(\mathcal{X},\mu)$, $(\mathcal{X},\mu_0)$, $(\mathcal{Y},\nu)$ and $(\mathcal{Y},\nu_0)$ be σ-finite measure spaces. Suppose that μ_0 is absolutely continuous with respect to μ and ν_0 is absolutely continuous with respect to ν. Let $\Phi \in \mathfrak{M}^{\mu,\nu}_{\mathcal{X},\mathcal{Y}}$. Then $\Phi \in \mathfrak{M}^{\mu_0,\nu_0}_{\mathcal{X},\mathcal{Y}}$ and $\|\Phi\|_{\mathfrak{M}^{\mu_0,\nu_0}_{\mathcal{X},\mathcal{Y}}} \leqslant \|\Phi\|_{\mathfrak{M}^{\mu,\nu}_{\mathcal{X},\mathcal{Y}}}$.*

Proof. By the Radon–Nikodym theorem, $d\mu_0 = \varphi\, d\mu$ and $d\nu_0 = \psi\, d\nu$ for nonnegative measurable functions φ and ψ on $\mathcal{X}$ and $\mathcal{Y}$. Let $k \in L^2(\mathcal{X}\times\mathcal{Y}, \mu_0\otimes\nu_0)$. Put

$$(Tk)(x,y) \stackrel{\text{def}}{=} k(x,y)\sqrt{\varphi(x)\psi(y)}.$$

Clearly, T is an isometric embedding from $L^2(\mathcal{X}\times\mathcal{Y}, \mu_0\otimes\nu_0)$ in $L^2(\mathcal{X}\times\mathcal{Y}, \mu\otimes\nu)$. Moreover, $\|Tk\|_{\mathcal{B}^{\mu,\nu}_{\mathcal{X},\mathcal{Y}}} = \|k\|_{\mathcal{B}^{\mu_0,\nu_0}_{\mathcal{X},\mathcal{Y}}}$. We have

$$\begin{aligned}\|\Phi k\|_{\mathcal{B}^{\mu_0,\nu_0}_{\mathcal{X},\mathcal{Y}}} &= \big\|T(\Phi k)\big\|_{\mathcal{B}^{\mu,\nu}_{\mathcal{X},\mathcal{Y}}} = \|\Phi Tk\|_{\mathcal{B}^{\mu,\nu}_{\mathcal{X},\mathcal{Y}}}\\ &\leqslant \|\Phi\|_{\mathfrak{M}^{\mu,\nu}_{\mathcal{X},\mathcal{Y}}}\|Tk\|_{\mathcal{B}^{\mu,\nu}_{\mathcal{X},\mathcal{Y}}} = \|\Phi\|_{\mathfrak{M}^{\mu,\nu}_{\mathcal{X},\mathcal{Y}}}\|k\|_{\mathcal{B}^{\mu_0,\nu_0}_{\mathcal{X},\mathcal{Y}}}\end{aligned}$$

for every $k \in L^2(\mathcal{X}\times\mathcal{Y}, \mu_0\otimes\nu_0)$. Hence, $\Phi \in \mathfrak{M}^{\mu_0,\nu_0}_{\mathcal{X},\mathcal{Y}}$ and $\|\Phi\|_{\mathfrak{M}^{\mu_0,\nu_0}_{\mathcal{X},\mathcal{Y}}} \leqslant \|\Phi\|_{\mathfrak{M}^{\mu,\nu}_{\mathcal{X},\mathcal{Y}}}$. $\square$

Note that if $\mathcal{X}$ and $\mathcal{Y}$ coincide with the set $\mathbb{Z}_+$ of nonnegative integers and μ and ν are the counting measure, the above definition coincides with the definition of Schur multipliers on the space of matrices: a matrix $A = \{a_{jk}\}_{j,k\geqslant 0}$ is called a Schur multiplier on the space of bounded matrices if

$$A \star B \quad \text{is a matrix of a bounded operator, whenever} \quad B \quad \text{is.}$$

Here we use the notation

$$A \star B = \{a_{jk}b_{jk}\}_{j,k\geqslant 0} \tag{2.2}$$

for the Schur–Hadamard product of the matrices $A = \{a_{jk}\}_{j,k\geqslant 0}$ and $B = \{b_{jk}\}_{j,k\geqslant 0}$.

Let $\mathcal{X}$ and $\mathcal{Y}$ be closed subsets of $\mathbb{R}$. We denote by $\mathfrak{M}_{\mathcal{X},\mathcal{Y}}$ the space of Borel Schur multipliers on $\mathcal{X}\times\mathcal{Y}$, i.e., the space of Borel functions Φ defined everywhere on $\mathcal{X}\times\mathcal{Y}$ such that

$$\|\Phi\|_{\mathfrak{M}_{\mathcal{X},\mathcal{Y}}} \stackrel{\text{def}}{=} \sup \|\Phi\|_{\mathfrak{M}^{\mu,\nu}_{\mathcal{X},\mathcal{Y}}} < \infty,$$

where the supremum is taken over all regular positive Borel measures μ and ν on $\mathcal{X}$ and $\mathcal{Y}$. It can be shown easily that

$$\sup_{(x,y)\in\mathcal{X}\times\mathcal{Y}} \big|\Phi(x,y)\big| \leqslant \|\Phi\|_{\mathfrak{M}_{\mathcal{X},\mathcal{Y}}}.$$

It is also easy to verify that if $\Phi_n \in \mathfrak{M}_{\mathcal{X},\mathcal{Y}}$, Φ is a bounded Borel function on $\mathcal{X}\times\mathcal{Y}$, and $\Phi_n(x,y)\to\Phi(x,y)$ for all $(x,y)\in\mathcal{X}\times\mathcal{Y}$, then

$$\|\Phi\|_{\mathfrak{M}_{\mathcal{X},\mathcal{Y}}} \leqslant \liminf_{n\to\infty}\|\Phi_n\|_{\mathfrak{M}_{\mathcal{X},\mathcal{Y}}}.$$

In particular, $\Phi \in \mathfrak{M}_{\mathcal{X},\mathcal{Y}}$ if $\liminf_{n\to\infty}\|\Phi_n\|_{\mathfrak{M}_{\mathcal{X},\mathcal{Y}}} < \infty$.

We are going to deal with functions f on $\mathcal{X}\times\mathcal{Y}$ that are continuous in each variable. It must be a well-known fact that such a function f has to be a Borel function. Indeed, one can construct an increasing sequence $\{\mathcal{Y}_n\}_{n=1}^\infty$ of discrete closed subsets of $\mathcal{Y}$ such that $\bigcup_{n=1}^\infty \mathcal{Y}_n$ is dense in $\mathcal{Y}$. Let us consider the function $f_n : \mathcal{X} \times \mathbb{R} \to \mathbb{C}$ such that $f \big| (\mathcal{X} \times \mathcal{Y}_n) = f_n \big| (\mathcal{X} \times \mathcal{Y}_n)$ and $f_n(x, \cdot)$ is a piecewise linear function with nodes in $\mathcal{Y}_n$ for all $x \in \mathcal{X}$. Clearly, the function f_n is defined uniquely if we require that $f_n(x, \cdot)$ is constant on each unbounded complimentary interval of $\mathcal{Y}_n$. It is easy to see that f_n is continuous on $\mathcal{X} \times \mathbb{R}$ and $\lim_{n\to\infty} f_n(x, y) = f(x, y)$ for all $(x, y) \in \mathcal{X} \times \mathcal{Y}$. Thus, f belongs to the first Baire class, and so it is Borel.

Theorem 2.2. *Let $\mathcal{X}$ and $\mathcal{Y}$ be closed subsets of $\mathbb{R}$ and let Φ be a function on $\mathcal{X} \times \mathcal{Y}$ that is continuous in each variables. Suppose that μ and μ_0 are positive regular Borel measures on $\mathcal{X}$, and ν and ν_0 are positive regular Borel measures on $\mathcal{Y}$. If* $\operatorname{supp}\mu_0 \subset \operatorname{supp}\mu$ *and* $\operatorname{supp}\nu_0 \subset \operatorname{supp}\nu$, *then* $\|\Phi\|_{\mathfrak{M}^{\mu_0,\nu_0}_{\mathcal{X},\mathcal{Y}}} \leqslant \|\Phi\|_{\mathfrak{M}^{\mu,\nu}_{\mathcal{X},\mathcal{Y}}}$.

We need two lemmata.

Lemma 2.3. *Let $\mathcal{X}$ and $\mathcal{Y}$ be compact subsets of $\mathbb{R}$ and let μ and ν be finite positive Borel measures on $\mathcal{X}$ and $\mathcal{Y}$. Suppose that $\{\nu_j\}_{j=1}^{\infty}$ is a sequence of finite positive Borel measures on $\mathcal{Y}$ that converges to ν in the weak-$*$ topology $\sigma((C(\mathcal{Y}))^*, C(\mathcal{Y}))$. If k is a bounded Borel function on $\mathcal{X} \times \mathcal{Y}$ such that $k(x, \cdot) \in C(\mathcal{Y})$ for every $x \in \mathcal{X}$, then*

$$\lim_{j\to\infty} \big\| \mathcal{I}_k^{\mu,\nu_j} \big\|_{\mathcal{B}^{\mu,\nu_j}_{\mathcal{X},\mathcal{Y}}} = \big\| \mathcal{I}_k^{\mu,\nu} \big\|_{\mathcal{B}^{\mu,\nu}_{\mathcal{X},\mathcal{Y}}}.$$

Proof. Clearly, $\mathcal{I}_k^{\mu,\nu_j}(\mathcal{I}_k^{\mu,\nu_j})^*$ is an integral operator on $L^2(\mathcal{X}, \mu)$ with kernel $l_j(x, y) = \int_{\mathcal{Y}} k(x, t)\overline{k(y, t)}\, d\nu_j(t)$. Besides, the sequence $\{l_j\}$ converges in $L^2(\mathcal{X} \times \mathcal{X}, \mu \otimes \mu)$ to the function l defined by $l(x, y) = \int_{\mathcal{Y}} k(x, t)\overline{k(y, t)}\, d\nu(t)$, which is the kernel of the integral operator $\mathcal{I}_k^{\mu,\nu}(\mathcal{I}_k^{\mu,\nu})^*$. Hence,

$$\begin{aligned}\lim_{j\to\infty} \big\| \mathcal{I}_k^{\mu,\nu_j} \big\|^2_{\mathcal{B}^{\mu,\nu_j}_{\mathcal{X},\mathcal{Y}}} &= \lim_{j\to\infty} \big\| \mathcal{I}_k^{\mu,\nu_j}\big(\mathcal{I}_k^{\mu,\nu_j}\big)^* \big\|_{\mathcal{B}^{\mu,\nu_j}_{\mathcal{X},\mathcal{Y}}} \\ &= \big\| \mathcal{I}_k^{\mu,\nu}\big(\mathcal{I}_k^{\mu,\nu}\big)^* \big\|_{\mathcal{B}^{\mu,\nu}_{\mathcal{X},\mathcal{Y}}} = \big\| \mathcal{I}_k^{\mu,\nu} \big\|^2_{\mathcal{B}^{\mu,\nu}_{\mathcal{X},\mathcal{Y}}}. \qquad \square\end{aligned}$$

Corollary 2.4. *Let $\mathcal{X}$ and $\mathcal{Y}$ be compact subsets of $\mathbb{R}$, and let μ and ν be finite positive Borel measures on $\mathcal{X}$ and $\mathcal{Y}$. Suppose that $\{\nu_j\}_{j=1}^{\infty}$ is a sequence of finite positive Borel measures on $\mathcal{Y}$ that converges to ν in $\sigma((C(\mathcal{Y}))^*, C(\mathcal{Y}))$. If Φ is a Borel function on $\mathcal{X} \times \mathcal{Y}$ such that $\Phi(x, \cdot) \in C(\mathcal{Y})$ for all $x \in \mathcal{X}$, then $\|\Phi\|_{\mathfrak{M}^{\mu,\nu}_{\mathcal{X},\mathcal{Y}}} \leqslant \liminf_{j\to\infty} \|\Phi\|_{\mathfrak{M}^{\mu,\nu_j}_{\mathcal{X},\mathcal{Y}}}$.*

Proof. It is easy to see that

$$\|\Phi\|_{\mathfrak{M}^{\mu,\nu}_{\mathcal{X},\mathcal{Y}}} = \sup\big\{ \|\Phi k\|_{\mathcal{B}^{\mu,\nu}_{\mathcal{X},\mathcal{Y}}} \colon k \in C(\mathcal{X} \times \mathcal{Y}),\ \|k\|_{\mathcal{B}^{\mu,\nu}_{\mathcal{X},\mathcal{Y}}} \leqslant 1 \big\}.$$

Let $k \in C(\mathcal{X} \times \mathcal{Y})$ with $\|k\|_{L^2(\mu\otimes\nu)} > 0$. Then

$$\begin{aligned}\|\Phi k\|_{\mathcal{B}^{\mu,\nu}_{\mathcal{X},\mathcal{Y}}} &= \lim_{j\to\infty} \|\Phi k\|_{\mathcal{B}^{\mu,\nu_j}_{\mathcal{X},\mathcal{Y}}} \leqslant \liminf_{j\to\infty} \big(\|\Phi\|_{\mathfrak{M}^{\mu,\nu_j}_{\mathcal{X},\mathcal{Y}}} \|k\|_{\mathcal{B}^{\mu,\nu_j}_{\mathcal{X},\mathcal{Y}}} \big) \\ &= \liminf_{j\to\infty} \|\Phi\|_{\mathfrak{M}^{\mu,\nu_j}_{\mathcal{X},\mathcal{Y}}} \lim_{j\to\infty} \|k\|_{\mathcal{B}^{\mu,\nu_j}_{\mathcal{X},\mathcal{Y}}} = \|k\|_{\mathcal{B}^{\mu,\nu}_{\mathcal{X},\mathcal{Y}}} \liminf_{j\to\infty} \|\Phi\|_{\mathfrak{M}^{\mu,\nu_j}_{\mathcal{X},\mathcal{Y}}}\end{aligned}$$

which implies the result. $\square$

We are going to use the following notation: for a measure μ and an integrable function φ, we write $\nu=\varphi\mu$ if ν is the (complex) measure defined by $d\nu=\varphi\,d\mu$.

The following fact can be proved very easily.

Lemma 2.5. *Let ν and ν_0 be finite Borel measures on $\mathbb{R}$ with compact supports. Suppose that* $\operatorname{supp}\nu_0\subset\operatorname{supp}\nu$. *Then there exists a sequence $\{\varphi_j\}_{j=1}^\infty$ in $C(\mathbb{R})$ such that $\varphi_j\geqslant 0$ everywhere on $\mathbb{R}$ for all j and $\nu_0=\lim_{j\to\infty}\varphi_j\nu$ in $\sigma((C(\operatorname{supp}\nu))^*,C(\operatorname{supp}\nu))$.*

Proof of Theorem 2.2. Put $\mathcal{X}_n\stackrel{\text{def}}{=}[-n,n]\cap X$ and $\mathcal{Y}_n\stackrel{\text{def}}{=}[-n,n]\cap Y$. Clearly, $\{\|\Phi\|_{\mathfrak{M}^{\mu,\nu}_{\mathcal{X}_n,\mathcal{Y}_n}}\}$ is a nondecreasing sequence and

$$\lim_{n\to\infty}\|\Phi\|_{\mathfrak{M}^{\mu,\nu}_{\mathcal{X}_n,\mathcal{Y}_n}}=\|\Phi\|_{\mathfrak{M}^{\mu,\nu}_{\mathcal{X},\mathcal{Y}}}.$$

This allows us to reduce the general case to the case when $\mathcal{X}$ and $\mathcal{Y}$ are compact. Besides, it suffices to consider the case where $\mu_0=\mu$. Indeed, the case $\nu_0=\nu$ can be reduced to the case $\mu_0=\mu$, and we have

$$\|\Phi\|_{\mathfrak{M}^{\mu_0,\nu_0}_{\mathcal{X},\mathcal{Y}}}\leqslant\|\Phi\|_{\mathfrak{M}^{\mu,\nu_0}_{\mathcal{X},\mathcal{Y}}}\leqslant\|\Phi\|_{\mathfrak{M}^{\mu,\nu}_{\mathcal{X},\mathcal{Y}}}.$$

Let $\mathcal{X}$ and $\mathcal{Y}$ be compact, and $\mu=\mu_0$. Applying Lemma 2.5, we can take a sequence $\{\varphi_j\}_{j=1}^\infty$ of nonnegative functions in $C(\mathbb{R})$ such that $\nu_0=\lim_{j\to\infty}\varphi_j\nu$ in the weak topology $\sigma((C(\mathcal{Y}))^*,C(\mathcal{Y}))$. Put $\nu_j\stackrel{\text{def}}{=}\varphi_j\nu$. By Theorem 2.1, $\|\Phi\|_{\mathfrak{M}^{\mu,\nu_j}_{\mathcal{X},\mathcal{Y}}}\leqslant\|\Phi\|_{\mathfrak{M}^{\mu,\nu}_{\mathcal{X},\mathcal{Y}}}$ for every $j\geqslant 1$. It remains to apply Corollary 2.4. $\square$

Theorem 2.2 implies the following fact:

Theorem 2.6. *Let $\mathcal{X}$ and $\mathcal{Y}$ be closed subsets of $\mathbb{R}$ and let Φ be a function on $\mathcal{X}\times\mathcal{Y}$ that is continuous in each variables. Suppose that μ and ν are positive regular Borel measures on $\mathcal{X}$ and $\mathcal{Y}$ such that* $\operatorname{supp}\mu=\mathcal{X}$ *and* $\operatorname{supp}\nu=\mathcal{Y}$. *Then $\|\Phi\|_{\mathfrak{M}_{\mathcal{X},\mathcal{Y}}}=\|\Phi\|_{\mathfrak{M}^{\mu,\nu}_{\mathcal{X},\mathcal{Y}}}$.*

The following result is well known.

Let $f\in C(\mathbb{R})$. Put $\Phi(x,y)\stackrel{\text{def}}{=}f(x-y)$. Then $\Phi\in\mathfrak{M}_{\mathbb{R},\mathbb{R}}$ if and only if f is the Fourier transform of a complex measure on $\mathbb{R}$. Moreover, $\|\Phi\|_{\mathfrak{M}_{\mathbb{R},\mathbb{R}}}=|\mu|(\mathbb{R})$.

A similar statement holds for any locally compact abelian group. In particular, it is true for the group $\mathbb{Z}$:

Let f be a function defined on $\mathbb{Z}$. Put $\Phi(m,n)\stackrel{\text{def}}{=}f(m-n)$. Then $\Phi\in\mathfrak{M}_{\mathbb{Z},\mathbb{Z}}$ if and only if $\{f(n)\}_{n\in\mathbb{Z}}$ are the Fourier coefficients of a complex Borel measure μ on the unit circle $\mathbb{T}$. Moreover, $\|\Phi\|_{\mathfrak{M}_{\mathbb{Z},\mathbb{Z}}}=|\mu|(\mathbb{T})$.

We need the following well-known fact.

Lemma 2.7. *Let*

$$H(m,n) \stackrel{\text{def}}{=} \begin{cases} \frac{1}{m-n}, & \text{if } m,n \in \mathbb{Z},\ m \neq n, \\ 0, & \text{if } m = n \in \mathbb{Z}. \end{cases}$$

Then $\|H\|_{\mathfrak{M}_{\mathbb{Z},\mathbb{Z}}} = \frac{\pi}{2}$.

Proof. It suffices to observe that

$$H(n,0) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{i}(\pi - t)e^{-\mathrm{i}nt}\,dt \quad \text{and} \quad \frac{1}{2\pi}\int_0^{2\pi} |\pi - t|\,dt = \frac{\pi}{2}. \qquad \square$$

3. Remarks on absolutely convergent Fourier integrals

In this section we collect elementary estimates of certain functions in the space of absolutely convergent Fourier integrals. Such estimates will be used in the next section for estimates of certain functions in the space of Schur multipliers.

We are going to deal with the space

$$\widehat{L}^1 = \widehat{L}^1(\mathbb{R}) \stackrel{\text{def}}{=} \mathscr{F}\big(L^1(\mathbb{R})\big), \qquad \|f\|_{\widehat{L}^1} = \|f\|_{\widehat{L}^1(\mathbb{R})} \stackrel{\text{def}}{=} \big\|\mathscr{F}^{-1}f\big\|_{L^1}.$$

Here we use the notation $\mathscr{F}$ for *Fourier transform*:

$$(\mathscr{F}f)(x) \stackrel{\text{def}}{=} \int_{\mathbb{R}} f(t)e^{-\mathrm{i}xt}\,dt, \quad f \in L^1(\mathbb{R}).$$

Unless otherwise stated, an interval throughout the paper means a closed nondegenerate (not necessarily finite) interval. For such an interval J, we consider the class $\widehat{L}^1(J)$ defined by $\widehat{L}^1(J) \stackrel{\text{def}}{=} \{f|J\colon f \in \widehat{L}^1\}$. If $f \in C(J)$, we put

$$\|\varphi\|_{\widehat{L}^1(J)} \stackrel{\text{def}}{=} \inf\big\{\|f\|_{\widehat{L}^1}\colon f|J = \varphi\big\}.$$

For $\varphi \in C(\mathbb{R})$, we put $\|\varphi\|_{\widehat{L}^1(J)} \stackrel{\text{def}}{=} \|\varphi|J\|_{\widehat{L}^1(J)}$. Clearly, $\|\varphi\|_{L^\infty(J)} \leqslant \|\varphi\|_{\widehat{L}^1(J)}$.

For an interval J, *we use the notation* $|J|$ *for its length.*

It is easy to see that the constant functions belong to the space $\widehat{L}^1(J)$ for bounded intervals J and $\|\mathbf{1}\|_{\widehat{L}^1(J)} = 1$. Moreover,

$$\widehat{L}^1(J) = \big\{(\mathscr{F}\mu)|J\colon \mu \in \mathscr{M}(\mathbb{R})\big\} \quad \text{and} \quad \|f\|_{\widehat{L}^1(J)} = \inf\big\{\|\mu\|_{\mathscr{M}}\colon (\mathscr{F}\mu)|J = f\big\}$$

for every bounded interval J, where $\mathscr{M}(\mathbb{R})$ denotes the space of (complex) Borel measures on $\mathbb{R}$.

In this section we are going to discuss (mostly known) estimates for $\|\cdot\|_{\widehat{L}^1(J)}$.

First, we recall the Pólya theorem, see [32].

Let f be an even continuous function such that $f\big|[0,\infty)$ is a decreasing convex function vanishing at the infinity. Then $f \in \widehat{L}^1$ and $\|f\|_{\widehat{L}^1} = f(0)$.

This theorem readily implies the following fact.

Lemma 3.1. *Let f be a continuous function on a closed ray J that vanishes at infinity. Suppose that f is monotone and convex* (*or concave*). *Then $f \in \widehat{L}^1(J)$ and $\|f\|_{\widehat{L}^1(J)} = \max_J |f|$.*

In what follows by a *locally absolutely continuous function* on $\mathbb{R}$ we mean a function whose restriction to any compact interval is absolutely continuous.

Lemma 3.2. *Let f be a locally absolutely continuous function in $L^2(\mathbb{R})$ such that $f' \in L^2(\mathbb{R})$. Then $f \in \widehat{L}^1(\mathbb{R})$ and $\|f\|^2_{\widehat{L}^1} \leqslant \|f\|_{L^2}\|f'\|_{L^2}$.*

Proof. Put $a = \|f\|_{L^2}$, $b = \|f'\|_{L^2}$. By Plancherel's theorem,

$$\|\mathscr{F}^{-1} f\|^2_{L^2} = \frac{a^2}{2\pi} \quad \text{and} \quad \|x\mathscr{F}^{-1} f\|^2_{L^2} = \frac{b^2}{2\pi}.$$

Hence,

$$\left\|\sqrt{b^2 + a^2x^2}\,\mathscr{F}^{-1} f\right\|^2_{L^2} = \frac{a^2b^2}{\pi}$$

and by the Cauchy–Bunyakovsky inequality,

$$\|\mathscr{F}^{-1} f\|_{L^1} \leqslant \frac{ab}{\sqrt{\pi}} \left\| \frac{1}{\sqrt{a^2x^2 + b^2}} \right\|_{L^2} = \sqrt{ab}. \qquad \square$$

Corollary 3.3. *Let $a > 0$. Put*

$$f_a(x) \stackrel{\text{def}}{=} \begin{cases} a^{-2}x, & \text{if } |x| \leqslant a, \\ x^{-1}, & \text{if } |x| \geqslant a. \end{cases}$$

Then $f_a \in \widehat{L}^1(\mathbb{R})$ and $\|f_a\|_{\widehat{L}^1} \leqslant \frac{2}{a}$.

Proof. It suffices to observe that $\|f_a\|^2_{L^2} = \frac{8}{3a}$, $\|f_a'\|^2_{L^2} = \frac{8}{3a^3}$, and $\sqrt{\frac{8}{3}} \leqslant 2$. $\square$

Lemma 3.4. *Let J be a bounded interval and let f be a Lipschitz function on $\mathbb{R}$ such that* $\operatorname{supp} f \subset J$. *Then $f \in \widehat{L}^1$ and*

$$\|f\|_{\widehat{L}^1} \leqslant \frac{1}{\sqrt[4]{12}} |J| \cdot \|f'\|_{L^\infty}.$$

Proof. Let $J = [-a, a]$. Clearly, $|f(x)| \leqslant (a - |x|)\|f'\|_{L^\infty}$ for all $x \in J$. Hence,

$$\|f\|^2_{L^2} \leqslant 2\|f'\|^2_{L^\infty} \int_0^a (a - t)^2\,dt = \frac{1}{12}\|f'\|^2_{L^\infty}|J|^3.$$

Using the obvious inequality $\|f'\|^2_{L^2} \leqslant \|f'\|^2_{L^\infty}|J|$, we get the desired estimate. $\square$

Corollary 3.5. *Let f be a Lipschitz function on $\mathbb{R}$ such that $f(0)=0$. Then*

$$\|f\|_{\widehat{L}^1(J)} \leqslant \frac{2}{\sqrt[4]{12}}|J| \cdot \|f'\|_{L^\infty}$$

for every bounded interval J that contains 0.

Proof. Put $2J \stackrel{\text{def}}{=} \{2x\colon x \in J\}$. Clearly, there exists a function f_J in $C(\mathbb{R})$ such that $f_J = f$ on J, $\operatorname{supp} f_J \subset 2J$, and $\|f_J'\|_{L^\infty} \leqslant \|f'\|_{L^\infty}$. $\square$

Lemma 3.6. *Let f be a locally absolutely continuous function on $\mathbb{R}$ such that $(1+|x|)f'(x) \in L^2(\mathbb{R})$. Suppose that $\lim_{x\to-\infty} f(x) = 0$ and $\lim_{x\to\infty} f(x) = 1$. Then*

$$\|f\|_{\widehat{L}^1(-\infty,a]} \leqslant \frac{1}{\sqrt{\pi}}\|f'\|_{L^2} + \sqrt{\frac{2}{\pi}}\|xf'\|_{L^2} + \frac{7}{2\pi} + \frac{2}{\pi}\log a$$

for every $a \geqslant 2$.

Proof. Put

$$f_a(x) \stackrel{\text{def}}{=} f(x) - a^{-1}\int_{-\infty}^{x} \chi_{[a,2a]}(t)\,dt.$$

Clearly, $\|f\|_{\widehat{L}^1(-\infty,a]} \leqslant \|f_a\|_{\widehat{L}^1}$.

We have

$$-\mathrm{i}x\mathscr{F}^{-1}f_a = \mathscr{F}^{-1}(f_a') = \mathscr{F}^{-1}(f') - \frac{e^{2a\mathrm{i}x} - e^{a\mathrm{i}x}}{2\pi a\mathrm{i}x}.$$

Put $h \stackrel{\text{def}}{=} \mathscr{F}^{-1}(f')$. Then

$$\begin{aligned}
\|f_a\|_{\widehat{L}^1} &= \int_{\mathbb{R}} \left| h(x) - \frac{e^{2a\mathrm{i}x} - e^{a\mathrm{i}x}}{2\pi a\mathrm{i}x}\right| \cdot \frac{dx}{|x|} \\
&\leqslant \int_{-1}^{1} \frac{|h(x)-h(0)|}{|x|}\,dx + \frac{1}{2\pi}\int_{-1}^{1}\left|\frac{e^{2a\mathrm{i}x} - e^{a\mathrm{i}x}}{a\mathrm{i}x} - 1\right| \cdot \frac{dx}{|x|} \\
&\quad + \int_{\{|x|\geqslant 1\}} \frac{|h(x)|}{|x|}\,dx + \frac{1}{2\pi a}\int_{\{|x|\geqslant 1\}} \frac{|e^{a\mathrm{i}x}-1|}{x^2}\,dx.
\end{aligned}$$

We have

$$\int_0^1 \frac{|h(x)-h(0)|}{x}\,dx \leqslant \int_0^1 \frac{1}{x}\left(\int_0^x |h'(t)|\,dt\right)dx = \int_0^1 |h'(t)| \cdot |\log t|\,dt.$$

Hence,

$$\int_{-1}^{1} \frac{|h(x)-h(0)|}{|x|}\,dx \leqslant \int_{-1}^{1} \left|h'(t)\right| \cdot \left|\log|t|\right|\,dt$$
$$\leqslant \left\|h'\right\|_{L^2}\left(\int_{-1}^{1} \log^2|t|\,dt\right)^{1/2} = \sqrt{\frac{2}{\pi}}\left\|xf'(x)\right\|_{L^2}$$

because $h' = \mathscr{F}^{-1}(\mathrm{i}xf')$.

By Taylor's formula for the function $e^{2\mathrm{i}x} - e^{\mathrm{i}x}$, we have

$$\left|e^{2\mathrm{i}x} - e^{\mathrm{i}x} - \mathrm{i}x\right| \leqslant \frac{5}{2}x^2.$$

Thus

$$\frac{1}{2\pi}\int_{-1}^{1}\left|\frac{e^{2a\mathrm{i}x}-e^{a\mathrm{i}x}}{a\mathrm{i}x} - 1\right| \cdot \frac{dx}{|x|} = \frac{1}{2\pi}\int_{-a}^{a}\left|\frac{e^{2\mathrm{i}x}-e^{\mathrm{i}x}}{\mathrm{i}x} - 1\right| \cdot \frac{dx}{|x|}$$
$$\leqslant \frac{1}{2\pi}\int_{-a}^{a}\min\left\{\frac{5}{2}, \frac{2}{|x|}\right\}dx \leqslant \frac{1}{2\pi}(5 + 4\log a).$$

Finally,

$$\int_{|x|\geqslant 1} \frac{|h(x)|}{|x|}\,dx \leqslant \sqrt{2}\|h\|_{L^2} = \frac{1}{\sqrt{\pi}}\left\|f'\right\|_{L^2}$$

by the Cauchy–Bunyakovsky inequality and

$$\frac{1}{2\pi a}\int_{\{|x|\geqslant 1\}} \frac{|e^{a\mathrm{i}x}-1|}{x^2}\,dx = \frac{1}{2\pi}\int_{\{|x|\geqslant a\}} \frac{|e^{\mathrm{i}x}-1|}{x^2}\,dx \leqslant \frac{2}{\pi a} \leqslant \frac{1}{\pi}$$

for $a \geqslant 2$. This implies the desired inequality. □

Theorem 3.7. *Let J be a bounded interval containing* 0. *Then*

$$\left\|\frac{e^x-1}{e^x+1}\right\|_{\widehat{L}^1(J)} \leqslant \frac{1}{\sqrt[4]{12}}|J| \leqslant \frac{3}{5}|J|. \tag{3.1}$$

Proof. Ii suffices to observe that $\|(\frac{e^x-1}{e^x+1})'\|_{L^\infty} = \frac{1}{2}$ and apply Corollary 3.5. □

Theorem 3.7 gives a sufficiently sharp estimate of the $\widehat{L}^1$-norm for little intervals J. For big intervals J, this estimate will be improved in Corollary 3.9.

Theorem 3.8. *Let $a \geqslant 2$. Then*

$$\left\|\frac{e^x}{1+e^x}\right\|_{\widehat{L}^1(-\infty,a]} \leqslant 2+\frac{2}{\pi}\log a.$$

Proof. We have

$$\left\|\left(\frac{e^x}{e^x+1}\right)'\right\|_{L^2}^2 = \int\limits_{\mathbb{R}} \frac{e^{2x}\,dx}{(e^x+1)^4} = \int\limits_0^\infty \frac{t\,dt}{(t+1)^4} = \frac{1}{6},$$

and[1]

$$\left\|x\left(\frac{e^x}{e^x+1}\right)'\right\|_{L^2}^2 = 2\int\limits_0^\infty \frac{x^2e^{2x}\,dx}{(e^x+1)^4} \leqslant 2\int\limits_0^\infty x^2e^{-2x}\,dx = \frac{1}{2},$$

whence for $a \geqslant 2$,

$$\left\|\frac{e^x}{1+e^x}\right\|_{\widehat{L}^1(-\infty,a]} \leqslant \frac{1}{\sqrt{6\pi}}+\frac{1}{\sqrt{\pi}}+\frac{7}{2\pi}+\frac{2}{\pi}\log a \leqslant 2+\frac{2}{\pi}\log a$$

by Lemma 3.6. □

Remark. Lemma 3.1 implies that

$$\left\|\frac{e^x}{1+e^x}\right\|_{\widehat{L}^1(-\infty,a]} \leqslant \frac{e^a}{1+e^a} \leqslant e^a$$

for $a \leqslant 0$ but we do not need this inequality.

Corollary 3.9. *Let J be a bounded interval containing* 0. *Then*

$$\left\|\frac{e^x-1}{e^x+1}\right\|_{\widehat{L}^1(J)} \leqslant 5+\frac{4}{\pi}\log\left(\frac{1}{2}|J|\right)$$

if $|J| \geqslant 4$.

Proof. We may assume that the center of J is nonpositive. Then $J \subset (-\infty, \frac{1}{2}|J|]$. We have

$$\left\|\frac{e^x-1}{e^x+1}\right\|_{\widehat{L}^1(J)} \leqslant 1+2\left\|\frac{e^x}{e^x+1}\right\|_{\widehat{L}^1(J)} \leqslant 5+\frac{4}{\pi}\log a = 5+\frac{4}{\pi}\log\left(\frac{1}{2}|J|\right). \qquad \square$$

[1] In fact, $\|x(\frac{e^x}{e^x+1})'\|_{L^2}^2 = \frac{\pi^2}{18}-\frac{1}{3}$.

4. Estimates of certain multiplier norms

In this section we are going to obtain sharp estimates for the Schur multiplier norms

$$\left\|\frac{e^x-e^y}{e^x+e^y}\right\|_{\mathfrak{M}_{J_1,J_2}}=\left\|\frac{e^{x-y}-1}{e^{x-y}+1}\right\|_{\mathfrak{M}_{J_1,J_2}} \tag{4.1}$$

for all intervals J_1 and J_2. First, we consider two special cases. In the first case $J_1=J_2$ while in the second case J_1 and J_2 do not overlap, i.e., their intersection has at most one point.

Theorem 4.1. *Let J_1 and J_2 be nonoverlapping intervals. Then*

$$\left\|\frac{e^x-e^y}{e^x+e^y}\right\|_{\mathfrak{M}_{J_1,J_2}}\leqslant 2.$$

Proof. Clearly, either $J_1-J_2\subset(-\infty,0]$ or $J_1-J_2\subset[0,\infty)$. It suffices to consider the case when $J_1-J_2\subset(-\infty,0]$. Then

$$\left\|\frac{e^x-e^y}{e^x+e^y}\right\|_{\mathfrak{M}_{J_1,J_2}}\leqslant 1+2\left\|\frac{e^{x-y}}{e^{x-y}+1}\right\|_{\mathfrak{M}_{J_1,J_2}}\leqslant 1+2\left\|\frac{e^{x}}{e^{x}+1}\right\|_{\widehat{L}^1(-\infty,0]}=2$$

by the Pólya theorem [32], see also Lemma 3.1. □

Theorem 4.2. *Let J be a bounded interval. Then*

$$\left\|\frac{e^x-e^y}{e^x+e^y}\right\|_{\mathfrak{M}_{J,J}}\leqslant \min\left\{\frac{6}{5}|J|,5+\frac{4}{\pi}\log_+|J|\right\}$$

and so

$$\left\|\frac{e^x-e^y}{e^x+e^y}\right\|_{\mathfrak{M}_{J,J}}\leqslant 4\log\bigl(1+|J|\bigr).$$

Proof. We have

$$\left\|\frac{e^x-e^y}{e^x+e^y}\right\|_{\mathfrak{M}_{J,J}}\leqslant\left\|\frac{e^x-1}{e^x+1}\right\|_{\widehat{L}^1(J-J)}.$$

Note that $|J-J|=2|J|$ and $0\in J-J$. The result follows now from Theorem 3.7 and Corollary 3.9. □

Theorem 4.3. *Let J_1 and J_2 be nonoverlapping intervals and let J be the convex hull of $J_1\cup J_2$. Then*

$$\frac{e-1}{e+1}\min\bigl\{1,|J|\bigr\}\leqslant\left\|\frac{e^x-e^y}{e^x+e^y}\right\|_{\mathfrak{M}_{J_1,J_2}}\leqslant\min\left\{2,\frac{6}{5}|J|\right\}.$$

Proof. The upper estimate follows readily from Theorems 4.1 and 4.2. Let us prove the lower estimate. We have

$$\left\| \frac{e^x - e^y}{e^x + e^y} \right\|_{\mathfrak{M}_{J_1,J_2}} \geqslant \sup_{x \in J_1,\, y \in J_2} \left| \frac{e^x - e^y}{e^x + e^y} \right| \geqslant \frac{e^{|J|} - 1}{e^{|J|} + 1} \geqslant \frac{e-1}{e+1} \min\bigl\{1, |J|\bigr\}$$

because the function $t \mapsto \frac{e^t - 1}{t(e^t+1)}$ decreases on $[0, \infty)$, while the function $t \mapsto \frac{e^t-1}{e^t+1}$ increases. $\square$

Theorem 4.4. *Let J be a bounded interval. Then*

$$\left\| \frac{e^x - e^y}{e^x + e^y} \right\|_{\mathfrak{M}_{J,J}} \geqslant \frac{1}{7} \min\bigl\{|J|, 1 + \log_+ |J|\bigr\}.$$

Proof. Put $Q_\varepsilon(t) \stackrel{\text{def}}{=} \frac{1}{\pi} \frac{t}{t^2 + \varepsilon^2}$, where $\varepsilon > 0$. Let us consider the convolution operator $\mathcal{C}_{Q_\varepsilon}$ on $L^2(\mathbb{R})$, $\mathcal{C}_{Q_\varepsilon} f \stackrel{\text{def}}{=} f * Q_\varepsilon$. Clearly, $\|\mathcal{C}_{Q_\varepsilon}\| = \|\mathscr{F} Q_\varepsilon\|_{L^\infty} = 1$, see, for example, [13, Chapter III, §1]. Note that $\mathcal{C}_{Q_\varepsilon}$ is an integral operator with kernel $Q_\varepsilon(x-y)$. We can define the integral operator $X_{J,\varepsilon}$ on $L^2(J)$ with kernel

$$\frac{1}{\pi} \frac{x-y}{(x-y)^2 + \varepsilon^2} \frac{e^x - e^y}{e^x + e^y}.$$

We have

$$\begin{aligned} |J| \cdot \|X_{J,\varepsilon}\| \geqslant (X_{J,\varepsilon} \chi_J, \chi_J) &= \frac{1}{\pi} \iint\limits_{J \times J} \frac{x-y}{(x-y)^2 + \varepsilon^2} \frac{e^x - e^y}{e^x + e^y} \, dx\, dy \\ &= \frac{2}{\pi} \int_0^{|J|} \frac{t}{t^2 + \varepsilon^2} \frac{e^t - 1}{e^t + 1} \bigl(|J| - t\bigr)\, dt \end{aligned}$$

and

$$\|X_{J,\varepsilon}\| \leqslant \|\mathcal{C}_{Q_\varepsilon}\| \cdot \left\| \frac{e^x - e^y}{e^x + e^y} \right\|_{\mathfrak{M}_{J,J}} = \left\| \frac{e^x - e^y}{e^x + e^y} \right\|_{\mathfrak{M}_{J,J}}.$$

Hence,

$$\left\| \frac{e^x - e^y}{e^x + e^y} \right\|_{\mathfrak{M}_{J,J}} \geqslant \frac{2}{\pi} \cdot \frac{1}{|J|} \int_0^{|J|} \frac{t}{t^2 + \varepsilon^2} \frac{e^t - 1}{e^t + 1} \bigl(|J| - t\bigr)\, dt$$

for every $\varepsilon > 0$, whence

$$\left\| \frac{e^x - e^y}{e^x + e^y} \right\|_{\mathfrak{M}_{J,J}} \geqslant \frac{2}{\pi} \int_0^{|J|} \frac{e^t - 1}{t(e^t + 1)} \left(1 - \frac{t}{|J|}\right) dt \geqslant \frac{1}{\pi} \int_0^{|J|} \frac{e^t - 1}{t(e^t + 1)}\, dt$$

because the function $t \mapsto \frac{e^t-1}{t(e^t+1)}$ decreases on $(0,\infty)$. It follows that

$$\left\| \frac{e^x - e^y}{e^x + e^y} \right\|_{\mathfrak{M}_{J,J}} \geqslant \frac{1}{\pi} \cdot \frac{e-1}{e+1} \int\limits_0^{|J|} \min\{1, t^{-1}\}\, dt.$$

This implies the desired estimate. □

Remark 1. Every rectangle $J_1 \times J_2$ is the union at most of three rectangles, each of which satisfies the hypotheses of either Theorem 4.2 or Theorem 4.3. This allows us to obtain a sharp estimate for the norms in (4.1) for every rectangle $J_1 \times J_2$.

Remark 2. Remark 1 and the change of variables $x \mapsto \log x$, $y \mapsto \log y$ allow us to obtain a sharp estimate for $\|\frac{x-y}{x+y}\|_{\mathfrak{M}_{J_1,J_2}}$, where J_1 and J_2 are intervals containing in $(0,\infty)$.

We proceed now to estimates of multiplier norms that will be used in this paper.

Theorem 4.5. *There exists a positive number C such that*

$$\left\| \frac{e^x - e^y}{e^x + e^y} \right\|_{\mathfrak{M}_{[a,\infty),(-\infty,b]}} \leqslant C \log\bigl(2 + (b-a)_+\bigr)$$

for all $a, b \in \mathbb{R}$.

Proof. The result follows from Theorems 4.1 if $a \geqslant b$. If $a < b$, then

$$[a,\infty) \times (-\infty,b] = \bigl([a,b] \times [a,b]\bigr) \cup \bigl([a,b] \times (-\infty,a]\bigr) \cup \bigl([b,\infty) \times (-\infty,b]\bigr),$$

and we can apply Theorem 4.2 to the first rectangle and Theorem 4.1 to the remaining rectangles. □

Theorem 4.6. *There exists a positive number C such that*

$$\left\| \frac{e^x - e^y}{e^x + e^y} \right\|_{\mathfrak{M}_{\mathbb{R},[a,b]}} \leqslant C \log(2 + b - a)$$

for all $a, b \in \mathbb{R}$ satisfying $a < b$.

Proof. We have

$$\mathbb{R} \times [a,b] = \bigl([a,b] \times [a,b]\bigr) \cup \bigl((-\infty,a] \times [a,b]\bigr) \cup \bigl([b,\infty) \times [a,b]\bigr).$$

It remains to apply Theorem 4.2 to the first rectangle and Theorem 4.1 to the remaining rectangles. □

Theorem 4.7. *There exists a positive number c such that*

$$\left\|\frac{x-y}{x+y}\right\|_{\mathfrak{M}_{[a,\infty),[0,b]}} \leqslant c\log\left(2+\log_+\frac{b}{a}\right)$$

for all $a, b \in (0,\infty)$.

Proof. Theorem 4.5 with the help of the change of variables $x \mapsto \log x$ and $y \mapsto \log y$ yields

$$\left\|\frac{x-y}{x+y}\right\|_{\mathfrak{M}_{[a,\infty),[\varepsilon,b+\varepsilon]}} \leqslant c\log\left(2+\log_+\frac{b+\varepsilon}{a}\right)$$

for every $\varepsilon > 0$, whence

$$\left\|\frac{x-y-\varepsilon}{x+y+\varepsilon}\right\|_{\mathfrak{M}_{[a,\infty),[0,b]}} \leqslant c\log\left(2+\log_+\frac{b+\varepsilon}{a}\right)$$

for every $\varepsilon > 0$. It remains to pass to the limit as $\varepsilon \to 0$. □

Theorem 4.8. *There exists a positive number c such that*

$$\left\|\frac{x-y}{x+y}\right\|_{\mathfrak{M}_{[a,b],[0,\infty)}} \leqslant c\log\left(2+\log\frac{b}{a}\right)$$

whenever $a, b \in (0,\infty)$ *and* $a < b$.

Proof. The result follows from Theorem 4.6 in the same way as Theorem 4.7 follows from Theorem 4.5. □

Theorem 4.9. *There exists a positive number c such that*

$$\left\|\frac{x-y}{x+y}\right\|_{\mathfrak{M}_{[a,b],[a,b]}} \geqslant c\log\left(1+\log\frac{b}{a}\right)$$

whenever $a, b \in (0,\infty)$ *and* $a < b$.

Proof. The result follows from Theorem 4.4 with the help of the change of variables $x \mapsto \log x$ and $y \mapsto \log y$. □

5. Operator Lipschitz functions and operator modulus of continuity

In this section we study operator Lipschitz functions on closed subsets of the real line. It is well known that a function f on $\mathbb{R}$ is operator Lipschitz if and only if it is *commutator Lipschitz*, i.e.,

$$\left\|f(A)R - Rf(A)\right\| \leqslant \text{const}\,\|AR - RA\|$$

for an arbitrary bounded operator R and an arbitrary self-adjoint operator A.

The same is true for functions on closed subsets of $\mathbb{R}$; moreover the operator Lipschitz constant coincides with the commutator Lipschitz constant. The following theorem was proved in [2, Theorem 10.1] in the case $\mathfrak{F} = \mathbb{R}$. The general case is analogous to the case $\mathfrak{F} = \mathbb{R}$. See also [19] where similar results for symmetric ideal norms are considered.

Theorem 5.1. *Let f be a continuous function defined on a closed subset $\mathfrak{F}$ of $\mathbb{R}$ and let $C \geqslant 0$. The following are equivalent*:

(i) $\|f(A) - f(B)\| \leqslant C\|A - B\|$ *for arbitrary self-adjoint operators A and B with spectra in $\mathfrak{F}$*;
(ii) $\|f(A)R - Rf(A)\| \leqslant C\|AR - RA\|$ *for all self-adjoint operators A with $\sigma(A) \subset \mathfrak{F}$ and all bounded operators R*;
(iii) $\|f(A)R - Rf(B)\| \leqslant C\|AR - RB\|$ *for arbitrary self-adjoint operators A and B with spectra in $\mathfrak{F}$ and for an arbitrary bounded operator R.*

A function $f \in C(\mathfrak{F})$ is said to be *operator Lipschitz* if it satisfies the equivalent statements of Theorem 5.1. We denote the set of operator Lipschitz functions on $\mathfrak{F}$ by $\mathrm{OL}(\mathfrak{F})$. For $f \in \mathrm{OL}(\mathfrak{F})$, we define $\|f\|_{\mathrm{OL}(\mathfrak{F})}$ to be the smallest constant satisfying the equivalent statements of Theorem 5.1. Put $\|f\|_{\mathrm{OL}(\mathfrak{F})} = \infty$ if $f \notin \mathrm{OL}(\mathfrak{F})$.

It is well known that every f in $\mathrm{OL}(\mathfrak{F})$ is differentiable at every nonisolated point of $\mathfrak{F}$, see [17]. Moreover, the same argument gives differentiability at ∞ in the following sense: there exists a finite limit $\lim_{|x|\to+\infty} x^{-1} f(x)$ provided $\mathfrak{F}$ is unbounded.

Let $f \in \mathrm{OL}(\mathfrak{F})$. Suppose that $\mathfrak{F}$ has no isolated points. Put

$$(\mathfrak{D} f)(x, y) \stackrel{\mathrm{def}}{=} \begin{cases} \frac{f(x)-f(y)}{x-y}, & \text{if } x, y \in \mathfrak{F},\ x \neq y, \\ f'(x), & \text{if } x \in \mathfrak{F},\ x = y. \end{cases}$$

The following equality holds:

$$\|f\|_{\mathrm{OL}(\mathfrak{F})} = \|\mathfrak{D} f\|_{\mathfrak{M}_{\mathfrak{F},\mathfrak{F}}}. \tag{5.1}$$

The inequality $\|f\|_{\mathrm{OL}(\mathfrak{F})} \leqslant \|\mathfrak{D} f\|_{\mathfrak{M}_{\mathfrak{F},\mathfrak{F}}}$ is an immediate consequence of the formula

$$f(A) - f(B) = \iint (\mathfrak{D} f)(x, y)\, dE_A(x)(A - B)\, dE_B(y), \tag{5.2}$$

where A and B are self-adjoint operators with bounded $A - B$ whose spectra are in $\mathfrak{F}$, and E_A and E_B are the spectral measures of A and B. The expression on the right is called a double operator integral. We refer the reader to [4–6] for the theory of double operator integrals elaborated by Birman and Solomyak. The validity of formula (5.2) under the assumption $\mathfrak{D} f \in \mathfrak{M}_{\mathfrak{F},\mathfrak{F}}$ and the inequality

$$\left\| \iint (\mathfrak{D} f)(x, y)\, dE_A(x)(A - B)\, dE_B(y) \right\| \leqslant \|\mathfrak{D}\|_{\mathfrak{M}_{\mathfrak{F},\mathfrak{F}}} \|A - B\|$$

was proved in [6]. The opposite inequality in (5.1) is going to be proved in Corollary 5.4.

In the general case for $f \in \mathrm{OL}(\mathfrak{F})$ we can define the function

$$(\mathfrak{D}_0 f)(x, y) \stackrel{\text{def}}{=} \begin{cases} \frac{f(x)-f(y)}{x-y}, & \text{if } x, y \in \mathfrak{F},\ x \neq y, \\ 0, & \text{if } x \in \mathfrak{F},\ x = y. \end{cases}$$

The following inequalities hold:

$$\|f\|_{\mathrm{OL}(\mathfrak{F})} \leqslant \|\mathfrak{D}_0 f\|_{\mathfrak{M}_{\mathfrak{F},\mathfrak{F}}} \leqslant 2\|f\|_{\mathrm{OL}(\mathfrak{F})}. \tag{5.3}$$

The first inequality in (5.3) follows from the formula

$$f(A) - f(B) = \iint (\mathfrak{D}_0 f)(x, y)\, dE_A(x)(A - B)\, dE_B(y), \tag{5.4}$$

whose validity can be verified in the same way as the validity of (5.2). The second inequality in (5.3) is going to be verified in Corollary 5.5.

Let f be a continuous function on a closed set $\mathfrak{F}$, $\mathfrak{F} \subset \mathbb{R}$. We define the operator modulus of continuity $\Omega_{f,\mathfrak{F}}$ as follows

$$\Omega_{f,\mathfrak{F}}(\delta) \stackrel{\text{def}}{=} \sup\bigl\{\bigl\|f(A) - f(B)\bigr\|:\ A = A^*,\ B = B^*,\ \sigma(A), \sigma(B) \subset \mathfrak{F},\ \|A - B\| \leqslant \delta\bigr\},$$

and the commutator modulus of continuity as follows

$$\Omega^{\flat}_{f,\mathfrak{F}}(\delta) \stackrel{\text{def}}{=} \sup\bigl\{\bigl\|f(A)R - Rf(A)\bigr\|:\ A = A^*,\ \sigma(A) \subset \mathfrak{F},\ \|R\| \leqslant 1,\ \|AR - RA\| \leqslant \delta\bigr\}.$$

One can prove that we get the same right-hand side if we require in addition that R is self-adjoint. On the other hand, $\|f(A)R - Rf(B)\| \leqslant \Omega^{\flat}_{f,\mathfrak{F}}(\|AR - RB\|)$ for all self-adjoint operators with $\sigma(A), \sigma(B) \subset \mathfrak{F}$ and for every bounded operator R with $\|R\| \leqslant 1$. Also, $\Omega_{f,\mathfrak{F}} \leqslant \Omega^{\flat}_{f,\mathfrak{F}} \leqslant 2\Omega_{f,\mathfrak{F}}$.

These results were obtained in [2] in the case $\mathfrak{F} = \mathbb{R}$. The same reasoning works in the general case.

Lemma 5.2. *Let $\mathfrak{F}$ be a closed subset of $\mathbb{R}$ and let μ and ν be regular positive Borel measures on $\mathfrak{F}$. Suppose that k is a function in $L^2(\mathfrak{F} \times \mathfrak{F}, \mu \otimes \nu)$ such that $k = 0$ on the diagonal $\Delta_{\mathfrak{F}} \stackrel{\text{def}}{=} \{(x, x):\ x \in \mathfrak{F}\}$ almost everywhere with respect to $\mu \otimes \nu$. Then*

$$\|k\mathfrak{D}_0 f\|_{\mathcal{B}^{\mu,\nu}_{\mathfrak{F},\mathfrak{F}}} \leqslant \|f\|_{\mathrm{OL}(\mathfrak{F})}\|k\|_{\mathcal{B}^{\mu,\nu}_{\mathfrak{F},\mathfrak{F}}}$$

for every continuous function f on $\mathfrak{F}$.

Proof. Let $\mathfrak{F}_n \stackrel{\text{def}}{=} \mathfrak{F} \cap [-n, n]$, and let μ_n and ν_n be the restrictions of μ and ν to $\mathfrak{F}_n$. Clearly,

$$\lim_{n\to\infty} \|k\|_{\mathcal{B}^{\mu_n,\nu_n}_{\mathfrak{F}_n,\mathfrak{F}_n}} = \|k\|_{\mathcal{B}^{\mu,\nu}_{\mathfrak{F},\mathfrak{F}}} \quad \text{for every } k \in L^2(\mathfrak{F} \times \mathfrak{F}, \mu \otimes \nu)$$

and

$$\lim_{n\to\infty} \|f\|_{\mathrm{OL}(\mathfrak{F}_n)} = \|f\|_{\mathrm{OL}(\mathfrak{F})} \quad \text{for every } f \in C(\mathfrak{F}).$$

Thus we may assume that $\mathfrak{F}$ is compact. It suffices to consider the case when k vanishes in a neighborhood of the diagonal $\Delta_{\mathfrak{F}}$. Put $l(x,y) \stackrel{\text{def}}{=} (x-y)^{-1}k(x,y)$. Denote by A and B multiplications by the independent variable on $L^2(\mathfrak{F},\mu)$ and $L^2(\mathfrak{F},\nu)$. Then $\mathcal{I}_k^{\mu,\nu} = A\mathcal{I}_l^{\mu,\nu} - \mathcal{I}_l^{\mu,\nu}B$ and $\mathcal{I}_{k\mathfrak{D}_0 f}^{\mu,\nu} = f(A)\mathcal{I}_l^{\mu,\nu} - \mathcal{I}_l^{\mu,\nu}f(B)$. It remains to observe that

$$\big\|f(A)\mathcal{I}_l^{\mu,\nu} - \mathcal{I}_l^{\mu,\nu}f(B)\big\| \leqslant \|f\|_{\mathrm{OL}(\mathfrak{F})}\big\|A\mathcal{I}_l^{\mu,\nu} - \mathcal{I}_l^{\mu,\nu}B\big\|,$$
$$\big\|A\mathcal{I}_l^{\mu,\nu} - \mathcal{I}_l^{\mu,\nu}B\big\| = \|k\|_{\mathcal{B}_{\mathfrak{F},\mathfrak{F}}^{\mu,\nu}},$$

and

$$\big\|f(A)\mathcal{I}_l^{\mu,\nu} - \mathcal{I}_l^{\mu,\nu}f(B)\big\| = \|k\mathfrak{D}_0 f\|_{\mathcal{B}_{\mathfrak{F},\mathfrak{F}}^{\mu,\nu}}. \qquad \square$$

Corollary 5.3. *Let $\mathfrak{F}$ be a closed subset of $\mathbb{R}$ with no isolated points, and let μ and ν be finite positive Borel measures on $\mathfrak{F}$. Suppose that f is a differentiable function on $\mathfrak{F}$ and $k \in L^2(\mathfrak{F}\times\mathfrak{F}, \mu\otimes\nu)$. If k vanishes $\mu\otimes\nu$-almost everywhere on the diagonal $\Delta_{\mathfrak{F}} \stackrel{\text{def}}{=} \{(x,x)\colon x\in\mathfrak{F}\}$, then*

$$\|k\mathfrak{D}f\|_{\mathcal{B}_{\mathfrak{F},\mathfrak{F}}^{\mu,\nu}} \leqslant \|f\|_{\mathrm{OL}(\mathfrak{F})}\|k\|_{\mathcal{B}_{\mathfrak{F},\mathfrak{F}}^{\mu,\nu}}.$$

Proof. It suffices to observe that $k\,\mathfrak{D}f = k\,\mathfrak{D}_0 f$ almost everywhere with respect to $\mu\otimes\nu$. $\square$

Corollary 5.4. *Let $\mathfrak{F}$ be a closed subset of $\mathbb{R}$ with no isolated points, and let μ and ν be finite positive Borel measures on $\mathfrak{F}$. If f is a differentiable function on $\mathfrak{F}$, then*

$$\|\mathfrak{D}f\|_{\mathfrak{M}_{\mathfrak{F},\mathfrak{F}}} \leqslant \|f\|_{\mathrm{OL}(\mathfrak{F})}.$$

Proof. Let μ be a regular Borel measure on $\mathfrak{F}$ with no atoms and such that $\operatorname{supp}\mu = \mathfrak{F}$. Then $(\mu\otimes\mu)(\Delta_{\mathfrak{F}}) = 0$ and Corollary 5.3 implies that

$$\|k\mathfrak{D}f\|_{\mathcal{B}_{\mathfrak{F},\mathfrak{F}}^{\mu,\mu}} \leqslant \|f\|_{\mathrm{OL}(\mathfrak{F})}\|k\|_{\mathcal{B}_{\mathfrak{F},\mathfrak{F}}^{\mu,\mu}}$$

for all $k \in L^2(\mathfrak{F}\times\mathfrak{F}, \mu\otimes\mu)$. Hence, $\|\mathfrak{D}f\|_{\mathfrak{M}_{\mathfrak{F},\mathfrak{F}}^{\mu,\mu}} \leqslant \|f\|_{\mathrm{OL}(\mathfrak{F})}$. It remains to apply Theorem 2.6. $\square$

Corollary 5.5. *Let $\mathfrak{F}$ be a closed subset of $\mathbb{R}$. Then*

$$\|\mathfrak{D}_0 f\|_{\mathfrak{M}_{\mathfrak{F},\mathfrak{F}}} \leqslant 2\|f\|_{\mathrm{OL}(\mathfrak{F})}$$

for every $f \in C(\mathfrak{F})$.

Proof. Let μ and ν be regular Borel measures on $\mathfrak{F}$. We have to verify that

$$\|k\mathfrak{D}_0 f\|_{\mathcal{B}_{\mathfrak{F},\mathfrak{F}}^{\mu,\nu}} \leqslant 2\|f\|_{\mathrm{OL}(\mathfrak{F})}\|k\|_{\mathcal{B}_{\mathfrak{F},\mathfrak{F}}^{\mu,\nu}}$$

for every $k \in L^2(\mathfrak{F}\times\mathfrak{F}, \mu\otimes\nu)$. Put $k_0 \stackrel{\text{def}}{=} \chi_{\Delta_{\mathfrak{F}}}k$ and $k_1 \stackrel{\text{def}}{=} k - k_0$. We have

$$\|k_0\|_{\mathcal{B}^{\mu,\nu}_{\mathfrak{F},\mathfrak{F}}} \leqslant \|k\|_{\mathcal{B}^{\mu,\nu}_{\mathfrak{F},\mathfrak{F}}}.$$

This inequality can be verified easily. We leave the verification to the reader.

It follows that $\|k_1\|_{\mathcal{B}^{\mu,\nu}_{\mathfrak{F},\mathfrak{F}}} \leqslant \|k_0\|_{\mathcal{B}^{\mu,\nu}_{\mathfrak{F},\mathfrak{F}}} + \|k\|_{\mathcal{B}^{\mu,\nu}_{\mathfrak{F},\mathfrak{F}}} \leqslant 2\|k\|_{\mathcal{B}^{\mu,\nu}_{\mathfrak{F},\mathfrak{F}}}$. It remains to observe that

$$\|k\mathfrak{D}_0 f\|_{\mathcal{B}^{\mu,\nu}_{\mathfrak{F},\mathfrak{F}}} = \|k_1\,\mathfrak{D}_0 f\|_{\mathcal{B}^{\mu,\nu}_{\mathfrak{F},\mathfrak{F}}} \leqslant \|f\|_{\mathrm{OL}(\mathfrak{F})}\|k_1\|_{\mathcal{B}^{\mu,\nu}_{\mathfrak{F},\mathfrak{F}}} \leqslant 2\|f\|_{\mathrm{OL}(\mathfrak{F})}\|k\|_{\mathcal{B}^{\mu,\nu}_{\mathfrak{F},\mathfrak{F}}}. \qquad \square$$

Let $\mathfrak{F}_1$ and $\mathfrak{F}_2$ be closed subsets of $\mathbb{R}$. We define the space $\mathrm{OL}(\mathfrak{F}_1,\mathfrak{F}_2)$ as the space of functions f in $C(\mathfrak{F}_1 \cup \mathfrak{F}_2)$ such that

$$\big\|f(A)R - Rf(B)\big\| \leqslant C\|AR - RB\| \tag{5.5}$$

for all bounded operator R and all self-adjoint operators A and B such that $\sigma(A) \subset \mathfrak{F}_1$ and $\sigma(B) \subset \mathfrak{F}_2$ with some positive number C. Denote by $\|f\|_{\mathrm{OL}(\mathfrak{F}_1,\mathfrak{F}_2)}$ the minimal constant satisfying (5.5). Clearly, $\|f\|_{\mathrm{OL}(\mathfrak{F}_1,\mathfrak{F}_2)} = \|f\|_{\mathrm{OL}(\mathfrak{F}_2,\mathfrak{F}_1)}$ and $\|f\|_{\mathrm{OL}(\mathfrak{F},\mathfrak{F})} = \|f\|_{\mathrm{OL}(\mathfrak{F})}$. As in the case $\mathfrak{F}_1 = \mathfrak{F}_2$, we can prove that

$$\|f\|_{\mathrm{OL}(\mathfrak{F}_1,\mathfrak{F}_2)} \leqslant \|\mathfrak{D}_0 f\|_{\mathfrak{M}_{\mathfrak{F}_1,\mathfrak{F}_2}} \leqslant 2\|f\|_{\mathrm{OL}(\mathfrak{F}_1,\mathfrak{F}_2)} \tag{5.6}$$

(cf. (5.3)).

Remark. In the case when $\mathfrak{F}_1 \neq \mathfrak{F}_2$ we cannot claim that the inequality

$$\big\|f(A) - f(B)\big\| \leqslant C\|A - B\| \tag{5.7}$$

for all self-adjoint A and B such that $\sigma(A) \subset \mathfrak{F}_1$ and $\sigma(B) \subset \mathfrak{F}_2$ implies (5.5).

Indeed, in the case $f(t) = |t|$, $\mathfrak{F}_1 = (-\infty, 0]$, and $\mathfrak{F}_2 = [0,\infty)$, inequality (5.7) holds with $C = 1$ because

$$\|A - B\| \leqslant \|A + B\|$$

for positive self-adjoint operators A and B. However, inequality (5.5) does not hold with any positive C. Indeed,

$$\left\|\frac{|x| - |y|}{x - y}\right\|_{\mathfrak{M}_{(-\infty,1],[1,\infty)}} = \left\|\frac{x - y}{x + y}\right\|_{\mathfrak{M}_{[1,\infty),[1,\infty)}} = \infty$$

by Theorem 4.9.

Theorem 5.6. *Suppose that inequality* (5.5) *holds for every bounded operator R and arbitrary self-adjoint operators A and B with simple spectra such that $\sigma(A) \subset \mathfrak{F}_1$ and $\sigma(B) \subset \mathfrak{F}_2$. Then $f \in \mathrm{OL}(\mathfrak{F}_1,\mathfrak{F}_2)$ and $\|f\|_{\mathrm{OL}(\mathfrak{F}_1,\mathfrak{F}_2)} \leqslant C$.*

Proof. We have to prove inequality (5.5) for arbitrary self-adjoint operators A and B with $\sigma(A) \subset \mathfrak{F}_1$ and $\sigma(B) \subset \mathfrak{F}_2$. It is convenient to think that the operators A and B act in different Hilbert spaces. Let A act in $\mathcal{H}_1$ and B in $\mathcal{H}_2$. Then R acts from $\mathcal{H}_2$ into $\mathcal{H}_1$. We are going to verify that

$$\big|\big(f(A)Ru, v\big) - \big(Rf(B)u, v\big)\big| = \big|\big(Ru, \overline{f}(A)v\big) - \big(f(B)u, R^*v\big)\big| \leqslant C\|AR - RB\|$$

for all unit vectors $u \in \mathcal{H}_2$ and $v \in \mathcal{H}_1$. Denote by $\mathcal{H}_1^0$ and $\mathcal{H}_2^0$ the invariant subspaces of A and B generated by v and u. Clearly, $A_0 \stackrel{\text{def}}{=} A|\mathcal{H}_1^0$ and $B_0 \stackrel{\text{def}}{=} B|\mathcal{H}_2^0$ are self-adjoint operators with simple spectra. Consider the operator $R_0 : \mathcal{H}_2^0 \to \mathcal{H}_1^0$, $R_0 h \stackrel{\text{def}}{=} PRh$ for $h \in \mathcal{H}_2$, where P is the orthogonal projection from $\mathcal{H}_1$ onto $\mathcal{H}_1^0$. Note that for $h \in \mathcal{H}_2^0$, we have $A_0R_0h = APRh = PARh$ and $R_0B_0h = PRBh$. Clearly, $\|A_0R_0 - R_0B_0\| \leqslant \|AR - RB\|$. Applying (5.5) to the operators A_0, B_0, and R_0, we obtain

$$\begin{aligned}
\big|\big(f(A)Ru, v\big) - \big(Rf(B)u, v\big)\big| &= \big|\big(Ru, \overline{f}(A)v\big) - \big(Rf(B)u, v\big)\big| \\
&= \big|\big(R_0u, \overline{f}(A_0)v\big) - \big(R_0f(B_0)u, v\big)\big| \\
&= \big|\big(f(A_0)R_0u, v\big) - \big(R_0f(B_0)u, v\big)\big| \\
&\leqslant C\|A_0R_0 - R_0B_0\| \leqslant C\|AR - RB\|. \qquad \square
\end{aligned}$$

Remark. Theorem 5.6 allows us to give alternative the proofs of (5.1), (5.3) and (5.6) that do not use double operator integrals.

Theorem 5.7. *Let f be a function defined on $\mathbb{Z}$. Then*

$$\Omega^{\flat}_{f,\mathbb{Z}}(\delta) = \delta\|f\|_{\mathrm{OL}(\mathbb{Z})}$$

for $\delta \in (0, \frac{2}{\pi}]$.

Proof. The inequality

$$\Omega^{\flat}_{f,\mathbb{Z}}(\delta) \leqslant \delta\|f\|_{\mathrm{OL}(\mathbb{Z})}, \quad \delta > 0,$$

is a consequence of Theorem 5.1. Let us prove the opposite inequality for $\delta \in (0, \frac{2}{\pi}]$. Fix $\varepsilon > 0$. There exists a self-adjoint operator A and a bounded operator R such that $\|AR - RA\| = 1$, $\sigma(A) \subset \mathbb{Z}$, and $\|f(A)R - Rf(A)\| \geqslant \|f\|_{\mathrm{OL}(\mathbb{Z})} - \varepsilon$. Put

$$R_A \stackrel{\text{def}}{=} \sum_{j \neq k} E_A\big(\{j\}\big)RE_A\big(\{k\}\big) = R - \sum_{j \in \mathbb{Z}} E_A\big(\{j\}\big)RE_A\big(\{j\}\big).$$

Clearly, $AR - RA = AR_A - R_AA$ and $f(A)R - Rf(A) = f(A)R_A - R_Af(A)$. Thus we may assume that $R = R_A$. Note that

$$AR - RA = \sum_{j \neq k}(j - k)E_A\big(\{j\}\big)RE_A\big(\{k\}\big).$$

Since

$$R = R_A = \sum_{j \neq k} \frac{1}{j-k}(j-k)E_A(\{j\})RE_A(\{k\}),$$

we have $R = H \star (AR - RA)$, where

$$H(j,k) \stackrel{\text{def}}{=} \begin{cases} \frac{1}{j-k}, & \text{if } j \neq k, \\ 0, & \text{if } j = k, \end{cases}$$

where $\star$ denotes Schur–Hadamard multiplication, see (2.2). It follows that

$$\|R\| \leqslant \|H\|_{\mathfrak{M}_{\mathbb{Z},\mathbb{Z}}}\|AR - RA\| = \|H\|_{\mathfrak{M}_{\mathbb{Z},\mathbb{Z}}} = \frac{\pi}{2}$$

by Lemma 2.7.

Let $\delta \in (0, \frac{2}{\pi}]$. Then $\|A(\delta R) - (\delta R)A\| = \delta$ and $\|\delta R\| \leqslant 1$. Hence,

$$\Omega^\flat_{f,\mathbb{Z}}(\delta) \geqslant \delta\big\|f(A)R - Rf(A)\big\| \geqslant \delta\big(\|f\|_{\mathrm{OL}(\mathbb{Z})} - \varepsilon\big).$$

Passing to the limit as $\varepsilon \to 0$, we obtain the desired result. $\square$

Let $\omega_{f,\mathfrak{F}}$ denote the usual scalar modulus of continuity of a continuous function f defined on $\mathfrak{F}$. Clearly, $\omega_{f,\mathfrak{F}} \leqslant \Omega_{f,\mathfrak{F}}$. Put $\omega_f \stackrel{\text{def}}{=} \omega_{f,\mathbb{R}}$ and $\Omega_f \stackrel{\text{def}}{=} \Omega_{f,\mathbb{R}}$. We are going to get some estimates for the commutator modulus of continuity $\Omega^\flat_{f,\mathfrak{F}}$. We consider first the case when $\mathfrak{F} = \mathbb{R}$. The following theorem is contained implicitly in [24].

Theorem 5.8. *Let f be a continuous function on $\mathbb{R}$. Then*

$$\Omega^\flat_f(\delta) \leqslant 2\omega_f(\delta/2) + 2\big\|f(\delta x)\big\|_{\mathrm{OL}(\mathbb{Z})}.$$

Proof. Let $\|AR - RA\| \leqslant \delta$ with $\|R\| = 1$. We can take a self-adjoint operator A_δ such that $A_\delta A = AA_\delta$, $\|A - A_\delta\| \leqslant \delta/2$ and $\sigma(A_\delta) \subset \delta\mathbb{Z}$. Then $\|f(A) - f(A_\delta)\| \leqslant \omega_f(\delta/2)$ and

$$\|A_\delta R - RA_\delta\| \leqslant \|A_\delta R - AR\| + \|AR - RA\| + \|RA - RA_\delta\| \leqslant 2\delta.$$

Hence,

$$\begin{aligned} \big\|f(A)R - Rf(A)\big\| &\leqslant \big\|f(A)R - f(A_\delta)R\big\| + \big\|f(A_\delta)R - Rf(A_\delta)\big\| + \big\|Rf(A_\delta) - Rf(A)\big\| \\ &\leqslant 2\omega_f(\delta/2) + \|A_\delta R - RA_\delta\| \cdot \|f\|_{\mathrm{OL}(\delta\mathbb{Z})} \leqslant 2\omega_f(\delta/2) + 2\delta\|f\|_{\mathrm{OL}(\delta\mathbb{Z})} \\ &= 2\omega_f(\delta/2) + 2\big\|f(\delta x)\big\|_{\mathrm{OL}(\mathbb{Z})}. \qquad \square \end{aligned}$$

Theorem 5.9. *Let f be a continuous function on $\mathbb{R}$. Then*

$$\Omega^\flat_f(\delta) \geqslant \max\left\{\omega_f(\delta), \frac{2}{\pi}\big\|f(\delta x)\big\|_{\mathrm{OL}(\mathbb{Z})}\right\}$$

for all $\delta > 0$.

Proof. Clearly, $\omega_f \leqslant \Omega_f \leqslant \Omega_f^\flat$. It remains to prove that $\|f(\delta x)\|_{\mathrm{OL}(\mathbb{Z})} \leqslant \frac{\pi}{2}\Omega_f^\flat(\delta)$. We have

$$\Omega_f^\flat(\delta) \geqslant \Omega_{f,\delta\mathbb{Z}}^\flat(\delta) = \Omega_{f(\delta x),\mathbb{Z}}^\flat(1) \geqslant \Omega_{f(\delta x),\mathbb{Z}}^\flat\left(\frac{2}{\pi}\right) = \frac{2}{\pi}\left\|f(\delta x)\right\|_{\mathrm{OL}(\mathbb{Z})}$$

by Theorem 5.7. □

We consider now similar estimates of $\Omega_{f,\mathfrak{F}}^\flat$ for an arbitrary closed subset $\mathfrak{F}$ of $\mathbb{R}$. Recall that a subset Λ of $\mathbb{R}$ is called a δ-*net* for $\mathfrak{F}$ if $\mathfrak{F} \subset \bigcup_{t\in\Lambda}[t-\delta, t+\delta]$.

Theorem 5.10. *Let f be a continuous function on a closed subset $\mathfrak{F}$ of $\mathbb{R}$. Suppose that $\mathfrak{F}_\delta$ is a subset of $\mathfrak{F}$ that forms a $(\delta/2)$-net of $\mathfrak{F}$. Then*

$$\Omega_{f,\mathfrak{F}}^\flat(\delta) \leqslant 2\omega_{f,\mathfrak{F}}(\delta/2) + 2\delta\|f\|_{\mathrm{OL}(\mathfrak{F}_\delta)}.$$

Proof. The proof is similar to the proof of Theorem 5.8. It suffices to replace the $(\delta/2)$-net $\delta\mathbb{Z}$ of $\mathbb{R}$ with the $(\delta/2)$-net $\mathfrak{F}_\delta$ of $\mathfrak{F}$. □

Theorem 5.11. *Let f be a continuous function on a closed subset $\mathfrak{F}$ of $\mathbb{R}$ and let $\delta > 0$. Suppose that Λ and M are closed subsets of $\mathfrak{F}$ such that $(\Lambda - \mathrm{M}) \cap (-\delta, \delta) \subset \{0\}$. Then*

$$\Omega_{f,\mathfrak{F}}^\flat(\delta) \geqslant \max\left\{\omega_{f,\mathfrak{F}}(\delta), \frac{\delta}{2}\|\mathfrak{D}_0 f\|_{\mathfrak{M}_{\Lambda,\mathrm{M}}}\right\}.$$

Proof. Clearly, $\omega_{f,\mathfrak{F}} \leqslant \Omega_{f,\mathfrak{F}} \leqslant \Omega_{f,\mathfrak{F}}^\flat$. Note that

$$\|\mathfrak{D}_0 f\|_{\mathfrak{M}_{\Lambda,\mathrm{M}}} = \sup_{a>0}\|\mathfrak{D}_0 f\|_{\mathfrak{M}_{\Lambda\cap[-a,a],\mathrm{M}\cap[-a,a]}}.$$

Thus it suffices to prove that

$$\Omega_{f,\mathfrak{F}}^\flat(\delta) \geqslant \frac{\delta}{2}\|\mathfrak{D}_0 f\|_{\mathfrak{M}_{\Lambda,\mathrm{M}}}$$

in the case when Λ and M are bounded.

Let $\varepsilon > 0$. There exist positive regular Borel measures λ on Λ, μ on M, and a function k in $L^2(\Lambda \times \mathrm{M}, \lambda \otimes \mu)$ such that $\|k\|_{\mathcal{B}_{\Lambda,\mathrm{M}}^{\lambda,\mu}} = 1$ and $\|k\mathfrak{D}_0 f\|_{\mathcal{B}_{\Lambda,\mathrm{M}}^{\lambda,\mu}} \geqslant \|\mathfrak{D}_0 f\|_{\mathfrak{M}_{\Lambda,\mathrm{M}}} - \varepsilon$. We define the function k_0 in $L^2(\Lambda \times \mathrm{M}, \lambda \otimes \mu)$ by

$$k_0(x,y) \stackrel{\mathrm{def}}{=} \begin{cases} k(x,y), & \text{if } x \neq y, \\ 0, & \text{if } x = y. \end{cases}$$

Then $k\mathfrak{D}_0 f = k_0\mathfrak{D}_0 f$ and $\|k_0\|_{\mathcal{B}_{\Lambda,\mathrm{M}}^{\lambda,\mu}} \leqslant 2$. Put $\Phi(x,y) \stackrel{\mathrm{def}}{=} f_\delta(x-y)$ where f_δ denotes the same as in Corollary 3.3. We define the self-adjoint operators $A: L^2(\Lambda,\lambda) \to L^2(\Lambda,\lambda)$ and $B: L^2(\mathrm{M},\mu) \to L^2(\mathrm{M},\mu)$ by $(Af)(x) \stackrel{\mathrm{def}}{=} xf(x)$ and $(Bg)(y) \stackrel{\mathrm{def}}{=} yg(y)$. Put

$$h(x,y) \stackrel{\mathrm{def}}{=} \Phi(x,y)k(x,y) = \Phi(x,y)k_0(x,y).$$

Clearly,

$$\|h\|_{\mathcal{B}^{\lambda,\mu}_{\Lambda,\mathrm{M}}} \leqslant \|\Phi\|_{\mathfrak{M}^{\lambda,\mu}_{\Lambda,\mathrm{M}}} \|k\|_{\mathcal{B}^{\lambda,\mu}_{\Lambda,\mathrm{M}}} \leqslant \|\Phi\|_{\mathfrak{M}_{\mathbb{R},\mathbb{R}}} \leqslant \frac{2}{\delta}$$

by Corollary 3.3.

Clearly, $A\mathcal{I}_h - \mathcal{I}_h B = \mathcal{I}_{k_0}$ and $f(A)\mathcal{I}_h - \mathcal{I}_h f(B) = \mathcal{I}_{k_0 \mathfrak{D}_0 f}$. (Recall that $\mathcal{I}_\varphi$ is the integral operator from $L^2(\mathrm{M}, \mu)$ into $L^2(\Lambda, \lambda)$ with kernel $\varphi \in L^2(\Lambda \times \mathrm{M}, \lambda \otimes \nu)$.) Then

$$\left\|\frac{\delta}{2}\mathcal{I}_h\right\| = \frac{\delta}{2}\|h\|_{\mathcal{B}^{\lambda,\mu}_{\Lambda,\mathrm{M}}} \leqslant 1,$$
$$\left\|A\left(\frac{\delta}{2}\mathcal{I}_h\right) - \left(\frac{\delta}{2}\mathcal{I}_h\right)B\right\| = \frac{\delta}{2}\|k_0\|_{\mathcal{B}^{\lambda,\mu}_{\Lambda,\mathrm{M}}} \leqslant \delta,$$

and

$$\left\|f(A)\left(\frac{\delta}{2}\mathcal{I}_h\right) - \left(\frac{\delta}{2}\mathcal{I}_h\right)f(B)\right\| = \frac{\delta}{2}\|k_0 \mathfrak{D}_0 f\|_{\mathcal{B}^{\lambda,\mu}_{\Lambda,\mathrm{M}}} \geqslant \frac{\delta}{2}\big(\|\mathfrak{D}_0 f\|_{\mathfrak{M}_{\Lambda,\mathrm{M}}} - \varepsilon\big).$$

Hence, $\Omega^{\flat}_{f,\mathfrak{F}}(\delta) \geqslant \frac{\delta}{2}(\|\mathfrak{D}_0 f\|_{\mathfrak{M}_{\Lambda,\mathrm{M}}} - \varepsilon)$ for every $\varepsilon > 0$. □

Theorem 5.11 allows us to obtain another proof of Theorem 4.17 in [3].

Theorem 5.12. *Let f be a continuous function on an unbounded closed subset $\mathfrak{F}$ of $\mathbb{R}$. Suppose that $\Omega_{f,\mathfrak{F}}(\delta) < \infty$ for $\delta > 0$. Then the function $t \mapsto t^{-1} f(t)$ has a finite limit as $|t| \to \infty$, $t \in \mathfrak{F}$.*

Proof. Assume the contrary. Then there exists a sequence $\{\lambda_n\}_{n=1}^\infty$ in $\mathfrak{F}$ such that $|\lambda_{n+1}| - |\lambda_n| > 1$ for all $n \geqslant 1$, $\lim_{n\to\infty} |\lambda_n| = \infty$ and the sequence $\{\lambda_n^{-1} f(\lambda_n)\}_{n=1}^\infty$ has no finite limit. Denote by Λ the image of the sequence $\{\lambda_n\}_{n=1}^\infty$. Then $\|f\|_{\mathrm{OL}(\Lambda)} = \infty$. This fact is contained implicitly in [17]. Indeed, Theorem 4.1 in [17] implies that every operator Lipschitz function f is differentiable at every nonisolated point. It is well known that the same argument gives us the differentiability at ∞ in the following sense: the function $t \mapsto t^{-1} f(t)$ has a finite limit as $|t| \to \infty$, provided the domain of f is unbounded. Applying Theorem 5.11 for $\mathrm{M} = \Lambda$ and $\delta = 1$, we find that $\Omega_{f,\mathfrak{F}}(1) = \infty$. □

We need the following known result, see [20]. We give the proof for the reader's convenience.

Theorem 5.13. *Let f be a bounded continuous function on a closed subset $\mathfrak{F}$ of $\mathbb{R}$. Suppose that $f \in \mathrm{OL}((-\infty, 1] \cap \mathfrak{F})$ and $f \in \mathrm{OL}([-1, \infty) \cap \mathfrak{F})$. Then $f \in \mathrm{OL}(\mathfrak{F})$ and*

$$\|f\|_{\mathrm{OL}(\mathfrak{F})} \leqslant C\Big(\|f\|_{\mathrm{OL}((-\infty,1]\cap\mathfrak{F})} + \|f\|_{\mathrm{OL}([-1,\infty)\cap\mathfrak{F})} + \sup_{\mathfrak{F}} |f|\Big),$$

where C is a numerical constant.

Proof. Put $\mathfrak{F}_1 \stackrel{\text{def}}{=} \mathfrak{F} \cap (-\infty, -1]$, $\mathfrak{F}_2 \stackrel{\text{def}}{=} \mathfrak{F} \cap [-1, 1]$, and $\mathfrak{F}_3 \stackrel{\text{def}}{=} \mathfrak{F} \cap [1, \infty)$. We have

$$\begin{aligned}\|f\|_{\mathrm{OL}(\mathfrak{F})} &\leqslant \|\mathfrak{D}_0 f\|_{\mathfrak{M}_{\mathfrak{F},\mathfrak{F}}} \leqslant \sum_{j=1}^{3}\sum_{k=1}^{3} \|\mathfrak{D}_0 f\|_{\mathfrak{M}_{\mathfrak{F}_j,\mathfrak{F}_k}} \\ &= \sum_{j=1}^{3} \|\mathfrak{D}_0 f\|_{\mathfrak{M}_{\mathfrak{F}_j,\mathfrak{F}_j}} + 2\|\mathfrak{D}_0 f\|_{\mathfrak{M}_{\mathfrak{F}_1,\mathfrak{F}_2}} + 2\|\mathfrak{D}_0 f\|_{\mathfrak{M}_{\mathfrak{F}_2,\mathfrak{F}_3}} + 2\|\mathfrak{D}_0 f\|_{\mathfrak{M}_{\mathfrak{F}_1,\mathfrak{F}_3}}.\end{aligned}$$

Each term $\|\mathfrak{D}_0 f\|_{\mathfrak{M}_{\mathfrak{F}_j,\mathfrak{F}_k}}$ except $\|\mathfrak{D}_0 f\|_{\mathfrak{M}_{\mathfrak{F}_1,\mathfrak{F}_3}}$ can be estimated in terms of $2\|f\|_{\mathrm{OL}(\mathfrak{F}_1\cup\mathfrak{F}_2)}$ or $2\|f\|_{\mathrm{OL}(\mathfrak{F}_2\cup\mathfrak{F}_3)}$.

Let us estimate $\|\mathfrak{D}_0 f\|_{\mathfrak{M}_{\mathfrak{F}_1,\mathfrak{F}_3}}$. We have

$$\begin{aligned}\|\mathfrak{D}_0 f\|_{\mathfrak{M}_{\mathfrak{F}_1,\mathfrak{F}_3}} &= \left\|\frac{f(x)-f(y)}{x-y}\right\|_{\mathfrak{M}_{\mathfrak{F}_1,\mathfrak{F}_3}} \\ &\leqslant \Bigl(\sup_{\mathfrak{F}_1}|f|\Bigr)\left\|\frac{1}{x-y}\right\|_{\mathfrak{M}_{\mathfrak{F}_1,\mathfrak{F}_3}} + \Bigl(\sup_{\mathfrak{F}_3}|f|\Bigr)\left\|\frac{1}{x-y}\right\|_{\mathfrak{M}_{\mathfrak{F}_1,\mathfrak{F}_3}} \\ &\leqslant 2\Bigl(\sup_{\mathfrak{F}}|f|\Bigr)\left\|\frac{1}{x-y}\right\|_{\mathfrak{M}_{\mathfrak{F}_1,\mathfrak{F}_3}} \leqslant 2\sup_{\mathfrak{F}}|f|\end{aligned}$$

because by Corollary 3.3,

$$\left\|\frac{1}{x-y}\right\|_{\mathfrak{M}_{\mathfrak{F}_1,\mathfrak{F}_3}} \leqslant \bigl\|f_2(x-y)\bigr\|_{\mathfrak{M}_{\mathbb{R},\mathbb{R}}} \leqslant 1,$$

where f_2 means the same as in Corollary 3.3.

Thus

$$\|f\|_{\mathrm{OL}(\mathfrak{F})} \leqslant 6\|f\|_{\mathrm{OL}(\mathfrak{F}_1\cup\mathfrak{F}_2)} + 4\|f\|_{\mathrm{OL}(\mathfrak{F}_2\cup\mathfrak{F}_3)} + 4\sup_{\mathfrak{F}}|f|. \qquad \square$$

6. The operator Lipschitz norm of the function $x \mapsto |x|$ on subsets of $\mathbb{R}$

In this section we obtain sharp estimates of the operator modulus of continuity of the function $x \mapsto |x|$ on certain subsets of the real line. This allows us to obtain sharp estimates of $\bigl\||S| - |T|\bigr\|$ for arbitrary bounded linear operators S and T. Note that our estimates considerably improve earlier results of [18].

Put $\mathrm{Abs}(x) \stackrel{\text{def}}{=} |x|$. For $J \subset [0, \infty)$, we put $\log(J) \stackrel{\text{def}}{=} \{\log t\colon\ t \in J,\ t > 0\}$.

Theorem 6.1. *There exist positive numbers C_1 and C_2 such that*

$$C_1 \log\bigl(2 + \bigl|\log(J_1 \cap J_2)\bigr|\bigr) \leqslant \|\mathrm{Abs}\|_{\mathrm{OL}((-J_1)\cup J_2)} \leqslant C_2 \log\bigl(2 + \bigl|\log(J_1 \cap J_2)\bigr|\bigr)$$

for all intervals J_1 and J_2 in $(0, \infty)$.

Proof. Put $J = J_1 \cap J_2$. Let us first establish the lower estimate. Note that $\|\mathrm{Abs}\|_{\mathrm{OL}((-J_1)\cup J_2)} \geqslant \|\mathrm{Abs}\|_{\mathrm{OL}(J_2)} = 1$. This proves the lower estimate in the case $|\log(J)| \leqslant 1$. In the case $|\log(J)| > 1$ we have

$$\|\mathrm{Abs}\|_{\mathrm{OL}((-J_1)\cup J_2)} \geqslant \|\mathrm{Abs}\|_{\mathrm{OL}((-J)\cup J)} \geqslant \left\| \frac{|x|-|y|}{x-y} \right\|_{\mathfrak{M}_{-J,J}} = \left\| \frac{x-y}{x+y} \right\|_{\mathfrak{M}_{J,J}}$$
$$\geqslant c_1 \log\bigl(1 + \bigl|\log(J)\bigr|\bigr) \geqslant c_2 \log\bigl(2 + \bigl|\log(J)\bigr|\bigr)$$

by Theorem 4.9.

We proceed now to the upper estimate. We consider first the case when $J = J_1$. Then

$$\|\mathrm{Abs}\|_{\mathrm{OL}((-J_1)\cup J_2)} \leqslant \|\mathrm{Abs}\|_{\mathrm{OL}((-J_1)\cup[0,\infty))} \leqslant 2 + 2\left\| \frac{x-y}{x+y} \right\|_{\mathfrak{M}_{J,[0,\infty)}}$$

and we can apply Theorem 4.8. The case $J = J_2$ is similar. Suppose that $J \neq J_1$ and $J \neq J_2$. Then $\inf J_1 \neq \inf J_2$. Let $\inf J_1 > \inf J_2$. Put $a \stackrel{\mathrm{def}}{=} \inf J_1$ and $b \stackrel{\mathrm{def}}{=} \sup J_2$. Then

$$\|\mathrm{Abs}\|_{\mathrm{OL}((-J_1)\cup J_2)} \leqslant \|\mathrm{Abs}\|_{\mathrm{OL}((-\infty_{-}a]\cup[0,b))} \leqslant 2 + 2\left\| \frac{x-y}{x+y} \right\|_{\mathfrak{M}_{[a,\infty),[0,b)}}$$

and the result follows from Theorem 4.7. $\square$

Let us state two special cases of Theorem 6.1.

Theorem 6.2. *There exist positive constants C_1 and C_2 such that*

$$C_1 \log\bigl(2 + \log\bigl(ba^{-1}\bigr)\bigr) \leqslant \|\mathrm{Abs}\|_{\mathrm{OL}((-\infty,0]\cup[a,b])} \leqslant C_2 \log\bigl(2 + \log\bigl(ba^{-1}\bigr)\bigr)$$

for all $a, b \in (0,\infty)$ with $a < b$.

Theorem 6.3. *There exist positive constants C_1 and C_2 such that*

$$C_1 \log\bigl(2 + \log_+\bigl(ba^{-1}\bigr)\bigr) \leqslant \|\mathrm{Abs}\|_{\mathrm{OL}((-b,0]\cup[a,\infty))} \leqslant C_2 \log\bigl(2 + \log_+\bigl(ba^{-1}\bigr)\bigr)$$

for all $a, b \in (0,\infty)$.

Theorem 6.4. *Let $\xi_a = \mathrm{Abs}|[-a,\infty)$ and $\eta_a = \mathrm{Abs}|[-a,a]$, where $a > 0$. Then there exist positive numbers C_1 and C_2 such that*

$$C_1 \delta \log\bigl(2 + \log\bigl(a\delta^{-1}\bigr)\bigr) \leqslant \Omega_{\eta_a}(\delta) \leqslant \Omega_{\xi_a}(\delta) \leqslant C_2 \delta \log\bigl(2 + \log\bigl(a\delta^{-1}\bigr)\bigr)$$

for $\delta \in (0,a]$,

$$C_1 \delta \leqslant \Omega_{\xi_a}(\delta) \leqslant C_2 \delta$$

for $\delta \in [a,\infty)$, and

$$C_1 a \leqslant \Omega_{\eta_a}(\delta) \leqslant C_2 a$$

for $\delta \in [a,\infty)$.

Proof. Put $\mathfrak{F}_\delta \stackrel{\text{def}}{=} [-a,\infty)\setminus(0,\delta)$. Clearly, $\mathfrak{F}_\delta$ is a $\delta/2$-net of $(-\infty,a]$. Hence, by Theorem 5.10 we have

$$\Omega_{\xi_a}(\delta) \leqslant \Omega^\flat_{\xi_a}(\delta) \leqslant \delta + 2\delta\|\xi_a\|_{\mathrm{OL}(\mathfrak{F}_\delta)}.$$

Applying Theorem 6.3, we obtain the desired upper estimate for Ω_{ξ_a}. Clearly, $\Omega_{\eta_a} \leqslant 2a$ everywhere because $0 \leqslant \eta_a \leqslant a$.

To obtain the lower estimates, we use Theorem 5.11. We consider first the case $\delta \in (0, \frac{a}{2})$. Put $\Lambda = [-a, 0]$ and $\mathrm{M} = [\delta, a]$. By Theorem 5.11,

$$\Omega_{\eta_a}(\delta) \geqslant \frac{1}{2}\Omega^\flat_{\eta_a}(\delta) \geqslant \frac{\delta}{4}\|\mathfrak{D}_0\eta_a\|_{\mathfrak{M}_{\Lambda,\mathrm{M}}}.$$

Theorem 4.9 implies now that $\Omega_{\eta_a}(\delta) \geqslant \mathrm{const}\,\delta\log(2+\log(a\delta^{-1}))$. The lower estimates in the case $\delta \in [\frac{a}{2},\infty)$ are trivial because $\Omega_{\eta_a} \geqslant \omega_{\eta_a}$ and $\Omega_{\xi_a} \geqslant \omega_{\xi_a}$. $\square$

Theorem 6.5. *There exists a positive number C such that*

$$\big\||A|-|B|\big\| \leqslant C\|A-B\|\log\left(2+\log\frac{\|A\|+\|B\|}{\|A-B\|}\right)$$

for all bounded self-adjoint operators A and B.

Proof. This is a special case of Theorem 6.4 that corresponds to $a = \|A\|+\|B\|$. $\square$

Theorem 6.4 also allows us to prove that the upper estimate in Theorem 6.5 is sharp.

Theorem 6.6. *Let $a>0$. There is a positive number c such that for every $\delta\in(0,a)$, there exist self-adjoint operators A and B such that $\|A\|+\|B\|\leqslant a$, $\|A-B\|\leqslant\delta$, but*

$$\big\||A|-|B|\big\| \geqslant c\delta\log\left(2+\log\frac{a}{\delta}\right).$$

We proceed now to the case of arbitrary (not necessarily self-adjoint) operators. Recall that for a bounded operator S on Hilbert space, its modulus $|S|$ is defined by

$$|S| \stackrel{\text{def}}{=} \big(S^*S\big)^{1/2}.$$

Theorem 6.7. *There exists a positive number C such that*

$$\big\||S|-|T|\big\| \leqslant C\|S-T\|\log\left(2+\log\frac{\|S\|+\|T\|}{\|S-T\|}\right)$$

for all bounded operators S and T.

Proof. Put

$$A = \begin{pmatrix} \mathbf{0} & S^* \\ S & \mathbf{0}\end{pmatrix} \quad\text{and}\quad B = \begin{pmatrix} \mathbf{0} & T^* \\ T & \mathbf{0}\end{pmatrix}.$$

Clearly, A and B are self-adjoint operators with

$$|A| = \begin{pmatrix} |S| & \mathbf{0} \\ \mathbf{0} & |S^*| \end{pmatrix} \quad \text{and} \quad |B| = \begin{pmatrix} |T| & \mathbf{0} \\ \mathbf{0} & |T^*| \end{pmatrix}.$$

Hence,

$$\big\| |S| - |T| \big\| \leqslant \big\| |A| - |B| \big\| \leqslant C\|A - B\| \log\left(2 + \log \frac{\|A\| + \|B\|}{\|A - B\|}\right)$$
$$= C\|S - T\| \log\left(2 + \log \frac{\|S\| + \|T\|}{\|S - T\|}\right). \qquad \square$$

Remark. Theorem 6.7 significantly improves Kato's inequality obtained in [18]:

$$\big\| |S| - |T| \big\| \leqslant \frac{1}{\pi}\|S - T\|\left(2 + \log \frac{\|S\| + \|T\|}{\|S - T\|}\right).$$

7. The operator modulus of continuity of a certain piecewise linear function

In this section we obtain a sharp estimate for the operator modulus of continuity of the piecewise linear function $\varkappa$ defined by

$$\varkappa(t) \stackrel{\text{def}}{=} \begin{cases} 1, & \text{if } t \geqslant 1, \\ t, & \text{if } -1 < t \leqslant 1. \\ -1, & \text{if } t > 1. \end{cases}$$

The results obtained in this section will be used in the next section to estimate the operator modulus of continuity of functions concave on $\mathbb{R}_+$.

It is easy to see that $\varkappa(t) = \frac{1}{2}(|1 + t| - |1 - t|)$.

Theorem 7.1. *There exist positive numbers C_1 and C_2 such that*

$$C_1 \log|\log \delta| \leqslant \|\varkappa\|_{\mathrm{OL}((-\infty,-1-\delta]\cup[-1,1]\cup[1+\delta,\infty))} \leqslant C_2 \log|\log \delta|$$

for every $\delta \in (0, \frac{1}{2})$.

Proof. Put $\varkappa_1 = \varkappa\big|((-\infty, -1 - \delta] \cup [-1, 1])$ and $\varkappa_2 = \varkappa\big|([-1, 1] \cup [1 + \delta, \infty))$. Note that

$$\varkappa_1(t) = \frac{1}{2}\big(|1 + t| - 1 + t\big) \quad \text{and} \quad \varkappa_2(t) = \frac{1}{2}\big(1 + t - |t - 1|\big).$$

It follows from Theorem 6.3 that

$$C_1 \log|\log \delta| \leqslant \|\varkappa_1\|_{\mathrm{OL}} \leqslant C_2 \log|\log \delta|$$

and

$$C_1 \log|\log \delta| \leqslant \|\varkappa_2\|_{\mathrm{OL}} \leqslant C_2 \log|\log \delta|.$$

Thus the desired lower estimate is evident and the desired upper estimate follows from Theorem 5.13. □

Theorem 7.2. *There exist positive numbers c_1 and c_2 such that*

$$c_1\delta\log\bigl(1+\log\bigl(1+\delta^{-1}\bigr)\bigr)\leqslant\Omega_\varkappa(\delta)\leqslant c_2\delta\log\bigl(1+\log\bigl(1+\delta^{-1}\bigr)\bigr)$$

for every $\delta>0$.

Proof. Note that $\lim_{t\to\infty}t\log(1+\log(1+t^{-1}))=1$. Thus it suffices to consider the case when $0<\delta\leqslant\frac{1}{2}$. Put $\mathfrak{F}_\delta\stackrel{\text{def}}{=}(-\infty,-1-\delta]\cup[-1,1]\cup[1+\delta,\infty)$. Clearly, $\mathfrak{F}_\delta$ is a δ-net for $\mathbb{R}$. Hence, by Theorem 5.10, we have

$$\Omega_\varkappa(\delta)\leqslant\Omega_\varkappa^\flat(\delta)\leqslant\delta+2\delta\|\varkappa\|_{\mathrm{OL}(\mathfrak{F}_\delta)}.$$

The desired upper estimate follows now from Theorem 7.1.

To obtain the lower estimate we can apply Theorem 6.4 because $\varkappa(t)=\frac{1}{2}(|1+t|-1+t)$ for $t\leqslant1$. □

8. Operator moduli of continuity of concave functions on $\mathbb{R}_+$

Recall that in [2] we proved that if f is a continuous function on $\mathbb{R}$, then its operator modulus of continuity Ω_f admits the estimate

$$\Omega_f(\delta)\leqslant\text{const}\,\delta\int_\delta^\infty\frac{\omega_f(t)}{t^2}\,dt=\text{const}\int_1^\infty\frac{\omega_f(t\delta)}{t^2}\,ds,\qquad\delta>0.$$

In this section we show that if f vanishes on $(-\infty,0]$ and is a concave nondecreasing function on $[0,\infty)$, then the above estimate can be considerably improved.

We also obtain several other estimates of operator moduli of continuity.

Theorem 8.1. *Suppose that $f''=\mu\in\mathscr{M}(\mathbb{R})$ (in the distributional sense), $\mu(\mathbb{R})=0$, and*

$$\int_{\mathbb{R}}\log\bigl(\log\bigl(|t|+3\bigr)\bigr)\,d|\mu|(t)<\infty.$$

Then

$$\Omega_f(\delta)\leqslant c\,\|\mu\|_{\mathscr{M}(\mathbb{R})}\delta\log\bigl(\log\bigl(\delta^{-1}+3\bigr)\bigr),$$

where c is a numerical constant.

Proof. Put

$$\varphi_s(t)\stackrel{\text{def}}{=}\frac{1}{2}\bigl(|t|+|s|\bigr)-\frac{|t-s|}{2},\qquad s,t\in\mathbb{R}.\tag{8.1}$$

It is easy to see that

$$\varphi_s(t) \stackrel{\text{def}}{=} \frac{|s|}{2}\varkappa\left(\frac{2t}{s}-1\right)+\frac{|s|}{2} \quad \text{for } s \neq 0.$$

Clearly,

$$\varphi_s'' = \delta_0 - \delta_s \quad \text{and} \quad \varphi_s(0) = 0. \tag{8.2}$$

Theorem 7.2 implies that

$$\begin{aligned}\Omega_{\varphi_s}(t) &\leqslant \operatorname{const} t \log\left(1+\log\left(1+\frac{|s|}{2t}\right)\right)\\ &\leqslant \operatorname{const} t \log\left(1+\log\left(1+\frac{|s|}{t}\right)\right), \quad t>0.\end{aligned} \tag{8.3}$$

It is easy to see that

$$t\log\bigl(1+\log\bigl(1+t^{-1}|s|\bigr)\bigr) \leqslant \operatorname{const}\bigl(\log\bigl(\log\bigl(|s|+3\bigr)\bigr)\bigr)t\log\bigl(\log\bigl(t^{-1}+3\bigr)\bigr).$$

To complete the proof, it suffices to observe that

$$f(t) = at+b-\int_{\mathbb{R}} \varphi_s(t)\,d\mu(s), \quad \text{for some } a, b \in \mathbb{C},$$

which follows easily from (8.2). □

The assumption that $\mu(\mathbb{R}) = 0$ in the hypotheses of Theorem 8.1 is essential. Moreover, the following result holds.

Theorem 8.2. *Suppose that $f'' = \mu \in \mathscr{M}(\mathbb{R})$ and $\mu(\mathbb{R}) \neq 0$. Then $\Omega_f(t) = \infty$ for every $t > 0$.*

Proof. Indeed, it is easy to see that there exists $c \in \mathbb{R}$ such that $f'(t) = c + \mu((-\infty, t))$ for almost all $t \in \mathbb{R}$. Hence,

$$\lim_{t\to\infty}\frac{f(t)}{t} = \lim_{t\to\infty} f'(t) = c+\mu(\mathbb{R}) \quad \text{and} \quad \lim_{t\to-\infty}\frac{f(t)}{t} = \lim_{t\to-\infty} f'(t) = c.$$

The result follows from Theorem 5.12. □

Let G be an open subset of $\mathbb{R}$. Denote by $\mathscr{M}_{\mathrm{loc}}(G)$ the set of all distributions on G that are locally (complex) measures.

Theorem 8.3. *Let $f \in C(\mathbb{R})$. Put $\mu \stackrel{\text{def}}{=} f''$ in the sense of distributions. Suppose that $\lim_{|t|\to\infty} t^{-1} f(t) = 0$, $\mu\big|(\mathbb{R}\setminus\{0\}) \in \mathscr{M}_{\mathrm{loc}}(\mathbb{R}\setminus\{0\})$ and*

$$\int_{\mathbb{R}\setminus\{0\}} \log\bigl(1+\log\bigl(1+|s|\bigr)\bigr)\,d|\mu|(s) < \infty.$$

Then

$$\Omega_f(\delta) \leqslant \operatorname{const} \delta \int\limits_{\mathbb{R} \setminus \{0\}} \log\bigl(1 + \log\bigl(1 + |s|\delta^{-1}\bigr)\bigr)\, d|\mu|(s).$$

Proof. Put

$$g(t) = -\int\limits_{\mathbb{R} \setminus \{0\}} \varphi_s(t)\, d\mu(s),$$

where φ_s is defined by (8.1). Inequality (8.3) implies that

$$\Omega_g(\delta) \leqslant \operatorname{const} \delta \int\limits_{\mathbb{R} \setminus \{0\}} \log\bigl(1 + \log\bigl(1 + |s|\delta^{-1}\bigr)\bigr)\, d|\mu|(s). \tag{8.4}$$

In particular, g is continuous on $\mathbb{R}$. Clearly, $g'' = f''$ on $\mathbb{R} \setminus \{0\}$. Hence, $f(x) - g(x) = a|x| + bx + c$ for some $a, b, c \in \mathbb{C}$. It follows from (8.4) that

$$\lim_{|t| \to \infty} \left| \frac{g(t)}{t} \right| \leqslant \lim_{t \to \infty} \frac{\omega_g(t)}{t} \leqslant \lim_{t \to \infty} \frac{\Omega_g(t)}{t} = 0 = \lim_{|t| \to \infty} \frac{f(t)}{t}$$

which implies that $f - g = \text{const}$. □

Corollary 8.4. *Let $a > 0$ and let f be a continuous function on $\mathbb{R}$ that is constant on $\mathbb{R} \setminus (-a, a)$. Put $\mu \stackrel{\text{def}}{=} f''$ in the sense of distributions. Suppose that $\mu \big| (\mathbb{R} \setminus \{0\}) \in \mathscr{M}_{\text{loc}}(\mathbb{R} \setminus \{0\})$ and*

$$C \stackrel{\text{def}}{=} \sup_{s>0} |\mu|\bigl([s, 2s] \cup [-2s, -s]\bigr) < \infty. \tag{8.5}$$

Then

$$\Omega_f(\delta) \leqslant C \operatorname{const} \delta \left(\log \frac{a}{\delta} \right) \log\left(\log \frac{a}{\delta} \right) \quad \text{for } \delta \in \left(0, \frac{a}{3}\right).$$

Proof. By Theorem 8.3,

$$\begin{aligned}
\Omega_f(\delta) &\leqslant \operatorname{const} \delta \left(\int\limits_0^a \log\bigl(1 + \log\bigl(1 + s\delta^{-1}\bigr)\bigr)\, d\bigl|\mu(s)\bigr| + \int\limits_0^a \log\bigl(1 + \log\bigl(1 + s\delta^{-1}\bigr)\bigr)\, d\bigl|\mu(-s)\bigr| \right) \\
&= \operatorname{const} \delta \sum_{n \geqslant 0} \int\limits_{2^{-n-1}a}^{2^{-n}a} \log\bigl(1 + \log\bigl(1 + s\delta^{-1}\bigr)\bigr)\, d|\mu|(s) \\
&\quad + \operatorname{const} \delta \sum_{n \geqslant 0} \int\limits_{2^{-n-1}a}^{2^{-n}a} \log\bigl(1 + \log\bigl(1 + s\delta^{-1}\bigr)\bigr)\, d|\mu|(-s).
\end{aligned}$$

It follows now from (8.5) and the inequality

$$\log\bigl(1+\log(1+\alpha x)\bigr) \leqslant 2\log\bigl(1+\log(1+x)\bigr), \quad 0<x<\infty,\ 1<\alpha\leqslant 2,$$

that

$$\begin{aligned}
\Omega_f(\delta) &\leqslant \operatorname{const}\delta \sum_{n\geqslant 0}\int\limits_{2^{-n-1}a}^{2^{-n}a} \log\bigl(1+\log\bigl(1+s\delta^{-1}\bigr)\bigr)\,\frac{ds}{s}\\
&= \operatorname{const}\delta \int\limits_0^a \log\bigl(1+\log\bigl(1+s\delta^{-1}\bigr)\bigr)\,\frac{ds}{s}\\
&= \operatorname{const}\delta \int\limits_0^{a/\delta} \log\bigl(1+\log(1+s)\bigr)\,\frac{ds}{s}\\
&\leqslant \operatorname{const}\delta + \operatorname{const}\delta \int\limits_1^{a/\delta} \log\bigl(1+\log(1+s)\bigr)\,\frac{ds}{s}\\
&= \operatorname{const}\delta\left(1+\bigl(\log\bigl(1+\log(1+s)\bigr)\log s\bigr)\Big|_1^{a/\delta} - \int\limits_1^{a/\delta}\frac{\log s\,ds}{(1+s)\log(1+\log(1+s))}\right)\\
&\leqslant \operatorname{const}\delta + \operatorname{const}\delta\bigl(\log\bigl(1+\log(1+s)\bigr)\log s\bigr)\Big|_1^{a/\delta}\\
&\leqslant \operatorname{const}\delta\left(\log\frac{a}{\delta}\right)\log\left(\log\frac{a}{\delta}\right)
\end{aligned}$$

for sufficiently small δ. $\square$

Corollary 8.5. *Let f be a continuous function on $\mathbb{R}$ that is constant on $\mathbb{R}\setminus(-a,a)$. Suppose that f is twice differentiable on $\mathbb{R}\setminus\{0\}$ and*

$$C \stackrel{\text{def}}{=} \sup_{s\neq 0}\bigl|sf''(s)\bigr| < \infty.$$

Then

$$\Omega_f(\delta) \leqslant \operatorname{const} C\delta\left(\log\frac{a}{\delta}\right)\log\left(\log\frac{a}{\delta}\right) \quad \textit{for } \delta\in\left(0,\frac{a}{3}\right).$$

The following result shows that in a sense Theorem 8.1 cannot be improved.

Theorem 8.6. *Let h be a positive continuous function on $\mathbb{R}$. Suppose that for every $f \in C(\mathbb{R})$ such that*

$$f'' = \mu \in \mathscr{M}(\mathbb{R}), \qquad \mu(\mathbb{R}) = 0, \quad \text{and} \quad \int_{\mathbb{R}} h(t)\,d|\mu|(t) < \infty,$$

we have $\Omega_f(\delta) < \infty$, $\delta > 0$. Then for some positive number c,

$$h(t) \geqslant c \log\bigl(\log\bigl(|t| + 3\bigr)\bigr), \quad t \in \mathbb{R}.$$

We need the following lemma, in which φ_s is the function defined by (8.1).

Lemma 8.7. *There is a positive number c such that for every $s \geqslant 10$, there exist self-adjoint operators A and B satisfying the conditions*:

$$\sigma(A), \sigma(B) \subset \left(\frac{s}{2}, \frac{3s}{2}\right), \qquad \|A - B\| \leqslant 1, \quad \text{and} \quad \bigl\|\varphi_s(A) - \varphi_s(B)\bigr\| \geqslant c \log\log s.$$

Proof. Clearly, it suffices to prove the lemma for sufficiently large s. By Theorem 6.4, there exist self-adjoint operators A_0 and B_0 such that $\|A_0\|, \|B_0\| < 1$, $\|A_0 - B_0\| \leqslant 2/s$, and $\bigl\||A_0| - |B_0|\bigr\| \geqslant \operatorname{const} s^{-1} \log(2 + \log s)$. Put $A \stackrel{\mathrm{def}}{=} sI + \frac{s}{2}A_0$ and $B \stackrel{\mathrm{def}}{=} sI + \frac{s}{2}B_0$. Then $\sigma(A), \sigma(B) \subset (\frac{s}{2}, \frac{3s}{2})$ and $\|A - B\| \leqslant 1$. Let us estimate $\|\varphi_s(A) - \varphi_s(B)\|$. Clearly,

$$\varphi_s(A) - \varphi_s(B) = \frac{s}{4}(A_0 - B_0) - \frac{s}{4}\bigl(|A_0| - |B_0|\bigr).$$

Hence,

$$\begin{aligned}\bigl\|\varphi_s(A) - \varphi_s(B)\bigr\| &\geqslant \frac{s}{4}\bigl\||A_0| - |B_0|\bigr\| - \frac{s}{4}\|A_0 - B_0\| \\ &\geqslant \operatorname{const} \log\log s - \frac{1}{2} \geqslant \operatorname{const} \log\log s\end{aligned}$$

for sufficiently large s. $\square$

Proof of Theorem 8.6. Assume the contrary. Then there exists a sequence $\{s_n\}$ of real numbers such that $\lim_{n\to\infty} |s_n| = \infty$ and $\lim_{n\to\infty} (\log(\log(|s_n|)))^{-1} h(s_n) = 0$. Passing to a subsequence, we can reduce the situation to the case when $s_n > 0$ for all n or $s_n < 0$ for all n. Without loss of generality we may assume that $s_n > 0$ for all n. Moreover, we may also assume that $s_1 \geqslant 10$, $s_{n+1} \geqslant 2s_n$ and $\log\log s_n \geqslant n^3(1 + h(s_n))$ for every $n \geqslant 1$. Put $\alpha_n \stackrel{\mathrm{def}}{=} n(\log\log s_n)^{-1}$ for $n \geqslant 1$ and $f(t) \stackrel{\mathrm{def}}{=} \sum_{n\geqslant 1} \alpha_n \varphi_{s_n}(t)$. Note that the series converges for every t because $\sigma \stackrel{\mathrm{def}}{=} \sum_{n\geqslant 1} \alpha_n < \infty$. Moreover,

$$f'' = \sigma\delta_0 - \sum_{n\geqslant 1} \alpha_n \delta_{s_n} \quad \text{and} \quad \sigma h(0) + \sum_{n\geqslant 1} \alpha_n h(s_n) < \infty.$$

By Lemma 8.7, there exist two sequences $\{A_n\}_{n\geqslant 1}$ and $\{B_n\}_{n\geqslant 1}$ of self-adjoint operators such that

$$\sigma(A_n), \sigma(B_n) \subset \left(\frac{s_n}{2}, \frac{3s_n}{2}\right), \qquad \|A_n - B_n\| \leqslant 1,$$

and

$$\left\|\varphi_{s_n}(A_n) - \varphi_{s_n}(B_n)\right\| \geqslant c \log\log s_n.$$

Note that $\varphi_{s_k}(A_n) = \varphi_{s_k}(B_n) = s_k I$ for $k < n$. Also, $\varphi_{s_k}(A_n) = A_n$ and $\varphi_{s_k}(B_n) = B_n$ for $k > n$. Hence,

$$f(A_n) - f(B_n) = \alpha_n\big(\varphi_{s_n}(A_n) - \varphi_{s_n}(B_n)\big) + \sum_{k>n} \alpha_k(A_n - B_n),$$

and so

$$\begin{aligned}\left\|f(A_n) - f(B_n)\right\| &\geqslant \alpha_n \left\|\varphi_{s_n}(A_n) - \varphi_{s_n}(B_n)\right\| - \sum_{k>n} \alpha_k \|A_n - B_n\| \\ &\geqslant C\alpha_n \log\log s_n - \sum_{k>n} \alpha_k \to \infty \quad \text{as } n \to \infty.\end{aligned}$$

Thus $\Omega_f(1) = \infty$ and we get a contradiction. $\square$

In [2] it was proved that

$$\Omega_f(\delta) \leqslant \int_1^\infty \frac{\omega_f(\delta s)}{s^2}\, ds$$

for every $f \in C(\mathbb{R})$. The following theorem shows that this estimate can be improved essentially for functions f concave on a ray.

Theorem 8.8. *Let f be a continuous nondecreasing function such that $f(t) = 0$ for $t \leqslant 0$, $\lim_{t\to\infty} t^{-1} f(t) = 0$, and f is concave on $[0, \infty)$. Then*

$$\Omega_f(\delta) \leqslant c \int_e^\infty \frac{f(\delta s)\, ds}{s^2 \log s},$$

where c is a numerical constant.

Proof. Let $\mu = -f''$ (in the distributional sense). Clearly, $\mu = 0$ on $(-\infty, 0)$ and μ is a positive regular measure on $(0, \infty)$ because f is concave on $(0, \infty)$. Hence, $\mu \in \mathscr{M}_{\mathrm{loc}}(\mathbb{R} \setminus \{0\})$. By Theorem 8.3, we have

$$\Omega_f(\delta) \leqslant \operatorname{const} \delta \int_0^\infty \log\bigl(1 + \log\bigl(1 + s\delta^{-1}\bigr)\bigr)\, d\mu(s).$$

To estimate this integral, we use the equality $f'(t) = \mu(t, \infty)$ for almost all $t > 0$ and apply the Tonelli theorem twice.

$$\begin{aligned}
&\delta \int_0^\infty \log\bigl(1 + \log\bigl(1 + s\delta^{-1}\bigr)\bigr)\, d\mu(s) \\
&\quad = \int_0^\infty \left(\int_0^s \frac{dt}{(1 + \log(1 + t\delta^{-1}))(1 + t\delta^{-1})} \right) d\mu(s) \\
&\quad = \int_0^\infty \frac{f'(t)\, dt}{(1 + \log(1 + t\delta^{-1}))(1 + t\delta^{-1})} \\
&\quad = \delta^{-1} \int_0^\infty \left(\int_t^\infty \frac{(2 + \log(1 + s\delta^{-1}))\, ds}{(1 + \log(1 + s\delta^{-1}))^2 (1 + s\delta^{-1})^2} \right) f'(t)\, dt \\
&\quad = \delta^{-1} \int_0^\infty \frac{2 + \log(1 + s\delta^{-1})}{(1 + \log(1 + s\delta^{-1}))^2 (1 + s\delta^{-1})^2} f(s)\, ds \\
&\quad = \int_0^\infty \frac{2 + \log(1 + s)}{(1 + \log(1 + s))^2 (1 + s)^2} f(s\delta)\, ds \\
&\quad \leqslant 2 \int_0^\infty \frac{1}{(1 + \log(1 + s))(1 + s)^2} f(s\delta)\, ds.
\end{aligned}$$

It remains to observe that

$$\begin{aligned}
\int_0^e \frac{1}{(1 + \log(1 + s))(1 + s)^2} f(s\delta)\, ds &\leqslant f(e\delta) \int_0^e \frac{1}{(1 + \log(1 + s))(1 + s)^2}\, ds \\
&\leqslant f(e\delta) \int_0^\infty \frac{ds}{(1 + s)^2} = f(e\delta) \\
&\leqslant \operatorname{const} \int_e^\infty \frac{f(s\delta)\, ds}{s^2 \log s}
\end{aligned}$$

and

$$\int_e^\infty \frac{1}{(1+\log(1+s))(1+s)^2} f(s\delta)\,ds \leqslant \int_e^\infty \frac{f(s\delta)\,ds}{s^2\log s}. \qquad \square$$

Corollary 8.9. *Suppose that under the hypotheses of Theorem* 8.8*, the function f is bounded and has finite right derivative at* 0*. Then*

$$\Omega_f(\delta) \leqslant \mathrm{const}\, a\delta \log\left(\log\frac{M}{a\delta}\right) \quad \text{for } \delta \in \left(0, \frac{M}{3a}\right),$$

where $a = f'_+(0)$ and $M = \sup f$.

Proof. Since $f(t) \leqslant \min\{at, M\}$, $t > 0$, the result follows from Theorem 8.8 and the following obvious facts:

$$\int_e^{\frac{M}{a\delta}} \frac{a\delta\,ds}{s\log s} = a\delta\log\left(\log\frac{M}{a\delta}\right) \quad \text{and} \quad \int_{\frac{M}{a\delta}}^\infty \frac{M\,ds}{s^2\log s} \leqslant \int_{\frac{M}{a\delta}}^\infty \frac{M\,ds}{s^2} = a\delta. \qquad \square$$

In [2] we proved that if f belongs to the Hölder class $\Lambda_\alpha(\mathbb{R})$, $0 \leqslant \alpha < 1$, then

$$\Omega_f(\delta) \leqslant \mathrm{const}(1-\alpha)^{-1}\|f\|_{\Lambda_\alpha}\delta^\alpha, \quad \delta > 0, \tag{8.6}$$

where

$$\|f\|_{\Lambda_\alpha} \stackrel{\text{def}}{=} \sup_{x\neq y} \frac{|f(x)-f(y)|}{|x-y|}.$$

The next result shows that if in addition to this f satisfies the hypotheses of Theorem 8.8, then the factor $(1-\alpha)^{-1}$ on the right-hand side of (8.6) can considerably be improved.

Corollary 8.10. *Suppose that under the hypotheses of Theorem* 8.8*, the function f belongs to $\Lambda_\alpha(\mathbb{R})$,* $0 \leqslant \alpha < 1$*. Then*

$$\Omega_f(\delta) \leqslant \mathrm{const}\left(\log\frac{2}{1-\alpha}\right)\|f\|_{\Lambda_\alpha}\delta^\alpha$$

for every $\delta > 0$.

Proof. Indeed,

$$\int_e^\infty \frac{ds}{s^{2-\alpha}\log s} = \int_1^\infty e^{(\alpha-1)t}\frac{dt}{t} = \int_{1-\alpha}^\infty \frac{e^{-t}\,dt}{t} \leqslant \mathrm{const}\log\frac{2}{1-\alpha}. \qquad \square$$

Remark. The function $x \mapsto 1 + \varkappa(x-1)$ satisfies the hypotheses of Corollary 8.9 with $a = 1$ and $M = 2$, and Corollary 8.9 yields a sharp result in this case. That means that Theorem 8.8 is also sharp in a sense.

The following theorem is a symmetrized version of Theorem 8.8.

Theorem 8.11. *Let f be a continuous function on $\mathbb{R}$ such that f is convex or concave on each of two rays $(-\infty, 0]$ and $[0, \infty)$. Suppose that there exists a finite limit $\lim_{|t|\to\infty} t^{-1} f(t) \stackrel{\text{def}}{=} a$. Then*

$$\Omega_f(\delta) \leqslant a\delta + c \int\limits_e^\infty \frac{|f(\delta s) - f(0) - \delta a s| + |f(-\delta s) - f(0) + \delta a s|}{s^2 \log s}\, ds,$$

where c is a numerical constant.

Proof. It suffices to consider the case where $f(0) = a = 0$. We assume first that $f(t) = 0$ for $t \leqslant 0$. To be definite, suppose that f is concave on $[0, \infty)$. Then f is a nondecreasing function because $\lim_{|t|\to\infty} t^{-1} f(t) = 0$, and so the result reduces to Theorem 8.8. The case $f(t) = 0$ for $t \geqslant 0$ follows from the considered case with the help of the change of variables $t \mapsto -t$. It remains to observe that each function f with $a = f(0) = 0$ can be represented in the form $f = g + h$ in such way that $g(t) = 0$ for $t \leqslant 0$, $h(t) = 0$ for $t \geqslant 0$, and the cases of the function g and h have been treated above. $\square$

Theorem 8.12. *Let f be a nonnegative continuous function on $\mathbb{R}$ such that $f(x) = 0$ for all $x \leqslant 0$ and the function $x \mapsto x^{-1} f(x)$ is nonincreasing on $(0, \infty)$. Suppose that $\Omega_f(\delta) < \infty$ for $\delta > 0$. Then*

$$f(x) \leqslant \text{const}\, \frac{x}{\log\log x}$$

for every $x \geqslant 4$.

Proof. By Theorem 5.11,

$$\Omega_f^\flat(1) \geqslant \frac{1}{2} \|\mathfrak{D}_0 f\|_{\mathfrak{M}_{[1,\infty),(-\infty,0]}}.$$

Making the change of variables $y \mapsto -y$ we get

$$\left\| \frac{f(x)}{x+y} \right\|_{\mathfrak{M}_{[1,\infty),[0,\infty)}} \leqslant 2\Omega_f^\flat(1).$$

Thus for every $a > 1$

$$\left\| \frac{x}{x+y} \right\|_{\mathfrak{M}_{[1,a],[1,a]}} \leqslant \max_{[1,a]} \left| \frac{x}{f(x)} \right| \cdot \left\| \frac{f(x)}{x+y} \right\|_{\mathfrak{M}_{[1,a],[1,a]}}$$
$$\leqslant \frac{a}{f(a)} \left\| \frac{f(x)}{x+y} \right\|_{\mathfrak{M}_{[1,\infty),[0,\infty)}} \leqslant \frac{2a\Omega_f^\flat(1)}{f(a)}.$$

It remains to apply Theorem 4.9. □

Remark. Let $x_0 > e$ and let g_α be a continuous function such that

$$g_\alpha(x) = \begin{cases} \frac{x}{\log^\alpha(\log x)}, & \text{if } x \geqslant x_0 > 0, \\ 0, & \text{if } x \leqslant 0. \end{cases}$$

Then $\Omega_{g_\alpha}(\delta) < \infty$ for $\alpha > 1$. Indeed, in this case g_α coincides with a function satisfying Theorem 8.8 outside a compact subset of $\mathbb{R}$. On the other hand, $\Omega_{g_\alpha}(\delta) = \infty$ for $\alpha < 1$. This follows from Theorem 8.12. Indeed, outside a compact subset of $\mathbb{R}$ the function g_α coincides with a function f, for which the function $x \mapsto x^{-1}f(x)$ is nonincreasing on $(0,\infty)$. The case $\alpha = 1$ is an open problem.

9. Lower estimates for operator moduli of continuity

Recall that it follows from (1.1) that if f is a function on $\mathbb{R}$ such that $\|f\|_{L^\infty} \leqslant 1$, $\|f\|_{\mathrm{Lip}} \leqslant 1$, then

$$\Omega_f(\delta) \leqslant \mathrm{const}\,\delta\left(1 + \log\frac{1}{\delta}\right), \quad \delta \in (0,1].$$

It is still unknown whether this estimate is sharp. In particular, the question whether one can replace the factor $(1 + \log\frac{1}{\delta})$ on the right-hand side with $(1 + \log\frac{1}{\delta})^s$ for some $s < 1$ is still open.

In Section 6 we established a lower estimate for the operator modulus of continuity of the function $x \mapsto |x|$ on finite intervals.

The main purpose of this section is to construct a C^∞ function f on $\mathbb{R}$ such that $\|f\|_{L^\infty} \leqslant 1$, $\|f\|_{\mathrm{Lip}} \leqslant 1$, and

$$\Omega_f(\delta) \geqslant \mathrm{const}\,\delta\sqrt{\log\frac{2}{\delta}}, \quad \delta \in (0,1].$$

Let $\sigma > 0$. Denote by $\mathscr{E}_\sigma$ the set of entire functions of exponential type at most σ.

Let $F \in \mathscr{E}_\sigma \cap L^2(\mathbb{R})$. Then

$$F(z) = \sum_{n\in\mathbb{Z}} \frac{\sin(\sigma z - \pi n)}{\sigma z - \pi n} F\left(\frac{\pi n}{\sigma}\right),$$

see, e.g., [21, Lecture 20.2, Theorem 1]. Let $f \in \mathscr{E}_\sigma \cap L^\infty(\mathbb{R})$. Then

$$f(z)\frac{\sin(\sigma(z-a))}{\sigma(z-a)} \in \mathscr{E}_{2\sigma} \cap L^2(\mathbb{R}).$$

Hence,

$$f(z)\frac{\sin(\sigma(z-a))}{\sigma(z-a)}=\sum_{n\in\mathbb{Z}}\frac{\sin(2\sigma z-\pi n)}{2\sigma z-\pi n}\cdot\frac{\sin(\sigma(\frac{\pi n}{2\sigma}-a))}{\sigma(\frac{\pi n}{2\sigma}-a)}f\left(\frac{\pi n}{2\sigma}\right)$$
$$=2\sum_{n\in\mathbb{Z}}\frac{\sin(2\sigma z-\pi n)\sin(\sigma a-\frac{\pi n}{2})}{(2\sigma z-\pi n)(2\sigma a-\pi n)}f\left(\frac{\pi n}{2\sigma}\right).$$

Substituting $z=a$, we obtain

$$f(z)=2\sum_{n\in\mathbb{Z}}\frac{\sin(2\sigma z-\pi n)\sin(\sigma z-\frac{\pi n}{2})}{(2\sigma z-\pi n)^2}f\left(\frac{\pi n}{2\sigma}\right)$$
$$=\sum_{n\in\mathbb{Z}}\frac{\sin^2(\sigma z-\frac{\pi n}{2})\cos(\sigma z-\frac{\pi n}{2})}{(\sigma z-\frac{\pi n}{2})^2}f\left(\frac{\pi n}{2\sigma}\right) \tag{9.1}$$

for $f\in\mathscr{E}_\sigma\cap L^\infty(\mathbb{R})$.

Denote by $\mathscr{E}_\sigma(\mathbb{C}^2)$ the set of all entire functions f on $\mathbb{C}^2$ such that the functions $z\mapsto f(z,\xi)$ and $z\mapsto f(\xi,z)$ belong to $\mathscr{E}_\sigma$ for every $\xi\in\mathbb{R}$ (or, which is the same, for all $\xi\in\mathbb{C}$). Equality (9.1) implies the following identity:

$$f(z,w)=\sum_{(m,n)\in\mathbb{Z}^2}\frac{\sin^2(\sigma z-\frac{\pi m}{2})\cos(\sigma z-\frac{\pi m}{2})\sin^2(\sigma w-\frac{\pi n}{2})\cos(\sigma w-\frac{\pi n}{2})}{(\sigma z-\frac{\pi m}{2})^2(\sigma w-\frac{\pi n}{2})^2}f\left(\frac{\pi m}{2\sigma},\frac{\pi n}{2\sigma}\right) \tag{9.2}$$

for every $f\in\mathscr{E}_\sigma(\mathbb{C}^2)\cap L^\infty(\mathbb{R}^2)$.

Theorem 9.1. *Let $\sigma>0$ and $\Phi\in\mathscr{E}_\sigma(\mathbb{C}^2)$. Suppose that $\Phi(\frac{\pi m}{2\sigma}+\alpha,\frac{\pi n}{2\sigma}+\beta)\in\mathfrak{M}_{\mathbb{Z},\mathbb{Z}}$ for some $\alpha,\beta\in\mathbb{R}$. Then $\Phi\in\mathfrak{M}_{\mathbb{R},\mathbb{R}}$ and*

$$\|\Phi(x,y)\|_{\mathfrak{M}_{\mathbb{R},\mathbb{R}}}\leqslant 2\left\|\Phi\left(\frac{\pi m}{2\sigma}+\alpha,\frac{\pi n}{2\sigma}+\beta\right)\right\|_{\mathfrak{M}_{\mathbb{Z},\mathbb{Z}}}.$$

Proof. Clearly, it suffices to consider the case when $\alpha=\beta=0$, $\sigma=\pi/2$ and $\|\Phi(m,n)\|_{\mathfrak{M}_{\mathbb{Z},\mathbb{Z}}}=1$. Then (see [31, Theorem 5.1]) there exist two sequences $\{\varphi_m\}_{m\in\mathbb{Z}}$ and $\{\psi_n\}_{n\in\mathbb{Z}}$ of vectors in the closed unit ball of a Hilbert space $\mathcal{H}$ such that $(\varphi_m,\psi_n)=\Phi(m,n)$. Put

$$g_x\stackrel{\text{def}}{=}\frac{4}{\pi^2}\sum_{m\in\mathbb{Z}}\frac{\sin^2(\frac{\pi}{2}(x-m))\cos(\frac{\pi}{2}(x-m))}{(x-m)^2}\varphi_m$$

and

$$h_y\stackrel{\text{def}}{=}\frac{4}{\pi^2}\sum_{n\in\mathbb{Z}}\frac{\sin^2(\frac{\pi}{2}(y-n))\cos(\frac{\pi}{2}(y-n))}{(y-n)^2}\psi_n.$$

We have

$$\begin{aligned}\|g_x\|_{\mathcal{H}} &\leqslant \frac{4}{\pi^2}\sum_{m\in\mathbb{Z}}\frac{\sin^2(\frac{\pi}{2}(x-m))|\cos(\frac{\pi}{2}(x-m))|}{(x-m)^2}\\ &= \frac{4}{\pi^2}\sum_{n\in\mathbb{Z}}\frac{\sin^2\frac{\pi x}{2}|\cos\frac{\pi x}{2}|}{(x-2n)^2}+\frac{4}{\pi^2}\sum_{n\in\mathbb{Z}}\frac{\sin^2(\frac{\pi x}{2}-\frac{\pi}{2})|\cos(\frac{\pi x}{2}-\frac{\pi}{2})|}{(x-2n-1)^2}\\ &= \left|\cos\frac{\pi x}{2}\right|+\left|\sin\frac{\pi x}{2}\right|\leqslant\sqrt{2}.\end{aligned}$$

In the same way, $\|h_y\|_{\mathcal{H}} \leqslant \sqrt{2}$ for all $y \in \mathbb{R}$. Clearly $|\Phi| \leqslant 1$ on $\mathbb{Z}^2$. The Cartwright theorem (see [21, Lecture 21, Theorem 4]) implies that Φ is bounded on $\mathbb{R}\times\mathbb{Z}$. Applying once more the Cartwright theorem, we find that $\Phi \in L^\infty(\mathbb{R}^2)$. Hence, we can apply formula (9.2) to the function Φ, whence $\Phi(x, y) = (g_x, h_y)$ for all $x, y \in \mathbb{R}$. It remains to observe that by Theorem 5.1 in [31],

$$\left\|\Phi(x,y)\right\|_{\mathfrak{M}_{\mathbb{R},\mathbb{R}}} \leqslant \sup_{x\in\mathbb{R}}\|g_x\|_{\mathcal{H}}\cdot\sup_{y\in\mathbb{R}}\|h_y\|_{\mathcal{H}} \leqslant 2. \qquad \square$$

Theorem 9.2. *Let $f \in \mathscr{E}_\sigma$. Then*

$$\Omega^\flat_f(\delta) \geqslant \frac{\delta}{2}\left\|\frac{f(x)-f(y)}{x-y}\right\|_{\mathfrak{M}_{\mathbb{R},\mathbb{R}}}$$

for every $\delta \in (0, \frac{1}{2\sigma}]$.

Proof. The general case easily reduces to the case $\sigma = \pi/4$. By Theorem 9.1, we have

$$\left\|\frac{f(x)-f(y)}{x-y}\right\|_{\mathfrak{M}_{\mathbb{R},\mathbb{R}}} \leqslant 2\left\|\frac{f(2m+1)-f(2n)}{2m-2n+1}\right\|_{\mathfrak{M}_{\mathbb{Z},\mathbb{Z}}} \leqslant 2\|f\|_{\mathrm{OL}(\mathbb{Z})}.$$

Hence, by Theorem 5.7,

$$\Omega^\flat_f(\delta) \geqslant \Omega^\flat_{f,\mathbb{Z}}(\delta) = \delta\|f\|_{\mathrm{OL}(\mathbb{Z})} \geqslant \frac{\delta}{2}\left\|\frac{f(x)-f(y)}{x-y}\right\|_{\mathfrak{M}_{\mathbb{R},\mathbb{R}}}$$

for $\delta \in (0, \frac{2}{\pi}]$. $\square$

Theorem 9.3. *Let $f \in \mathscr{E}_\sigma$. Then*

$$\Omega_f(\delta) \geqslant \frac{\delta}{4}\left\|\frac{f(x)-f(y)}{x-y}\right\|_{\mathfrak{M}_{\mathbb{R},\mathbb{R}}}$$

for every $\delta \in (0, \frac{1}{2\sigma}]$.

Proof. It suffices to observe that $\Omega^\flat_f(\delta) \leqslant 2\Omega_f(\delta)$ by Theorem 10.2 in [2]. $\square$

Theorem 9.4. *For every $\delta \in (0,1]$, there exists an entire function $f \in \mathscr{E}_{1/\delta}$ such that $\|f\|_{L^\infty(\mathbb{R})} \leqslant 1$, $\|f'\|_{L^\infty(\mathbb{R})} \leqslant 1$ and $\Omega_f(\delta) \geqslant C\,\delta\sqrt{\log\frac{2}{\delta}}$, where C is a positive numerical constant.*

We need some lemmata.

Lemma 9.5. *For every positive integer n, there exists a trigonometric polynomial f of degree n such that $\|f\|_{L^\infty} \leqslant 1$, $\|f'\|_{L^\infty} \leqslant 1$, and*

$$\left\|\frac{f(x)-f(y)}{e^{\mathrm{i}x}-e^{\mathrm{i}y}}\right\|_{\mathfrak{M}_{[0,2\pi],[0,2\pi]}} \geqslant c\sqrt{\log n}.$$

Proof. It follows from the results of [27] that for every function h in $C^1(\mathbb{T})$,

$$\left\|\frac{h(e^{\mathrm{i}x})-h(e^{\mathrm{i}y})}{e^{\mathrm{i}x}-e^{\mathrm{i}y}}\right\|_{\mathfrak{M}_{[0,2\pi],[0,2\pi]}} \geqslant \operatorname{const}\|h\|_{B_1^1}, \tag{9.3}$$

where B_1^1 is a Besov space (see [30] for the definition) of functions on $\mathbb{T}$. Note that this result was deduced in [27] from the nuclearity criterion for Hankel operators (see [26] and [30, Chapter 6]). It is easy to see from the definition of $B_1^1(\mathbb{T})$ (see, e.g., [30]) that

$$\|h\|_{B_1^1} \geqslant \operatorname{const}\sum_{j\geqslant 0} 2^j\left|\hat{h}\left(2^j\right)\right|. \tag{9.4}$$

It is well known (see, for example, [12]) that for every positive integer n, there exists an analytic polynomial h such that

$$h(0)=0, \qquad \deg h = n, \qquad \left\|h'\right\|_{L^\infty(\mathbb{T})} = 1, \quad \text{and} \quad \sum_{j\geqslant 0} 2^j\left|\hat{h}\left(2^j\right)\right| \geqslant d\sqrt{\log n},$$

where d is a positive numerical constant. Then inequality (9.3) implies that

$$\left\|\frac{h(e^{\mathrm{i}x})-h(e^{\mathrm{i}y})}{e^{\mathrm{i}x}-e^{\mathrm{i}y}}\right\|_{\mathfrak{M}_{[0,2\pi],[0,2\pi]}} \geqslant \operatorname{const}\sqrt{\log n}.$$

Put $f(x) \stackrel{\text{def}}{=} h(e^{\mathrm{i}x})$. It remains to observe that $\|f'\|_{L^\infty} = \|h'\|_{L^\infty(\mathbb{T})} = 1$ and $\|f\|_{L^\infty} = \|h\|_{L^\infty(\mathbb{T})} \leqslant 1$. $\square$

Lemma 9.6. *Let $n \in \mathbb{Z}$. Then*

$$\left\|\frac{x-y-2\pi n}{e^{\mathrm{i}x}-e^{\mathrm{i}y}}\right\|_{\mathfrak{M}_{J_1,J_2}} \leqslant \frac{3\sqrt{2}\pi}{4}$$

for all intervals J_1 and J_2 with $J_1 - J_2 \subset [(2n-\frac{3}{2})\pi, (2n+\frac{3}{2})\pi]$.

Proof. We can restrict ourselves to the case $n=0$. We have

$$\left\|\frac{x-y}{e^{\mathrm{i}x}-e^{\mathrm{i}y}}\right\|_{\mathfrak{M}_{J_1,J_2}}=\left\|\frac{x-y}{e^{\mathrm{i}(x-y)}-1}\right\|_{\mathfrak{M}_{J_1,J_2}}$$
$$\leqslant\left\|\frac{t}{e^{\mathrm{i}t}-1}\right\|_{\widehat{L}^1([-\frac{3\pi}{2},\frac{3\pi}{2}])}=\left\|\frac{t}{2\sin(t/2)}\right\|_{\widehat{L}^1([-\frac{3\pi}{2},\frac{3\pi}{2}])}.$$

Consider the 3π-periodic function ξ such that $\xi(t)=\frac{t}{2\sin(t/2)}$ for $t\in[-\frac{3\pi}{2},\frac{3\pi}{2}]$. We can expand ξ in Fourier series

$$\xi(t)=\sum_{n\in\mathbb{Z}}a_n e^{\frac{2}{3}n\mathrm{i}t}.$$

Note that $a_n=a_{-n}\in\mathbb{R}$ for all $n\in\mathbb{Z}$ because ξ is even and real. Moreover, ξ is convex on $[-\frac{3\pi}{2},\frac{3\pi}{2}]$. Hence, by Theorem 35 in [16], $(-1)^n a_n\geqslant 0$ for all $n\in\mathbb{Z}$. It follows that

$$\left\|\frac{t}{2\sin(t/2)}\right\|_{\widehat{L}^1([-\frac{3\pi}{2},\frac{3\pi}{2}])}\leqslant\sum_{n\in\mathbb{Z}}|a_n|=\xi\left(\frac{3\pi}{2}\right)=\frac{3\sqrt{2}\pi}{4}. \qquad \square$$

Corollary 9.7. *Let $J_1=[\pi j,\pi j+\pi]$ and $J_2=[\pi k-\frac{\pi}{2},\pi k+\frac{\pi}{2}]$, where $j,k\in\mathbb{Z}$. Then*

$$\left\|\frac{x-y-2\pi n}{e^{\mathrm{i}x}-e^{\mathrm{i}y}}\right\|_{\mathfrak{M}_{J_1,J_2}}\leqslant\frac{3\sqrt{2}\pi}{4}$$

for some $n\in\mathbb{Z}$.

Proof. We have $J_1-J_2=[\pi(j-k)-\frac{\pi}{2},\pi(j-k)+\frac{3\pi}{2}]$. If $j-k$ is even, then we can apply Lemma 9.6 with $n=\frac{1}{2}(j-k)$. If $j-k$ is odd, then we can apply Lemma 9.6 with $n=\frac{1}{2}(j-k+1)$. $\square$

Lemma 9.8. *Let g be a 2π-periodic function in $C^1(\mathbb{R})$. Then*

$$\left\|\frac{g(x)-g(y)}{e^{\mathrm{i}x}-e^{\mathrm{i}y}}\right\|_{\mathfrak{M}_{[0,2\pi],[0,2\pi]}}\leqslant 3\sqrt{2}\pi\left\|\frac{g(x)-g(y)}{x-y}\right\|_{\mathfrak{M}_{\mathbb{R},\mathbb{R}}}.$$

Proof. Note that

$$\left\|\frac{g(x)-g(y)}{x-y}\right\|_{\mathfrak{M}_{\mathbb{R},\mathbb{R}}}=\left\|\frac{g(x)-g(y)}{x-y-2\pi n}\right\|_{\mathfrak{M}_{\mathbb{R},\mathbb{R}}}$$

for all $n\in\mathbb{Z}$ and

$$\left\|\frac{g(x)-g(y)}{e^{\mathrm{i}x}-e^{\mathrm{i}y}}\right\|_{\mathfrak{M}_{[0,2\pi],[0,2\pi]}}=\left\|\frac{g(x)-g(y)}{e^{\mathrm{i}x}-e^{\mathrm{i}y}}\right\|_{\mathfrak{M}_{[0,2\pi],[-\frac{\pi}{2},\frac{3\pi}{2}]}}.$$

Now we can represent the square $[0,2\pi]\times[-\frac{\pi}{2},\frac{3\pi}{2}]$ as the union of four squares with sides of length π, each of which satisfies the hypotheses of Corollary 9.7. $\square$

Proof of Theorem 9.4. It suffices to consider the case when $\delta\in(0,\frac{1}{2}]$. Then $\delta\in[\frac{1}{n+1},\frac{1}{n}]$ for an integer $n\geqslant 2$. By Lemma 9.5, there exists a trigonometric polynomial f of degree n such that $\|f\|_{L^\infty}\leqslant 1$, $\|f'\|_{L^\infty}\leqslant 1$ and

$$\left\|\frac{f(x)-f(y)}{e^{\mathrm{i}x}-e^{\mathrm{i}y}}\right\|_{\mathfrak{M}_{[0,2\pi],[0,2\pi]}}\geqslant c\sqrt{\log n}.$$

Hence,

$$\left\|\frac{f(x)-f(y)}{x-y}\right\|_{\mathfrak{M}_{\mathbb{R},\mathbb{R}}}\geqslant c\sqrt{\log n}$$

by Lemma 9.8. Clearly, $g\in\mathscr{E}_n\subset\mathscr{E}_{1/\delta}$. Applying Theorem 9.3, we obtain

$$\varOmega_f(t)\geqslant \text{const}\sqrt{\log n}\;t,\quad 0<t\leqslant\frac{1}{2n}.$$

Hence,

$$\varOmega_f(\delta)\geqslant\varOmega_f\left(\frac{1}{2n}\right)\geqslant C_0\frac{\sqrt{\log n}}{n}\geqslant C\delta\sqrt{\log\left(\frac{2}{\delta}\right)}$$

for some positive numbers C_0 and C. $\square$

Theorem 9.9. *There exist a positive number c and a function $f\in C^\infty(\mathbb{R})$ such that $\|f\|_{L^\infty}\leqslant 1$, $\|f'\|_{L^\infty}\leqslant 1$, and $\varOmega_f(\delta)\geqslant c\delta\sqrt{\log\frac{2}{\delta}}$ for every $\delta\in(0,1]$.*

Proof. Applying Theorem 9.4 for $\delta=2^{-n}$, we can construct a sequence of functions $\{f_n\}_{n\geqslant1}$ and two sequences of bounded self-adjoint operators $\{A_n\}_{n\geqslant1}$ and $\{B_n\}_{n\geqslant1}$ such that $\|f_n\|_{L^\infty}\leqslant 1$, $\|f_n'\|_{L^\infty}\leqslant1$, $\|A_n-B_n\|\leqslant 2^{-n}$ and $\|f_n(A_n)-f_n(B_n)\|\geqslant C\sqrt{n}2^{-n}$ for all $n\geqslant1$. Denote by Δ_n the convex hull of $\sigma(A_n)\cup\sigma(B_n)$. Using the translations $f_n\mapsto f_n(x-a_n)$, $A_n\mapsto A_n+a_nI$, $B_n\mapsto B_n+a_nI$ and $\Delta_n\mapsto a_n+\Delta_n$ for a suitable sequence $\{a_n\}_{n=1}^\infty$ in $\mathbb{R}$, we can achieve the condition that the intervals Δ_n are disjoint and $\operatorname{dist}(\Delta_m,\Delta_n)>2$ for $m\neq n$. We can construct a function $f\in C^\infty(\mathbb{R})$ such that $\|f\|_{L^\infty}\leqslant1$, $\|f'\|_{L^\infty}\leqslant1$ and $f|\Delta_n=f_n|\Delta_n$ for all $n\geqslant1$. Clearly, $\varOmega_f(2^{-n})\geqslant C\sqrt{n}2^{-n}$ for all $n\geqslant1$ and some positive C which easily implies the result. $\square$

To obtain the lower estimate in Theorem 9.9, we used the inequality

$$\left\|\frac{f(e^{\mathrm{i}x})-f(e^{\mathrm{i}y})}{e^{\mathrm{i}x}-e^{\mathrm{i}y}}\right\|_{\mathfrak{M}_{[0,2\pi],[0,2\pi]}}\geqslant\text{const}\sum_{j\geqslant0}2^j\big|\hat f\big(2^j\big)\big|,\tag{9.5}$$

which in turn implies that there exists a positive number C such that for every positive integer n there exists a polynomial f of degree n such that

$$\left\| \frac{f(e^{\mathrm{i}x}) - f(e^{\mathrm{i}y})}{e^{\mathrm{i}x} - e^{\mathrm{i}y}} \right\|_{\mathfrak{M}_{[0,2\pi],[0,2\pi]}} \geqslant C\sqrt{\log n}\|f\|_{\mathrm{Lip}}. \tag{9.6}$$

We do not know whether Theorem 9.9 can be improved. It would certainly be natural to try to improve (9.6). The best known lower estimate for the norm of divided differences in the space of Schur multipliers was obtained in [27]. To state it, we need some definitions.

Let $f \in L^1(\mathbb{T})$. Denote by $\mathscr{P}f$ the Poisson integral of f,

$$(\mathscr{P})f(z) \stackrel{\text{def}}{=} \int_{\mathbb{T}} \frac{(1-|z|^2)f(\zeta)}{|z-\zeta|^2}\, d\boldsymbol{m}(\zeta), \quad z \in \mathbb{D},$$

where $\boldsymbol{m}$ is normalized Lebesgue measure on $\mathbb{T}$.

For $t \in \mathbb{R}$ and $\delta \in (0,1)$, we define the Carleson domain $Q(t,\delta)$ by

$$Q(t,\delta) \stackrel{\text{def}}{=} \bigl\{re^{\mathrm{i}s}\colon\ 0 < 1-r < h,\ |s-t| < \delta\bigr\}.$$

A positive Borel measure on μ on $\mathbb{D}$ is said to be a *Carleson measure* if

$$\mathscr{C}(\mu) \stackrel{\text{def}}{=} \mu(\mathbb{D}) + \sup\bigl\{\delta^{-1}\mu\bigl(Q(t,\delta)\bigr)\colon\ t \in \mathbb{R},\ \delta \in (0,1)\bigr\} < \infty.$$

If ψ is a nonnegative measurable function on $\mathbb{D}$, we put

$$\mathscr{C}(\psi) \stackrel{\text{def}}{=} \mathscr{C}(\mu), \quad \text{where } d\mu \stackrel{\text{def}}{=} \psi\, d\boldsymbol{m}_2.$$

Here $\boldsymbol{m}_2$ is planar Lebesgue measure.

It follows from results of [27] (see also [29]) that

$$\left\| \frac{f(e^{\mathrm{i}x}) - f(e^{\mathrm{i}y})}{e^{\mathrm{i}x} - e^{\mathrm{i}y}} \right\|_{\mathfrak{M}_{[0,2\pi],[0,2\pi]}} \geqslant \mathrm{const}\,\|f\|_{\mathscr{L}}, \tag{9.7}$$

where

$$\|f\|_{\mathscr{L}} \stackrel{\text{def}}{=} \mathscr{C}\bigl(\bigl\|\mathrm{Hess}(\mathscr{P}f)\bigr\|\bigr),$$

where for a function φ of class C^2, its Hessian $\mathrm{Hess}(\varphi)$ is the matrix of its second order partial derivatives.

It turns out, however, that for a trigonometric polynomial f of degree n,

$$\|f\|_{\mathscr{L}} \leqslant \mathrm{const}\sqrt{\log(1+n)}\|f\|_{\mathrm{Lip}}, \tag{9.8}$$

and so even if instead of inequality (9.5) we use inequality (9.7), we cannot improve Theorem 9.9.

Inequality (9.8) is an immediate consequence of the following fact:

Theorem 9.10. *For a trigonometric polynomial f of degree n, $n \geqslant 2$, the following inequality holds*:

$$\mathscr{C}\bigl(\bigl|\nabla(\mathscr{P}f)\bigr|\bigr) \leqslant \mathrm{const}\sqrt{\log n}\,\|f\|_{L^\infty}.$$

We are going to use the well-known fact that a function f in $L^1(\mathbb{T})$ belongs to the space $\mathrm{BMO}(\mathbb{T})$ if and only if the measure μ defined by $d\mu = |\nabla(\mathscr{P}f)|^2(1-|z|)\,d\boldsymbol{m}_2$ is a Carleson measure. We refer to [13] for Carleson measures and the space BMO.

Proof of Theorem 9.10. Suppose that $\|f\|_{L^\infty} = 1$. We have to prove that

$$\int\limits_{Q(t,\delta)} \bigl|\nabla(\mathscr{P}f)\bigr|\,dx\,dy \leqslant \operatorname{const}\delta\sqrt{\log n}\,. \tag{9.9}$$

Note that $|\nabla(\mathscr{P}f)| \leqslant 2n$ by Bernstein's inequality. Hence,

$$\begin{aligned}\int\limits_{\{1-n^{-1}<|\zeta|<1\}\cap Q(t,\delta)} \bigl|\nabla(\mathscr{P}f)\bigr|\,d\boldsymbol{m}_2 &\leqslant 2n\boldsymbol{m}_2\bigl(\bigl\{\zeta\colon\ 1-n^{-1}<|\zeta|<1\bigr\}\cap Q(t,\delta)\bigr)\\ &= 2n\delta\bigl(1-\bigl(1-n^{-1}\bigr)^2\bigr) \leqslant 4\delta.\end{aligned}$$

This proves (9.9) in the case $\delta \geqslant 1-n^{-1}$. In the case $\delta < 1-n^{-1}$ it remains to estimate the integral over the set $\{\zeta\colon\ |\zeta| < 1-n^{-1}\}\cap Q(t,\delta)$. Note that $\|f\|_{\mathrm{BMO}} \leqslant \operatorname{const}\|f\|_{L^\infty}$. Hence, there exists a constant C such that

$$\int\limits_{Q(t,\delta)} \bigl|\nabla(\mathscr{P}f)\bigr|^2\bigl(1-|\zeta|\bigr)\,d\boldsymbol{m}_2(\zeta) \leqslant C\delta.$$

Thus

$$\begin{aligned}&\int\limits_{\{|\zeta|<1-n^{-1}\}\cap Q(t,\delta)} \bigl|\nabla(\mathscr{P}f)\bigr|\,d\boldsymbol{m}_2\\ &\quad\leqslant \Biggl(\int\limits_{Q(t,\delta)} \bigl|\nabla(\mathscr{P}f)\bigr|^2\bigl(1-|\zeta|\bigr)\,d\boldsymbol{m}_2(\zeta)\Biggr)^{1/2}\Biggl(\int\limits_{\{|\zeta|<1-n^{-1}\}\cap Q(t,\delta)} \bigl(1-|\zeta|\bigr)^{-1}\,d\boldsymbol{m}_2(\zeta)\Biggr)^{1/2}\\ &\quad\leqslant \operatorname{const}\delta\bigl(\log(n\delta)\bigr)^{1/2} \leqslant \operatorname{const}\delta(\log n)^{1/2}. \qquad \square\end{aligned}$$

10. Lower estimates in the case of unitary operators

The purpose of this section is to obtain lower estimates for the operator modulus of continuity for functions on the unit circle.

We define an operator modulus of continuity of a continuous function f on $\mathbb{T}$ by

$$\Omega_f(\delta) \stackrel{\text{def}}{=} \sup\bigl\{\bigl\|f(U)-f(V)\bigr\|\colon\ U \text{ and } V \text{ are unitary},\ \|U-V\| \leqslant \delta\bigr\}.$$

As in the case of self-adjoint operators (see [2]), one can prove that

$$\bigl\|f(U)R - Rf(V)\bigr\| \leqslant 2\Omega_f\bigl(\|UR-RV\|\bigr)$$

for all unitary operators U, V and an operator R of norm 1. We define the space $\mathrm{OL}(\mathbb{T})$ as the set of $f \in C(\mathbb{T})$ such that

$$\|f\|_{\mathrm{OL}(\mathbb{T})} \stackrel{\mathrm{def}}{=} \sup_{\delta>0} \delta^{-1} \Omega_f(\delta) < \infty.$$

Given a closed subset $\mathfrak{F}$ of $\mathbb{T}$, we can also introduce the operator modulus of continuity $\Omega_{f,\mathfrak{F}}$ and define the space $\mathrm{OL}(\mathfrak{F})$ of operator Lipschitz functions on $\mathfrak{F}$.

For closed subsets $\mathfrak{F}_1$ and $\mathfrak{F}_2$ of $\mathbb{T}$, the space $\mathfrak{M}_{\mathfrak{F}_1,\mathfrak{F}_2}$ of Schur multipliers can be defined by analogy with the self-adjoint case. Note that the analogues of (5.1) and (5.3) for functions on closed subsets of $\mathbb{T}$ can be proved as in Section 5.

Let $f \in C(\mathbb{T})$. We put $f_\spadesuit(t) \stackrel{\mathrm{def}}{=} f(e^{\mathrm{i}t})$. It is clear that $\Omega_{f_\spadesuit} \leqslant \Omega_f$. Hence, $\|f_\spadesuit\|_{\mathrm{OL}(\mathbb{R})} \leqslant \|f\|_{\mathrm{OL}(\mathbb{T})}$. Lemma 9.8 implies that $\|f\|_{\mathrm{OL}(\mathbb{T})} \leqslant 3\sqrt{2}\pi \|f_\spadesuit\|_{\mathrm{OL}(\mathbb{R})}$. One can prove that $\Omega_f \leqslant \mathrm{const}\, \Omega_{f_\spadesuit}$.

Recall that it follows from results of [27] that for $f \in C(\mathbb{T})$,

$$\|f\|_{\mathrm{OL}(\mathbb{T})} \geqslant \mathrm{const} \|f\|_{B_1^1};$$

actually we used this estimate in Section 9, see inequality (9.3).

We would like to remind also that for each positive integer n, there exists an analytic polynomial f such that $\deg f = n$, $\|f'\|_{L^\infty(\mathbb{T})} = 1$, and $\|f\|_{\mathrm{OL}(\mathbb{T})} \geqslant \mathrm{const}\sqrt{\log n}$; see Lemma 9.5.

Put

$$\mathfrak{d}_n(z) \stackrel{\mathrm{def}}{=} \frac{1}{n}\frac{z^n - 1}{z - 1} = \frac{1}{n}\sum_{k=0}^{n-1} z^k.$$

It is easy to see that

$$\mathfrak{d}_n\big(\zeta z^{-1}\big) = z^{1-n}\frac{z^n - \zeta^n}{n(z-\zeta)} = z^{1-n}\zeta^{n-1}\mathfrak{d}_n\big(z\zeta^{-1}\big).$$

Denote by $\mathbb{T}_n$ the set of nth roots of 1, i.e., $\mathbb{T}_n \stackrel{\mathrm{def}}{=} \{\zeta \in \mathbb{T}\colon \zeta^n = 1\}$.

Let f be an analytic polynomial of degree less that n. Then

$$f(z) = \sum_{\zeta \in \tau\mathbb{T}_n} f(\zeta)\mathfrak{d}_n\big(z\zeta^{-1}\big) \quad \text{for every } \tau \in \mathbb{T}.$$

If f is a trigonometric polynomial and $\deg f \leqslant n$, then for every $\xi \in \mathbb{T}$, the function $z^n f(z)\mathfrak{d}_{2n}(z\xi^{-1})$ is an analytic polynomial of degree less than $4n$. Hence,

$$z^n f(z)\mathfrak{d}_{2n}\big(z\xi^{-1}\big) = \sum_{\zeta\in\tau\mathbb{T}_{4n}} f(\zeta)\mathfrak{d}_{2n}\big(\zeta\xi^{-1}\big)\mathfrak{d}_{4n}\big(z\zeta^{-1}\big).$$

Substituting $\xi = z$ we get

$$f(z) = z^{-n}\sum_{\zeta\in\tau\mathbb{T}_{4n}} f(\zeta)\mathfrak{d}_{2n}\big(\zeta z^{-1}\big)\mathfrak{d}_{4n}\big(z\zeta^{-1}\big) = \sum_{\zeta\in\tau\mathbb{T}_{4n}} f(\zeta)F_n(z,\zeta) \tag{10.1}$$

for every $\tau \in \mathbb{T}$, where

$$F_n(z,\zeta) \stackrel{\mathrm{def}}{=} z^{1-3n}\zeta^{1-4n}\frac{(z^{2n}-\zeta^{2n})(z^{4n}-\zeta^{4n})}{8n^2(z-\zeta)^2}.$$

Denote by $\mathcal{P}_n(\mathbb{T}^2)$ the set of all trigonometric polynomial f on $\mathbb{T}^2$ such that the functions $z \mapsto f(z,\xi)$ and $z \mapsto f(\xi,z)$ are trigonometric polynomials on $\mathbb{T}$ of degree at most n for every $\xi \in \mathbb{T}$. Equality (10.1) implies the following identity:

$$f(z,w) = \sum_{\zeta\in\tau_1\mathbb{T}_{4n}}\sum_{\xi\in\tau_2\mathbb{T}_{4n}} f(\zeta,\xi)F_n(z,\zeta)F_n(w,\xi) \tag{10.2}$$

for every $f \in \mathcal{P}_n(\mathbb{T}^2)$ and for arbitrary τ_1 and τ_2 in $\mathbb{T}$.

Theorem 10.1. *Let $\Phi \in \mathcal{P}_n(\mathbb{T}^2)$. Then*

$$\|\Phi\|_{\mathfrak{M}_{\mathbb{T},\mathbb{T}}} \leqslant 2\|\Phi\|_{\mathfrak{M}_{\tau_1\mathbb{T}_{4n},\tau_2\mathbb{T}_{4n}}}$$

for all $\tau_1, \tau_2 \in \mathbb{T}$.

Proof. Clearly, it suffices to consider the case when $\tau_1 = \tau_2 = 1$ and $\|\Phi\|_{\mathfrak{M}_{\mathbb{T}_{4n},\mathbb{T}_{4n}}} = 1$. Then (see [31, Theorem 5.1]) there exist two sequences $\{\varphi_\zeta\}_{\zeta\in\mathbb{T}_{4n}}$ and $\{\psi_\xi\}_{\xi\in\mathbb{T}_{4n}}$ of vectors in the closed unit ball of a Hilbert space $\mathcal{H}$ such that $(\varphi_\zeta, \psi_\xi) = \Phi(\zeta,\xi)$. Put

$$g_z \stackrel{\mathrm{def}}{=} \sum_{\zeta\in\mathbb{T}_{4n}} F_n(z,\zeta)\varphi_\zeta \quad \text{and} \quad h_w \stackrel{\mathrm{def}}{=} \sum_{\xi\in\mathbb{T}_{4n}} F_n(w,\xi)\psi_\xi.$$

Taking into account that for $z \in \mathbb{T}$,

$$\frac{1}{2n}\sum_{\zeta\in\mathbb{T}_{2n}}\left|\frac{z^{2n}-\zeta^{2n}}{z-\zeta}\right|^2 = \frac{1}{2n}\sum_{\zeta\in\mathbb{T}_{4n}\setminus\mathbb{T}_{2n}}\left|\frac{z^{2n}-\zeta^{2n}}{z-\zeta}\right|^2 = \int_{\mathbb{T}}\left|\frac{z^{2n}-\zeta^{2n}}{z-\zeta}\right|^2 d\boldsymbol{m}(\zeta) = 2n,$$

we obtain

$$\begin{aligned}
\|g_z\|_{\mathcal{H}} &\leqslant \sum_{\zeta\in\mathbb{T}_{4n}}\left|F_n(z,\zeta)\right| \\
&\leqslant \frac{|z^{2n}+1|}{8n^2}\sum_{\zeta\in\mathbb{T}_{2n}}\left|\frac{z^{2n}-\zeta^{2n}}{z-\zeta}\right|^2 + \frac{|z^{2n}-1|}{8n^2}\sum_{\zeta\in\mathbb{T}_{4n}\setminus\mathbb{T}_{2n}}\left|\frac{z^{2n}-\zeta^{2n}}{z-\zeta}\right|^2 \\
&= \frac{|z^{2n}+1|+|z^{2n}-1|}{2} \leqslant \sqrt{2}.
\end{aligned}$$

In the same way, $\|h_w\|_{\mathcal{H}} \leqslant \sqrt{2}$ for every $w \in \mathbb{T}$. By (10.2), we have $\Phi(z,w) = (g_z, h_w)$ for all $z, w \in \mathbb{T}$. It remains to observe that by Theorem 5.1 in [31],

$$\left\|\Phi(z,w)\right\|_{\mathfrak{M}_{\mathbb{T},\mathbb{T}}} \leqslant \sup_{z\in\mathbb{T}}\|g_z\|_{\mathcal{H}} \cdot \sup_{w\in\mathbb{T}}\|h_w\|_{\mathcal{H}} \leqslant 2. \qquad \square$$

We need the following version of Theorem 5.7:

Theorem 10.2. *Let f be a function on $\mathbb{T}_n$. Then*

$$\Omega^{\flat}_{f,\mathbb{T}_n}(\delta)=\delta\|f\|_{\mathrm{OL}(\mathbb{T}_n)}$$

for every $\delta\in(0,\frac{4}{n}]$.

To prove Theorem 10.2, we need a lemma. Put

$$\lambda(z)\stackrel{\mathrm{def}}{=}\begin{cases} z^{-1}, & \text{if } z\in\mathbb{C},\ z\neq 0,\\ 0, & \text{if } z=0.\end{cases}$$

Lemma 10.3. *Let n be a positive integer. Then*

$$\left\|\lambda(z-w)\right\|_{\mathfrak{M}_{\mathbb{T}_n,\mathbb{T}_n}}=\begin{cases}\frac{n}{4}, & \text{if } n \text{ is even},\\ \frac{n^2-1}{4n}, & \text{if } n \text{ is odd}.\end{cases}$$

Proof. It is easy to verify that

$$\sum_{k=1}^{n}\left(k-\frac{n+1}{2}\right)z^k=\frac{nz^n}{z-1}-\frac{z^n-1}{(z-1)^2}-\frac{n+1}{2}z\frac{z^n-1}{z-1}=n\lambda(z-1)$$

for $z\in\mathbb{T}_n$. Hence,

$$\lambda(z-w)=w^{-1}\lambda\bigl(zw^{-1}-1\bigr)=\frac{1}{n}\sum_{k=1}^{n}\left(k-\frac{n+1}{2}\right)z^k w^{-k-1}. \tag{10.3}$$

Thus

$$\left\|\lambda(z-w)\right\|_{\mathfrak{M}_{\mathbb{T}_n,\mathbb{T}_n}}\leqslant\frac{1}{n}\sum_{k=1}^{n}\left|k-\frac{n+1}{2}\right|=\begin{cases}\frac{n}{4}, & \text{if } n \text{ is even},\\ \frac{n^2-1}{4n}, & \text{if } n \text{ is odd}.\end{cases}$$

The opposite inequality is also true. It can be deduced from the observation that equality (10.3) means that the function $\lambda(z-1)$ on the group $\mathbb{T}_n$ is the Fourier transform of the n-periodic sequence $\{a_k\}_{k\in\mathbb{Z}}$ defined by $a_k=k-\frac{n+1}{2}$ for $k=1,2,\dots,n$. Here we identify the group dual to $\mathbb{T}_n$ with the group $\mathbb{Z}/n\mathbb{Z}$. We omit details because we need only the upper estimate. □

Proof of Theorem 10.2. The inequality

$$\Omega^{\flat}_{f,\mathbb{T}_n}(\delta)\leqslant\delta\|f\|_{\mathrm{OL}(\mathbb{T}_n)},\quad \delta>0,$$

is a consequence of a unitary version of Theorem 5.1, which can be proved in the same way as the self-adjoint version, see also [3, Theorem 4.13].

Let us prove the opposite inequality for $\delta \in (0, \frac{4}{n}]$. Fix $\varepsilon > 0$. There exists a unitary operator U and bounded operator R such that $\|UR - RU\| = 1$, $\sigma(U) \subset \mathbb{T}_n$, and $\|f(U)R - Rf(U)\| \geqslant \|f\|_{\mathrm{OL}(\mathbb{T}_n)} - \varepsilon$. Put

$$R_U \stackrel{\mathrm{def}}{=} \sum_{\zeta,\xi\in\mathbb{T}_n,\, \zeta\neq\xi} E_U\big(\{\zeta\}\big)RE_U\big(\{\xi\}\big) = R - \sum_{\zeta\in\mathbb{T}_n} E_U\big(\{\zeta\}\big)RE_U\big(\{\zeta\}\big).$$

Clearly, $UR - RU = UR_U - R_UU$ and $f(U)R - Rf(U) = f(U)R_U - R_Uf(U)$. Thus we may assume that $R = R_U$. Note that

$$UR - RU = \sum_{\zeta,\xi\in\mathbb{T}_n,\ \zeta\neq\xi} (\zeta - \xi)E_U\big(\{\zeta\}\big)RE_U\big(\{\xi\}\big).$$

Since

$$R = R_U = \sum_{\zeta,\xi\in\mathbb{T}_n,\ \zeta\neq\xi} (\zeta - \xi)\lambda(\zeta - \xi)E_U\big(\{\zeta\}\big)RE_U\big(\{\xi\}\big),$$

we have $R = H_n \star (UR - RU)$, where $H_n(\zeta, \xi) = \lambda(\zeta - \xi)$, where $\zeta, \xi \in \mathbb{T}_n$. Thus by Lemma 10.3,

$$\|R\| \leqslant \|H_n\|_{\mathfrak{M}_{\mathbb{T}_n,\mathbb{T}_n}} \|UR - RU\| = \|H_n\|_{\mathfrak{M}_{\mathbb{T}_n,\mathbb{T}_n}} \leqslant \frac{n}{4}.$$

Let $\delta \in (0, \frac{4}{n}]$. Then $\|U(\delta R) - (\delta R)U\| = \delta$ and $\|\delta R\| \leqslant 1$. Hence,

$$\Omega^\flat_{f,\mathbb{T}_n}(\delta) \geqslant \delta\big\|f(U)R - Rf(U)\big\| \geqslant \delta\big(\|f\|_{\mathrm{OL}(\mathbb{T}_n)} - \varepsilon\big).$$

Passing to the limit as $\varepsilon \to 0$, we obtain the desired result. □

Theorem 10.4. *Let f be a trigonometric polynomial of degree $n \geqslant 1$. Then*

$$\Omega^\flat_{f,\mathbb{T}}(\delta) \geqslant \frac{\delta}{2}\|f\|_{\mathrm{OL}(\mathbb{T})}$$

for $\delta \in (0, \frac{1}{n}]$.

Proof. Applying Theorems 10.1 and 10.2, we obtain

$$\left\|\frac{f(z) - f(w)}{z - w}\right\|_{\mathfrak{M}_{\mathbb{T},\mathbb{T}}} \leqslant 2\left\|\frac{f(z) - f(w)}{z - w}\right\|_{\mathfrak{M}_{\mathbb{T}_{4n},\mathbb{T}_{4n}}} = 2\delta^{-1}\Omega^\flat_{f,\mathbb{T}_{4n}}(\delta) \leqslant 2\delta^{-1}\Omega^\flat_{f,\mathbb{T}}(\delta)$$

for $\delta \in (0, \frac{1}{n}]$. □

Theorem 10.5. *Let f be a trigonometric polynomial of degree $n \geqslant 1$. Then*

$$\Omega_{f,\mathbb{T}}(\delta) \geqslant \frac{\delta}{4}\|f\|_{\mathrm{OL}(\mathbb{T})}$$

for $\delta \in (0, \frac{1}{n}]$.

Proof. It suffices to observe that $\Omega^{\flat}_{f,\mathbb{T}}(\delta) \leqslant 2\Omega_{f,\mathbb{T}}(\delta)$. □

Theorem 10.6. *Let $f \in C(\mathbb{T})$. Then*

$$\Omega_f\big(2^{-n}\big) \geqslant C\,2^{-n}\sum_{k=0}^{n-1} 2^k\big(\big|\hat{f}\big(2^k\big)\big| + \big|\hat{f}\big(-2^k\big)\big|\big),$$

where C is a positive constant.

Proof. Applying the convolution with the de la Vallée Poussin kernel, we can find an analytic polynomial f_n such that $\deg f_n < 2^n$, $\hat{f}_n(k) = \hat{f}(k)$ for $k \leqslant 2^{n-1}$ and $\Omega_{f_n} \leqslant 3\Omega_f$. Applying inequalities (9.3) and (9.4), we obtain

$$\|f_n\|_{\mathrm{OL}(\mathbb{T})} \geqslant \mathrm{const}\sum_{k=0}^{n-1} 2^k\big(\big|\hat{f}\big(2^k\big)\big| + \big|\hat{f}\big(-2^k\big)\big|\big).$$

It remains to apply Theorem 10.5 for $\delta = 2^{-n}$. □

In the following theorem we use the notation C_A for the disk-algebra:

$$C_A \stackrel{\mathrm{def}}{=} \big\{f \in C(\mathbb{T}):\ \hat{f}(n) = 0 \text{ for } n < 0\big\}.$$

Theorem 10.7. *Let $\omega:(0,2] \to \mathbb{R}$ be a positive continuous function. Suppose that $\omega(2t) \leqslant \mathrm{const}\,\omega(t)$, the function $t \mapsto t^{-1}(\log\frac{4}{t})^{-1}\omega(t)$ is nondecreasing, and*

$$\int_0^2 \frac{\omega^2(t)\,dt}{t^3\log^2\frac{4}{t}} < \infty. \tag{10.4}$$

Then there exists a function $f \in C_A$ such that $f' \in C_A$ and $\Omega_f(\delta) \geqslant \omega(\delta)$ for all $\delta \in (0,2]$.

Proof. Note that the inequality $\Omega_f(\delta) \geqslant \omega(\delta)$ for $\delta = 2^{-n}$ implies that $\Omega_f(\delta) \geqslant \mathrm{const}\,\omega(\delta)$ for all $\delta \in (0,2]$. Thus it suffices to obtain the desired estimate for $\delta = 2^{-n}$. Taking Theorem 10.6 into account, we can reduce the result to the problem to construct a function $g \in C_A$ such that

$$a_n \stackrel{\mathrm{def}}{=} \frac{2^n\omega(2^{-n})}{n} \leqslant \frac{1}{n}\sum_{k=0}^{n-1}\big|\hat{g}\big(2^k\big)\big|$$

for all nonnegative integer n.

Indeed, in this case the function f defined by

$$f(z)=\int_0^z \frac{g(\zeta)-g(0)}{\zeta}\,d\zeta$$

satisfies the inequality

$$a_n \leqslant \frac{1}{n}\sum_{k=0}^{n-1} 2^k\big|\hat f\big(2^k\big)\big|.$$

Condition (10.4) implies that $\{a_n\}_{n\geqslant 0}\in \ell^2$. Moreover, $\{a_n\}_{n\geqslant 0}$ is a nonincreasing sequence because the function $t\mapsto t^{-1}(\log\frac{4}{t})^{-1}\omega(t)$ is nondecreasing.

We can find a function $g\in C_A$ such that $\hat g(2^k)=a_k$ for all $k\geqslant 0$, see, for example, [12]. Then

$$\frac{1}{n}\sum_{k=0}^{n-1}\big|\hat g\big(2^k\big)\big| = \frac{1}{n}\sum_{k=0}^{n-1} a_k \geqslant a_{n-1}\geqslant a_n. \qquad \square$$

Remark. Theorem 10.7 remains valid if we replace the assumption that the function $t\mapsto t^{-1}(\log\frac{4}{t})^{-1}\omega(t)$ is nondecreasing with the assumption that there exists a positive constant C such that

$$\frac{\omega(t)}{t\log\frac{4}{t}}\leqslant C\frac{\omega(s)}{s\log\frac{4}{s}}, \quad \text{whenever } 0<t<s\leqslant 2.$$

11. Self-adjoint operators with finite spectrum. Estimates in terms of the ε-entropy of the spectrum

In this section we obtain sharp estimates of the quasicommutator norms $\|f(A)R-Rf(B)\|$ in the case when A has finite spectrum. This allows us to obtain sharp estimates of the operator Lipschitz norm in terms of the Lipschitz norm in the case of operators on finite-dimensional spaces in terms of the dimension.

Moreover, we obtain a more general result (see Theorem 11.5) in terms of ε-entropy of the spectrum of A, where $\varepsilon=\|AR-RA\|$. This leads to an improvement of inequality (1.1).

Note that the results of this section improve some results of [10] and [11].

Let $\mathfrak{F}$ be a closed subset of $\mathbb{R}$. Denote by $\mathrm{Lip}(\mathfrak{F})$ the set of Lipschitz functions on $\mathfrak{F}$. Put

$$\|f\|_{\mathrm{Lip}(\mathfrak{F})}\stackrel{\mathrm{def}}{=}\inf\big\{C>0:\ \big|f(x)-f(y)\big|\leqslant C|x-y|\ \forall x,y\in\mathfrak{F}\big\}.$$

Let $\{s_j(T)\}_{j=0}^{\infty}$ be the sequence of singular values of a bounded operator T. We use the notation $\boldsymbol{S}_\omega$ for the Matsaev ideal,

$$\boldsymbol{S}_\omega\stackrel{\mathrm{def}}{=}\left\{T:\ \|T\|_{\boldsymbol{S}_\omega}\stackrel{\mathrm{def}}{=}\sum_{j=0}^{\infty}(1+j)^{-1}s_j(T)<\infty\right\}.$$

We need the following statement which is contained implicitly in [23].

Theorem 11.1. *Let f be a Lipschitz function on a closed subset $\mathfrak{F}$ of $\mathbb{R}$. Then for every nonempty finite subset Λ in $\mathfrak{F}$,*

$$\|\mathfrak{D}_0 f\|_{\mathfrak{M}_{\Lambda,\mathfrak{F}}} \leqslant C\bigl(1+\log\bigl(\operatorname{card}(\Lambda)\bigr)\bigr)\|f\|_{\mathrm{Lip}(\mathfrak{F})},$$

where C is a numerical constant.

Proof. Let $k \in L^2(\mu \otimes \nu)$, where μ and ν are Borel measures on Λ and $\mathfrak{F}$. Clearly, $\operatorname{rank} \mathcal{I}_k^{\mu,\nu} \leqslant \operatorname{card}(\Lambda)$. Hence, $\|\mathcal{I}_k^{\mu,\nu}\|_{S_\omega} \leqslant (1+\log(\operatorname{card}(\Lambda)))\|\mathcal{I}_k^{\mu,\nu}\|$. Now Theorem 2.3 in [23] implies that

$$\bigl\|\mathcal{I}_{k\mathfrak{D}_0 f}^{\mu,\nu}\bigr\| \leqslant \operatorname{const}\bigl(1+\log\bigl(\operatorname{card}(\Lambda)\bigr)\bigr)\bigl\|\mathcal{I}_k^{\mu,\nu}\bigr\| \cdot \|f\|_{\mathrm{Lip}(\mathfrak{F})}. \qquad \square$$

Theorem 11.2. *Let A and B be self-adjoint operators. Suppose that $\sigma(A)$ is finite. Then*

$$\bigl\|f(A)R - Rf(B)\bigr\| \leqslant C\bigl(1+\log\bigl(\operatorname{card}\bigl(\sigma(A)\bigr)\bigr)\bigr)\|f\|_{\mathrm{Lip}(\sigma(A)\cup\sigma(B))}\|AR-RB\|$$

for all bounded operators R and $f \in \mathrm{Lip}(\sigma(A)\cup\sigma(B))$, where C is a numerical constant.

Proof. The result follows from Theorem 11.1 if we take into account the following generalizations of (5.2) and (5.4) (see [7]):

$$f(A)R - Rf(B) = \iint\limits_{\sigma(A)\times\sigma(B)} (\mathfrak{D}_0 f)(x,y)\,dE_A(x)(AR-RB)\,dE_B(y)$$

and

$$\Biggl\|\ \iint\limits_{\sigma(A)\times\sigma(B)} (\mathfrak{D}_0 f)(x,y)\,dE_A(x)(AR-RB)\,dE_B(y)\Biggr\| \leqslant \|\mathfrak{D}_0 f\|_{\mathfrak{M}(\sigma(A)\times\sigma(B))}\|AR-RB\|$$

which proves the result. $\square$

Corollary 11.3. *Let A, B be self-adjoint operators and let R be a linear operator on $\mathbb{C}^n$. Then*

$$\bigl\|f(A)R - Rf(B)\bigr\| \leqslant C(1+\log n)\|f\|_{\mathrm{Lip}(\sigma(A)\cup\sigma(B))}\|AR-RB\| \tag{11.1}$$

for every function f on $\sigma(A)\cup\sigma(B)$, where C is a numerical constant.

Remark 1. Note that in the special case $f(t) = |t|$ inequality (11.1) is well-known, see, e.g., [8]. This special case also follows from Matsaev's theorem, see [15, Chapter III, Theorem 4.2] (see also [14] where a finite-dimensional improvement of Matsaev's theorem was obtained).

Remark 2. We also would like to note that inequality (11.1) is sharp. Indeed, it follows immediately from Lemma 15 of [8] that for each positive integer n there exist $n \times n$ self-adjoint matrices A and R such that

$$\bigl\||A|R - R|A|\bigr\| \geqslant \operatorname{const}\log(1+n)\|AR-RA\| \quad \text{and} \quad AR - RA \neq \mathbf{0}. \tag{11.2}$$

We also refer the reader to [22] where inequality (11.2) is essentially contained. Moreover, (11.2) can be deduced from the results of Matsaev and Gohberg mentioned above.

The following result is a special case of Corollary 11.3 that corresponds to $R = I$.

Theorem 11.4. *Let A, B be self-adjoint operators on $\mathbb{C}^n$. Then*

$$\big\|f(A) - f(B)\big\| \leqslant C(1 + \log n)\|f\|_{\mathrm{Lip}(\sigma(A)\cup\sigma(B))}\|A - B\|$$

for every function f on $\sigma(A) \cup \sigma(B)$, where C is an absolute constant.

Remark. The estimate in Theorem 11.4 is also sharp. Indeed, for each positive integer n there exist $n \times n$ self-adjoint matrices A and B such that $A \neq B$ and

$$\big\||A| - |B|\big\| \geqslant \mathrm{const}\log(1 + n)\|A - B\|.$$

This follows easily from (11.2), see the proof of Theorem 10.1 in [2].

Definition. Let $\mathfrak{F}$ be a nonempty compact subset of $\mathbb{R}$. Recall that for $\varepsilon > 0$, the *ε-entropy $K_\varepsilon(\mathfrak{F})$ of $\mathfrak{F}$* is defined as

$$K_\varepsilon(\mathfrak{F}) \stackrel{\mathrm{def}}{=} \inf\log\big(\mathrm{card}(\Lambda)\big),$$

where the infimum is taken over all $\Lambda \subset \mathbb{R}$ such that Λ is an ε-net of $\mathfrak{F}$. The following result is a generalization of Theorem 11.2. On the other hand, it improves inequality (1.1) obtained in [2].

Theorem 11.5. *Let A and B be self-adjoint operators and let R be bounded operator with $\|R\| \leqslant 1$. Suppose that $\sigma(A) \subset \mathfrak{F}$, where $\mathfrak{F}$ is a closed subset of $\mathbb{R}$. Then for every $f \in \mathrm{Lip}(\sigma(A) \cup \sigma(B))$,*

$$\big\|f(A)R - Rf(B)\big\| \leqslant \mathrm{const}\big(1 + K_\varepsilon(\mathfrak{F})\big)\|f\|_{\mathrm{Lip}(\sigma(A)\cup\sigma(B))}\|AR - RB\|,$$

where $\varepsilon \stackrel{\mathrm{def}}{=} \|AR - RB\|$.

Proof. We repeat the argument of the proof of Theorem 5.8. Clearly, f can be extended to a Lipschitz function on $\mathbb{R}$ with the same Lipschitz constant. We can find a self-adjoint operator A_ε such that $A_\varepsilon A = AA_\varepsilon$, $\|A - A_\varepsilon\| \leqslant \varepsilon$, and $\log(\mathrm{card}(\sigma(A_\varepsilon))) \leqslant K_\varepsilon(\mathfrak{F})$. Then

$$\begin{aligned}\big\|f(A_\varepsilon)R - Rf(B)\big\| &\leqslant \mathrm{const}\big(1 + K_\varepsilon(\mathfrak{F})\big)\|f\|_{\mathrm{Lip}(\sigma(A)\cup\sigma(B))}\|A_\varepsilon R - RB\| \\ &\leqslant 2\,\mathrm{const}\,\delta\big(1 + K_\varepsilon(\mathfrak{F})\big)\|f\|_{\mathrm{Lip}(\sigma(A)\cup\sigma(B))}\end{aligned}$$

by Theorem 11.2. It remains to observe that since A commutes wit A_ε, we have

$$\begin{aligned}\big\|f(A)R - Rf(B)\big\| &\leqslant \big\|f(A) - f(A_\varepsilon)\big\| + \big\|f(A_\varepsilon)R - Rf(B)\big\| \\ &\leqslant \varepsilon\|f\|_{\mathrm{Lip}(\sigma(A))} + \big\|f(A_\varepsilon)R - Rf(B)\big\|. \qquad \square\end{aligned}$$

Corollary 11.6. *Let A and B be self-adjoint operators and let $\sigma(A) \subset \mathfrak{F}$, where $\mathfrak{F}$ is a closed subset of $\mathbb{R}$. Then for every $f \in \mathrm{Lip}(\sigma(A) \cup \sigma(B))$,*

$$\left\| f(A) - f(B) \right\| \leqslant \mathrm{const}\bigl(1 + K_\varepsilon(\mathfrak{F})\bigr) \| f \|_{\mathrm{Lip}(\sigma(A)\cup\sigma(B))} \| A - B \|,$$

where $\varepsilon \stackrel{\mathrm{def}}{=} \|A - B\|$.

Proof. It suffices to put $R = I$. $\square$

If we apply Theorem 11.5 to the case $K = [a, b]$, we obtain the following estimate, which improves inequality (1.1) in the special case $R = I$.

Corollary 11.7. *Let $f \in \mathrm{Lip}(\mathbb{R})$. Let A be a self-adjoin t operator with $\sigma(A) \subset [a, b]$. and $\|R\| \leqslant 1$. Then for every self-adjoint operator B,*

$$\left\| f(A)R - Rf(B) \right\| \leqslant \mathrm{const} \| f \|_{\mathrm{Lip}} \log\left(2 + \frac{b - a}{\|AR - RB\|}\right) \|AR - RB\|.$$

Note that we do not impose any assumptions on the spectrum of B.

Corollary 11.8. *Let $f \in \mathrm{Lip}(\mathbb{R})$. Let A be a self-adjoint operator with $\sigma(A) \subset [a, b]$. Then for every self-adjoint operator B,*

$$\left\| f(A) - f(B) \right\| \leqslant \mathrm{const} \| f \|_{\mathrm{Lip}} \log\left(2 + \frac{b - a}{\|A - B\|}\right) \|A - B\|.$$

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