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Canonical Heights on Projective Space

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Let V be a non-singular projective variety defined over a number field K and let $\phi: V \to V$ be a morphism defined over K. Suppose there is a divisor class η on V which is an eigenclass for ϕ with eigenvalue $\alpha > 1$. Call and Silverman [2] have shown that there is a unique canonical height $\hat{h}_{V,\eta,\phi}$ on $V(\bar{K})$ characterized by the properties that $\hat{h}_{V,\eta,\phi}$ is a Weil height for the divisor class η and

$$\hat{h}_{V,\eta,\phi}(\phi P) = \alpha \hat{h}_{V,\eta,\phi}(P) \qquad \text{for all} \quad P \in V(\bar{K}).$$

The construction in [2] generalizes the work of Néron [12] and Tate (see [8]), who have proven the existence of canonical heights on abelian varieties, and of Silverman [19], who constructed canonical heights on K3 surfaces.

Suppose $\phi: \mathbf{P}^n \to \mathbf{P}^n$ is a morphism of degree $d > 1$. If η is any divisor class on $\mathbf{P}^n$, then $\phi^*\eta = d\eta$, so every divisor class is an eigenclass for ϕ. Since the divisor class group on $\mathbf{P}^n$ is isomorphic to $\mathbf{Z}$, we will restrict our attention to the divisor class on $\mathbf{P}^n$ of degree one. In this case, by the construction in [2], there is a unique canonical height $\hat{h}_\phi$ which differs from the standard Weil height on $\mathbf{P}^n$ by a bounded amount and satisfies

$$\hat{h}_\phi(\phi P) = d\hat{h}_\phi(P) \qquad \text{for all} \quad P \in \mathbf{P}^n(\bar{K}).$$

The purpose of this paper is to give explicit formulas for computing these canonical heights $\hat{h}_\phi$ on $\mathbf{P}^n$ and to begin the study of their arithmetic properties. We now describe the contents of the paper in more detail.

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In Section 1 we recall Tate's averaging procedure to describe how the canonical height $\hat{h}_\phi$ may be obtained as a limit of Weil heights on the points in the ϕ-orbit of P. Although constructions of this type are well-known (Cf. [8], pp. 82–83), it was only recently observed in [2] that for all $P \in \mathbf{P}^n(\bar{K})$, $\hat{h}_\phi(P) = 0$ if and only if the ϕ-orbit of P is finite (i.e., P is pre-periodic under ϕ). Combining this remark with the non-degeneracy of the height $\hat{h}_\phi$ yields Northcott's theorem (see [13] and Lewis [9]) that ϕ can have at most finitely many K-rational pre-periodic points. Furthermore, this suggests that by obtaining formulas for the canonical height $\hat{h}_\phi$ we may be able to count the number of K-rational pre-periodic points of ϕ. The related question of bounding the maximum period of K-rational periodic points for polynomial maps on $\mathbf{P}^n$ is studied by Morton and Silverman [10], Narkiewicz [11], and Pezda [14] and [15]. In addition, Silverman [20] has used canonical heights to bound the number of integer points in a ϕ-orbit. We will turn to the question of counting pre-periodic points for polynomial maps in Section 6.

In Section 2 we describe how the canonical height $\hat{h}_\phi$ may be decomposed into a sum of canonical local heights. Specifically, for each hyperplane H on $\mathbf{P}^n$ and each rational function $f \in K(\mathbf{P}^n)$ such that

$$\phi^* H = dH + \operatorname{div}(f),$$

it follows from [2] that there is a unique canonical local height $\hat{\lambda}_{H,\phi,f}$ which is a Weil local height for the divisor H and satisfies

$$\hat{\lambda}_{H,\phi,f}(\phi P, v) = d\hat{\lambda}_{H,\phi,f}(P, v) + v(f(P))$$

for all absolute values $v \in M_{\bar{K}}$ and all $P \in \mathbf{P}^n(\bar{K})$ such that $P, \phi P \notin |H|$. (If E is a number field or the field of algebraic numbers $\bar{K}$, we use M_E to denote the set of standard absolute values on E described in Sections 1 and 2.) Then for all number fields L and all points $P \in \mathbf{P}^n(L) \backslash |H|$,

$$\hat{h}_\phi(P) = \frac{1}{[L : \mathbf{Q}]} \sum_{v \in M_L} [L_v : \mathbf{Q}_v] \, \hat{\lambda}_{H,\phi,f}(P, v).$$

With this decomposition formula for $\hat{h}_\phi(P)$ in hand, we focus our attention on developing methods for computing the canonical local heights $\hat{\lambda}_{H,\phi,f}(P, v)$.

If $v \in M_K$ is non-archimedean and π is the prime ideal of K determined by v, then we say that ϕ has *good reduction at* v if ϕ extends to a morphism $\mathbf{P}^n/R_\pi \to \mathbf{P}^n/R_\pi$ of schemes over the local ring R_π. Thus ϕ has good reduction at v for all but finitely many non-archimedean $v \in M_K$. At each v where ϕ has good reduction, we show in Theorem 2.2 and Lemma 2.3 how any canonical local height $\hat{\lambda}_{H,\phi,f}(P, v)$ may be computed via a simple formula.

For the finitely many remaining absolute values $v \in M_K$, including the archimedean v, we give both series and sequence formulas for $\hat{\lambda}_{H,\phi,f}(P, v)$ in Section 3. In particular, given any $v \in M_K$ and any $P \in \mathbf{P}^n(K_v)$, we can explicitly describe a series which rapidly converges to $\hat{\lambda}_{H,\phi,f}(P, v)$. If the choice of homogeneous polynomials defining ϕ is fixed, then this series can be shown to telescope. If, in addition, the orbit of P under ϕ does not intersect $|H|$, then we obtain a limit formula for $\hat{\lambda}_{H,\phi,f}(P, v)$ which is analogous to the limit formula for $\hat{h}_\phi(P)$ derived in Section 1. We conclude Section 3 by computing the canonical local heights $\hat{\lambda}_{H,\tau_e,f}(P, v)$ at all non-archimedean v for a family of degree two morphisms on τ_e on $\mathbf{P}^1$.

Motivated by the limit formula obtained in Section 3, we restrict our attention beginning in Section 4 to those morphisms $\phi: \mathbf{P}^n \to \mathbf{P}^n$ which have a hyperplane eigendivisor W; i.e., a hyperplane W such that $\phi^* W = dW$ as a divisor on $\mathbf{P}^n$. For such a morphism ϕ with a hyperplane eigendivisor W, Theorem 2.1 shows that there is a unique canonical local height $\hat{\lambda}_{W,\phi}$ such that

$$\hat{\lambda}_{W,\phi}(\phi P, v) = d\hat{\lambda}_{W,\phi}(P, v),$$

for all $P \in \mathbf{P}^n \backslash |W|$. In Section 4 we show that given any Weil local height λ_W, for all v and all $P \notin |W|$, $\hat{\lambda}_{W,\phi}(P, v)$ is the limit of the values of $d^{-m}\lambda_W(\phi^m P, v)$ as m goes to infinity. It follows that a point P will be a pre-periodic point of ϕ if and only if $\hat{\lambda}_{W,\phi}(P, v) = 0$ for all $v \in M_{\bar{K}}$. This provides us with a strong criterion for determining the pre-periodic points of ϕ. To facilitate our computations, it is convenient to describe how canonical local heights are affected by a change of coordinates. This is accomplished in Section 4 and applied to give exact formulas for all canonical local heights, and thus all canonical heights, associated to diagonalizable morphisms (i.e., morphisms on $\mathbf{P}^n$ which possess $n+1$ distinct hyperplane eigendivisors whose intersection is empty).

A hyperplane eigendivisor for a morphism $\phi: \mathbf{P}^1 \to \mathbf{P}^1$ is simply a totally ramified fixed point, so the morphisms on $\mathbf{P}^1$ which possess hyperplane eigendivisors are precisely the polynomial maps. Beginning in Section 5 we focus on polynomial maps on $\mathbf{P}^1$. For any polynomial map ϕ with hyperplane eigendivisor W, we demonstrate how to compute the canonical local heights $\hat{\lambda}_{W,\phi}(P, v)$ for all $P \in \mathbf{P}^1(K) \backslash |W|$ and all non-archimedean $v \in M_K$. In the special case where the *Newton polygon* of ϕ is a straight line which intersects the line $x = 1$ below the x-axis, our method can be refined to give an explicit formula for $\hat{\lambda}_{W,\phi}(P, v)$. As a corollary we obtain explicit formulas for the canonical local heights associated to any degree two polynomial map ϕ on $\mathbf{P}^1$ and non-archimedean $v \in M_K$.

In Section 6 we develop the connection between the v-adic dynamics of polynomial maps on $\mathbf{P}^1$ and the v-adic canonical local heights associated to them. Suppose that an embedding of $\bar{K}$ into $\mathbf{C}$ is fixed and v_∞ denotes the standard archimedean absolute value on $\mathbf{C}$. Recall that the filled Julia set of ϕ as defined in complex dynamical systems (Cf. Devaney [4], Section 3.8) consists of those points $P \in \mathbf{P}^1(\mathbf{C})$ whose orbit under ϕ is bounded with respect to v_∞. We extend this definition to all $v \in M_{\bar{K}}$ by defining the *v-adic filled Julia set of ϕ* to be the set of points $P \in \mathbf{P}^1(\bar{K}_v)$ whose orbit under ϕ is v-adically bounded. If W is a hyperplane eigendivisor of the polynomial map ϕ and $|W| \neq \{P\}$, we prove that, for each $v \in M_{\bar{K}}$, $\hat{\lambda}_{W,\phi}(P, v) = 0$ if and only if P lies in the v-adic filled Julia set of ϕ. Then we deduce that the pre-periodic points of ϕ are precisely the points which lie in the intersection of all the v-adic filled Julia sets of ϕ. Combining the calculations of the non-archimedean canonical local heights made in Section 5 with some archimedean estimates, we conclude the paper by using this criterion to bound the number of rational pre-periodic points of any quadratic $\mathbf{Q}$-polynomial map in terms of the number of primes dividing the denominators of its coefficients.

Canonical heights have proven to be invaluable tools in the study of abelian varieties and their applications throughout arithmetic geometry. The recently developed theory of canonical heights on K3 surfaces (Cf. [3] and [19]) has already produced several interesting arithmetic applications and open conjectures. It is our hope that canonical heights on projective space will likewise prove to be useful in the study of the geometry and dynamics of morphisms on $\mathbf{P}^n$.

In this course of this research we have been fortunate to receive many valuable comments. We would like to thank David Cox for helpful conversations on geometry and dynamical systems, and Barry Mazur for sharing his insight into the connection between canonical local heights and v-adic dynamics. We gratefully acknowledge the care the referee took in reviewing the paper and, in particular, his/her comments which improved Theorem 6.1 and the results of Section 5. We also appreciate John Tate's generous suggestions which included showing us how to refine Propositions 3.4 and 3.5. Finally, we would like to extend special thanks to Joe Silverman, whose continuing help and numerous comments have had much to do with the success of our work.

1. CANONICAL GLOBAL HEIGHTS

We set the following notation for use throughout the paper.

K	an algebraic number field.
$\mathbf{P}^n$	$\mathbf{P}^n(\bar{K})$, where $\bar{K}$ is the field of algebraic numbers.

Our starting point is the standard height on $\mathbf{P}^n$.

DEFINITION. The *set of standard absolute values over* $\mathbf{Q}$ is the set $M_{\mathbf{Q}} = \{| \; |_\infty\} \cup \{| \; |_p : p \text{ a prime}\}$, where $| \; |_\infty$ is the ordinary archimedean absolute value on $\mathbf{Q}$ and the non-archimedean p-adic absolute values $| \; |_p$ are normalized so that $|p|_p = 1/p$. The *set of standard absolute values over* K is the set

$$M_K = \{| \; |_v : \text{the restriction of } | \; |_v \text{ to } \mathbf{Q} \text{ is in } M_{\mathbf{Q}}\}.$$

For each $| \; |_v \in M_K$, define a map $v: K^* \to \mathbf{R}$ by $v(x) = -\log |x|_v$. Since we will use logarithmic heights throughout this paper, it will be convenient to abuse notation and write $v \in M_K$ instead of $| \; |_v \in M_K$. When it is necessary to distinguish between them, we will refer to v as a valuation and to $| \; |_v$ as the absolute value associated to v.

DEFINITION. The *standard height on* $\mathbf{P}^n(K)$ is the map $h: \mathbf{P}^n(K) \to \mathbf{R}$ where

$$\begin{aligned} h([x_0, ..., x_n]) &= \frac{1}{[K:\mathbf{Q}]} \sum_{v \in M_K} [K_v : \mathbf{Q}_v] \log \max\{|x_0|_v, ..., |x_n|_v\} \\ &= \frac{-1}{[K:\mathbf{Q}]} \sum_{v \in M_K} [K_v : \mathbf{Q}_v] \min\{v(x_0), ..., v(x_n)\}. \end{aligned}$$

The product formula assures that this function yields the same value for any choice of homogeneous coordinates. It is worth noting that when we are working with $P \in \mathbf{P}^n(\mathbf{Q})$, we can pick coordinates $x_0, ..., x_n$ for P that are integers with no common prime factors and thus eliminate all the non-archimedean terms in the standard height formula; in this case, $h(P) = \log \max\{|x_0|_\infty, ..., |x_n|_\infty\}$. Also note that the standard height on $\mathbf{P}^n(K)$ is *non-degenerate*, i.e., for any $C > 0$, the set $\{P \in \mathbf{P}^n(K) \mid h(P) \leqslant C\}$ has only finitely many elements.

The standard height on $\mathbf{P}^n(K)$ can be extended to $\mathbf{P}^n(\bar{K})$ as follows; for any $[x_0, ..., x_n] \in \mathbf{P}^n(\bar{K})$, take an algebraic number field L which contains $x_0, ..., x_n$ and compute $h([x_0, ..., x_n])$ over L with the definition given above. The extension formula (Cf. Lang [7], Cor. 1, p. 39) guarantees that this computation yields the same height independent of the choice of L. The non-degeneracy of h extends to $\bar{K}$ in the following manner; for any $C, D > 0$, the set

$$\{P \in \mathbf{P}^n \mid P \in \mathbf{P}^n(L) \text{ for some } L \text{ such that } [L : K] \leqslant D \text{ and } h(P) \leqslant C\}$$

is finite (Cf. Silverman [17], Thm. 5.11, pp. 213–214).

The following theorem, which is essential to our canonical height construction, is a well-known consequence of the Hilbert Nullstellensatz (Cf. Lang [8], p. 81).

THEOREM 1.1. *If* $\phi : \mathbf{P}^n \to \mathbf{P}^m$ *is a morphism of degree d, then there exists* $C > 0$ *such that for all* $P \in \mathbf{P}^n$, $|h(\phi P) - dh(P)| \leqslant C$.

If $n = m$, the existence of the canonical height on $\mathbf{P}^n$ may be established using a limit argument due to Tate.

THEOREM 1.2. *For any morphism* $\phi : \mathbf{P}^n \to \mathbf{P}^n$ *of degree* $d > 1$, *there exists a unique function* $\hat{h}_\phi : \mathbf{P}^n \to \mathbf{R}$, *called the* canonical height associated with ϕ, *which has the following properties*:

(i) *For all* $P \in \mathbf{P}^n$, $\hat{h}_\phi(\phi P) = d\hat{h}_\phi(P)$.

(ii) *The function* $h - \hat{h}_\phi$ *is bounded on* $\mathbf{P}^n$; *i.e.,* $\hat{h}_\phi$ *is a Weil height on* $\mathbf{P}^n$. *Furthermore, the canonical height* $\hat{h}_\phi$ *satisfies*

$$\hat{h}_\phi(P) = \lim_{k \to \infty} \frac{h(\phi^k P)}{d^k}, \tag{1}$$

for all $P \in \mathbf{P}^n$.

Proof. This result follows directly from Theorem 1.1 and Proposition 1.2 of [2] by taking the variety V to be $\mathbf{P}^n$ and the divisor class η to be the degree one class on $\mathbf{P}^n$. ▮

Note that the standard height h on $\mathbf{P}^n$ is a canonical height. In particular, if $\rho : \mathbf{P}^n \to \mathbf{P}^n$ is the morphism of degree $d > 1$ defined by $\rho[x_0, x_1, ..., x_n] = [x_0^d, x_1^d, ..., x_n^d]$, then $h(\rho P) = dh(P)$ for all $P \in \mathbf{P}^n$ by the definition of h. By Theorem 1.2, the canonical height associated with ρ is unique, so $h = \hat{h}_\rho$.

Recall that if $\phi : \mathbf{P}^n \to \mathbf{P}^n$ is a morphism and $P \in \mathbf{P}^n$, the ϕ-*orbit of P* is the set $\{\phi^k P \mid k \geqslant 0\}$, and also that P is called a *pre-periodic point of* ϕ if the ϕ-orbit of P is finite. Since h is non-degenerate and the difference between h and $\hat{h}_\phi$ is bounded, $\hat{h}_\phi$ is clearly also non-degenerate. Combining these remarks with the limit formula (1) produces the following corollary.

COROLLARY 1.3. *Let* $\phi : \mathbf{P}^n \to \mathbf{P}^n$ *be a morphism of degree* $d > 1$ *defined on K.*

(a) *For all* $P \in \mathbf{P}^n$, *P is pre-periodic if and only if* $\hat{h}_\phi(P) = 0$.

(b) *For any* $D \geqslant 1$, *the set*

$$\{P \in \mathbf{P}^n \mid P \text{ is a pre-periodic point of } \phi \text{ and } P \in \mathbf{P}^n(L) \text{ where } [L:K] \leqslant D\}$$

is finite.

Proof. This is Corollary 1.1.1 of [2] with $V = \mathbf{P}^n$ and η the divisor class of degree one on $\mathbf{P}^n$. ∎

If $d > 1$ and $n \geqslant 1$ are fixed, the following conjecture of Morton and Silverman [10] asserts that the number of pre-periodic points of a degree d morphism ϕ on $\mathbf{P}^n$ is bounded independently of ϕ.

Boundedness Conjecture. For any positive integers D, n, and $d > 1$, there exists a constant $C(D, n, d) > 0$ such that for any field K with $[K : Q] = D$ and any morphism $\phi\colon \mathbf{P}^n \to \mathbf{P}^n$ of degree d defined over K,

$$\#\{P \in \mathbf{P}^n(K) \mid P \text{ is a pre-periodic point of } \phi\} \leqslant C(D, n, d).$$

We will return to the special case $D = n = 1$ and $d = 2$ of the Boundedness Conjecture in Section 6.

2. CANONICAL LOCAL HEIGHTS

In this section we will specialize the theory developed in [2] to construct canonical local heights on $\mathbf{P}^n$. Then, in Theorem 2.2, we will show that whenever ϕ has good reduction at $v \in M_K$, the canonical local heights associated to v are standard Weil local heights. This provides us with a simple method for computing any canonical local height at all but finitely many $v \in M_K$. We will derive a formula for the difference between any two canonical local heights and recall from [2] that the canonical global height $\hat{h}_\phi$ may be constructed by summing canonical local heights over the absolute values v.

To the notation established in Section 1, we add the following.

$M = M_{\bar{K}}$ — the set of absolute values on $\bar{K}$ extending those in M_K.

ϕ — a morphism $\phi\colon \mathbf{P}^n \to \mathbf{P}^n$ of degree $d \geqslant 2$ defined over K.

l — a linear form $l(x_0, ..., x_n) \in K[x_0, ..., x_n]$.

H — the hyperplane defined by $l(x_0, ..., x_n) = 0$, considered as a divisor on $\mathbf{P}^n$. We write $H = \operatorname{div}(l)$.

λ_l — a Weil local height function $\lambda_l\colon (\mathbf{P}^n(\bar{K}) \setminus |H|) \times M \to \mathbf{R}$ associated to H, defined by

$$\lambda_l(P, v) = \max\left\{v\left(\frac{l(P)}{x_0(P)}\right), ..., v\left(\frac{l(P)}{x_n(P)}\right)\right\}.$$

We will say that a Weil local height λ_W associated to the hyperplane W is *standard* if there is a linear form g such that $\operatorname{div}(g) = W$ and

$$\lambda_W(P, v) = \max\left\{v\left(\frac{g(P)}{x_0(P)}\right), ..., v\left(\frac{g(P)}{x_n(P)}\right)\right\}.$$

Thus λ_l is a standard Weil local height. Note that if λ_W is any standard Weil local height, and L is any number field, then by the product formula

$$h(P) = \frac{1}{[L:\mathbf{Q}]} \sum_{v \in M_L} [L_v : \mathbf{Q}_v]\, \lambda_W(P, v),$$

for all points $P \in \mathbf{P}^n(L) \backslash |W|$, where h is the standard height on $\mathbf{P}^n$ and M_L is the set of standard absolute values on L.

To aid the reader we recall some basic definitions and terminology (Cf. [8]). An M_K-*constant* is a function $\kappa: M_K \to \mathbf{R}$ such that $\kappa(w) = 0$ for all but finitely many $w \in M_K$. Given an M_K-constant κ, we may extend κ to an M-*constant* by setting $\kappa(v) = \kappa(w)$ if $v \in M$ extends $w \in M_K$. It is useful to think of κ as a family of constants $\{c(v)\}$ parametrized by M. A function $\beta: \mathbf{P}^n \times M \to \mathbf{R}$ is said to be M-*bounded* if there is an M-constant κ such that

$$|\beta(P, v)| \leqslant \kappa(v)$$

for all $(P, v) \in \mathbf{P}^n \times M$. β is said to be M-*continuous* if for all $v \in M$ the map

$$P \mapsto \beta(P, v)$$

is continuous with respect to the v-topology on $\mathbf{P}^n$.

Let $f \in K(\mathbf{P}^n)$ be a rational function such that

$$\phi^* H = dH + \operatorname{div}(f). \tag{2}$$

By the functoriality property of Weil local heights ([8], Proposition 2.6, p. 258), there is an M-bounded and M-continuous function $\gamma: \mathbf{P}^n \times M \to \mathbf{R}$ such that

$$\lambda_l(\phi P, v) = d\lambda_l(P, v) + v(f(P)) + \gamma(P, v), \tag{3}$$

for all $v \in M$ and all $P \in \mathbf{P}^n \backslash (|H| \cup |\phi^* H|)$. It follows from Theorem 2.1(b) of [2] (with $V = \mathbf{P}^n$ and divisor $E = H$) that ϕ and f uniquely determine a canonical local height $\hat{\lambda}_{H, \phi, f}$ associated to H which satisfies the functoriality relation (3) without the γ term; i.e., we have

THEOREM 2.1. *Let $f \in K(\mathbf{P}^n)$ be a rational function such that*

$$\phi^* H = dH + \operatorname{div}(f).$$

Then there exists a unique function $\hat{\lambda}_{H,\phi,f}$ which is a Weil local height for the divisor H and which satisfies

$$\hat{\lambda}_{H,\phi,f}(\phi P, v) = d\hat{\lambda}_{H,\phi,f}(P, v) + v(f(P)), \tag{4}$$

for all $v \in M$ and all $P \in \mathbf{P}^n$ such that $P, \phi P \notin |H|$.

Remark. Fix $v \in M$ and let $\bar{K}_v$ denote the completion of $\bar{K}$ at v. Then it follows from the proof of Theorem 2.1(b) of [2] that the domain of the canonical local height may be extended to $\mathbf{P}^n(\bar{K}_v) \backslash |H|$. In particular, there is a unique function $\hat{\lambda}_v \colon \mathbf{P}^n(\bar{K}_v) \backslash |H| \to \mathbf{R}$ such that

(i) the difference $\hat{\lambda}_v(P) - \lambda_l(P, v)$ is bounded and v-continuous on $\mathbf{P}^n(\bar{K}_v)$, and

(ii) $\hat{\lambda}_v(\phi P) = d\hat{\lambda}_v(P) + v(f(P))$, for all $P \in \mathbf{P}^n(\bar{K}_v)$ such that $P, \phi P \notin |H|$.

Of course, by uniqueness, we have $\hat{\lambda}_v(P) = \hat{\lambda}_{H,\phi,f}(P, v)$ for all $P \in \mathbf{P}^n(\bar{K})$, $P \notin |H|$.

Suppose $v \in M_K$ is non-archimedean and let $\pi = \pi_v$ denote the prime ideal of K determined by v. We will say that $\phi \colon \mathbf{P}^n/K \to \mathbf{P}^n/K$ has *good reduction at* v if ϕ extends to a morphism $\mathbf{P}^n/R_\pi \to \mathbf{P}^n/R_\pi$ of schemes over the local ring R_π. In particular, let $\mathbf{F}_\pi$ denote the residue field of R_π and let $\bar{\mathbf{F}}_\pi$ denote an algebraic closure of $\mathbf{F}_\pi$. Then to say that ϕ has good reduction at v means that ϕ can be written in the form $\phi = [f_0(x_0, ..., x_n), ..., f_n(x_0, ..., x_n)]$, where $f_0, ..., f_n \in K[x_0, ..., x_n]$ are homogeneous polynomials of degree d with v-integral coefficients, so that the reduced polynomials $\tilde{f}_0 \pmod{\pi}, ..., \tilde{f}_n \pmod{\pi}$ in $R_\pi[x_0, ..., x_n]$ have no common roots in $\mathbf{P}^n(\bar{\mathbf{F}}_\pi)$. Furthermore, as our next result shows, the polynomials $f_0, ..., f_n$ may be used to define a rational function $f \in K(\mathbf{P}^n)$ such that $\hat{\lambda}_{H,\phi,f}(P, v) = \lambda_l(P, v)$ for all $P \in \mathbf{P}^n(K) \backslash |H|$.

THEOREM 2.2. *Suppose $v \in M_K$ is non-archimedean and ϕ has good reduction at v. Write $\phi = [f_0, ..., f_n]$ as above, where $f_0, ..., f_n \in K[x_0, ..., x_n]$ are homogeneous polynomials with v-integral coordinates whose reductions modulo π have no common roots in $\mathbf{P}^n(\bar{\mathbf{F}}_\pi)$. Define $f \in K(\mathbf{P}^n)$ by $f(P) = l(f_0(P), ..., f_n(P))/l(P)^d$. Then $\phi^* H = dH + \operatorname{div}(f)$ and*

$$\hat{\lambda}_{H,\phi,f}(P, v) = \lambda_l(P, v),$$

for all $P \in \mathbf{P}^n(K) \backslash |H|$.

Proof. Since $\operatorname{div}(l)=H$, $\operatorname{div}(f)=\phi^*H-dH$ by the definition of f.

By Theorem 2.1, $\hat{\lambda}_{H,\phi,f}$ is the unique Weil local height for H satisfying (4). Since $v\in M_K$ is fixed, by the remark following Theorem 2.1 it suffices to show that $\gamma(P,v)=\lambda_l(\phi P,v)-d\lambda_l(P,v)-v(f(P))=0$ for all $P\in\mathbf{P}^n(\bar{K}_v)$ such that $P,\phi P\notin|H|$. Using the definitions of λ_l and f, and the fact that $\phi P=[f_0(P),...,f_n(P)]$, we compute

$$\begin{aligned}\gamma(P,v)&=\lambda_l(\phi P,v)-d\lambda_l(P,v)-v(f(P))\\&=\max\left\{v\left(\frac{l(f_0(P),...,f_n(P))}{f_0(P)}\right),...,v\left(\frac{l(f_0(P),...,f_n(P))}{f_n(P)}\right)\right\}\\&\quad-d\max\left\{v\left(\frac{l(P)}{x_0(P)}\right),...,v\left(\frac{l(P)}{x_n(P)}\right)\right\}-v(f(P))\\&=\max\{-v(f_0(P)),...,-v(f_n(P))\}\\&\quad-d\max\{-v(x_0(P)),...,-v(x_n(P))\}\\&=d\min\{v(x_0(P)),...,v(x_n(P))\}-\min\{v(f_0(P)),...,v(f_n(P))\}\end{aligned}$$

for all P in the Zariski open set $\mathbf{P}^n(\bar{K}_v)\backslash(|H|\cup|\phi^*H|)$. However, since v-continuous functions on $\mathbf{P}^n(\bar{K}_v)$ which agree on a Zariski open subset must be identically equal (Cf. [8], Lemma 1.4, p. 251), we conclude

$$\gamma(P,v)=d\min\{v(x_0(P)),...,v(x_n(P))\}-\min\{v(f_0(P)),...,v(f_n(P))\}$$

for all $P\in\mathbf{P}^n(\bar{K}_v)$.

Given $P\in\mathbf{P}^n(\bar{K}_v)$ choose v-integral coordinates $z_0,...,z_n$ for P such that $\min\{v(z_0),...,v(z_n)\}=0$. Then since $f_0,...,f_n$ have v-integral coefficients,

$$\gamma(P,v)=-\min\{v(f_0(z_0,...,z_n)),...,v(f_n(z_0,...,z_n))\}\leqslant 0.$$

But if $\gamma(P,v)<0$, the reduction of $[z_0,...,z_n]$ modulo π would be a common root of $\tilde{f}_0,...,\tilde{f}_n$ in $\mathbf{P}^n(\bar{\mathbf{F}}_\pi)$. Thus $\gamma(P,v)$ must be zero for all $P\in\mathbf{P}^n(\bar{K}_v)$. ∎

We next observe that the difference between any two canonical local heights associated to ϕ equals the valuation of a rational function plus an M-constant. In particular, suppose that ϕ has good reduction at v and $g\in K(\mathbf{P}^n)$ is any rational function such that $\phi^*H=dH+\operatorname{div}(g)$. Then the following lemma combined with Theorem 2.2 gives a simple formula for the canonical local height $\hat{\lambda}_{H,\phi,g}(P,v)$.

LEMMA 2.3. *Let $k \in K[x_0, ..., x_n]$ be a linear form and set $W = \mathrm{div}(k)$. Suppose $f, g \in K(\mathbf{P}^n)$ satisfy*

$$\phi^* H = dH + \mathrm{div}(f) \qquad \text{and} \qquad \phi^* W = dW + \mathrm{div}(g). \tag{5}$$

Then

$$c = \left(\frac{f}{g}\right)\left(\frac{k}{l} \circ \phi\right)\left(\frac{l}{k}\right)^d \tag{6}$$

is a non-zero constant in K^*, and the canonical local heights $\hat{\lambda}_{H, \phi, f}$ and $\hat{\lambda}_{W, \phi, g}$ satisfy

$$\hat{\lambda}_{H, \phi, f}(P, v) = \hat{\lambda}_{W, \phi, g}(P, v) + v\left(\frac{l}{k}(P)\right) + \frac{1}{1-d} v(c)$$

for all $P \in \mathbf{P}^n \backslash (|H| \cup |W|)$ and all $v \in M$.

Proof. We first check that c is a non-zero constant by computing its divisor,

$$\mathrm{div}(c) = [\mathrm{div}(f) - \mathrm{div}(g)] + [\phi^* W - \phi^* H] + d[H - W] = 0,$$

by (6) and (5). Hence, $c \in K^*$ since f, g, l, k and ϕ are defined over K.

Let

$$\sigma(P, v) = \hat{\lambda}_{W, \phi, g}(P, v) + v\left(\frac{l}{k}(P)\right) + \frac{1}{1-d} v(c).$$

Since $v((l/k)(P))$ is a local height associated to the divisor $H - W$, and $(1/(1-d))\, v(c)$ is an M-constant, σ has a unique M-continuous extension $\bar{\sigma}$ to $(\mathbf{P}^n \backslash |H|) \times M$. Furthermore, $\bar{\sigma}$ is a Weil local height associated to the divisor H. To show that $\bar{\sigma}$ is in fact the canonical local height $\hat{\lambda}_{H, \phi, f}$, we compute

$$\begin{aligned}
&\bar{\sigma}(\phi P, v) - d\bar{\sigma}(P, v) - v(f(P)) \\
&\quad = \hat{\lambda}_{W, \phi, g}(\phi P, v) + v\left(\frac{l}{k}(\phi P)\right) + v(c) \\
&\qquad - d\left[\hat{\lambda}_{W, \phi, g}(P, v) + v\left(\frac{l}{k}(P)\right)\right] - v(f(P)) \\
&\quad = v(c) - \left[v(f(P)) - v(g(P)) + v\left(\frac{k}{l}(\phi P)\right) + dv\left(\frac{l}{k}(P)\right)\right] = 0
\end{aligned}$$

for all $v \in M$ and all P in a Zariski open subset of $\mathbf{P}^n$. Since any Weil local height which is zero on a Zariski open subset of $\mathbf{P}^n$ must be identically zero ([8], Corollary 2.3, p. 257), $\bar{\sigma}(\phi P, v) - d\bar{\sigma}(P, v) - v(f(P)) = 0$ for all $v \in M$ and all $P \in \mathbf{P}^n$ such that $P, \phi P \notin |H|$. Therefore, $\bar{\sigma} = \hat{\lambda}_{H, \phi, f}$ by Theorem 2.1. ▮

We conclude this section by observing that the canonical global height $\hat{h}_\phi$ described in Section 1 may be computed by summing the canonical local heights $\hat{\lambda}_{H, \phi, f}$ over the valuations v.

THEOREM 2.4. *Let $f \in K(\mathbf{P}^n)$ be any rational function such that $\phi^* H = dH + \operatorname{div}(f)$. Then for all number fields L and all points $P \in \mathbf{P}^n(L) \backslash |H|$,*

$$\hat{h}_\phi(P) = \frac{1}{[L : \mathbf{Q}]} \sum_{v \in M_L} [L_v : \mathbf{Q}_v] \, \hat{\lambda}_{H, \phi, f}(P, v),$$

where M_L is the set of standard absolute values on L.

Proof. This is Theorem 2.3 of [2] with variety $V = \mathbf{P}^n$ and divisor $E = H$. ▮

3. CONVERGENT SERIES AND SEQUENCES FOR CANONICAL LOCAL HEIGHTS

For those non-archimedean absolute values v at which the morphism ϕ has good reduction, combining Theorem 2.2 and Lemma 2.3 gives a simple method for computing the canonical local height $\hat{\lambda}_{H, \phi, f}(P, v)$. This leaves the task of determining the canonical local height at the finitely many absolute values which are either archimedean or non-archimedean absolute values at which ϕ has bad reduction. In this section we will describe a special case of the rapidly converging series given in [2] for computing the canonical local height at any absolute value v. Then we will show that when we fix a choice of polynomials which define ϕ, the series formula for $\hat{\lambda}_{H, \phi, f}(P, v)$ telescopes. In the special case where the orbit of P under ϕ does not intersect $|H|$, this enables us to derive a limit formula for $\hat{\lambda}_{H, \phi, f}(P, v)$ which is analogous to the limit formula for the canonical global height given in Theorem 1.2.

To the notation of the preceding sections, we add the following.

H_i the coordinate hyperplane $\operatorname{div}(x_i)$, for $i = 0, ..., n$.

λ_i a fixed standard Weil local height associated to H_i, defined by

$$\begin{aligned}\lambda_i(P, v) &= \max\left\{ v\left(\frac{x_i(P)}{x_0(P)}\right), ..., v\left(\frac{x_i(P)}{x_n(P)}\right)\right\} \\ &= v(x_i(P)) - \min\{v(x_0(P)), ..., v(x_n(P))\}.\end{aligned}$$

Throughout this section, and only within this section, we will fix a single valuation v. As in Section 2, we let $f \in K(\mathbf{P}^n)$ denote a rational function satisfying

$$\phi^* H = dH + \operatorname{div}(f).$$

Let $\bar{K}_v$ denote the completion of $\bar{K}$ at v. Recall that given any Weil local height λ_W, the map $P \mapsto \lambda_W(P, v)$ may be defined for all points $P \in \mathbf{P}^n(\bar{K}_v) \setminus |W|$. Our goal is to give a series for $\hat{\lambda}_{H,\phi,f}(P, v)$ which converges for all $P \in \mathbf{P}^n(\bar{K}_v) \setminus |H|$. Tate [21] originally described a series for computing canonical heights on elliptic curves. The idea of switching coordinates to ensure convergence in all cases was first introduced by Silverman [18]. The series for $\hat{\lambda}_{H,\phi,f}(P, v)$ will be constructed using the following rational functions. For all integers i, j, with $0 \leqslant i, j \leqslant n$, define $t_i, s_{ij} \in K(\mathbf{P}^n)$ by

$$t_i = \frac{l}{x_i} \qquad \text{and} \qquad s_{ij} = f \cdot \frac{t_i^d}{t_j \circ \phi}. \tag{7}$$

THEOREM 3.1. *Let $P \in \mathbf{P}^n(\bar{K}_v) \setminus |H|$. Choose any sequence of indices $i_0, i_1, i_2, \ldots$ (depending on P) such that*

$$v(x_{i_k}(\phi^k P)) = \min_{0 \leqslant j \leqslant n} \{v(x_j(\phi^k P))\}. \tag{8}$$

(a) *For every $k \geqslant 0$, the function $s_{i_k i_{k+1}}$ is defined at $\phi^k P$. Furthermore, the sequence of real numbers*

$$c_k = -v(s_{i_k i_{k+1}}(\phi^k P)), \qquad k = 0, 1, 2, \ldots \tag{9}$$

is bounded independently of k and P.

(b) *The canonical local height satisfies*

$$\hat{\lambda}_{H,\phi,f}(P, v) = v(t_{i_0}(P)) + \sum_{k=0}^{N-1} d^{-k-1} c_k + O(d^{-N}), \tag{10}$$

where the big-O constant is independent of both P and N.

Proof. This is Theorem 5.3 of [2] with variety $V = \mathbf{P}^n$, divisors $E = H$ and $D_i = H_i$, and local heights $\lambda_{|D_i|} = \lambda_i$. Theorem 5.3 of [2] assumes that the sequence of indices $i_0, i_1, i_2, \ldots$ satisfies

$$\lambda_{i_k}(\phi^k P, v) = \min_{0 \leqslant i \leqslant n} \lambda_i(\phi^k P, v).$$

Note that this is equivalent to (8) since, by the definition of the standard local heights λ_i, for all $Q \in \mathbf{P}^n(\bar{K}_v)$ we have

$$\min_{0 \leqslant i \leqslant n} \lambda_i(Q, v) = 0,$$

and $\lambda_j(Q, v) = 0$ if and only if $v(x_j(Q)) = \min_{0 \leqslant i \leqslant n} \{v(x_i(Q))\}$. ∎

To simplify our series formula for $\hat{\lambda}_{H, \phi, f}(P, v)$, we begin by fixing homogeneous polynomials $f_0, f_1, ..., f_n \in K[x_0, ..., x_n]$ such that

$$\phi[x_0, ..., x_n] = [f_0(x_0, ..., x_n), ..., f_n(x_0, ..., x_n)].$$

Let $\mathbf{0}$ denote the $(n+1)$-tuple of zeros in $\bar{K}^{n+1}$. Thus $f_0, ..., f_n$ have no common zeros in $\bar{K}^{n+1} \backslash \{\mathbf{0}\}$, because ϕ is a morphism. Define a map $\Phi: (\bar{K}^{n+1} \backslash \{\mathbf{0}\}) \to (\bar{K}^{n+1} \backslash \{\mathbf{0}\})$ by

$$\Phi(a_0, ..., a_n) = (f_0(a_0, ..., a_n), ..., f_n(a_0, ..., a_n)).$$

Similarly, let $X_0, ..., X_n$ denote the coordinate functions on $\bar{K}^{n+1}$ corresponding to $x_0, ..., x_n$ respectively. Note that $\phi[x_0, ..., x_n] = [\Phi(X_0, ..., X_n)]$, so $\phi^k[x_0, ..., x_n] = [\Phi^k(X_0, ..., X_n)]$ for all $k \geqslant 0$. As in Theorem 2.3, define $f \in K(\mathbf{P}^n)$ by

$$f = \frac{l(f_0(x_0, ..., x_n), ..., f_n(x_0, ..., x_n))}{l(x_0, ..., x_n)^d}. \tag{11}$$

PROPOSITION 3.2. *Let $P \in \mathbf{P}^n \backslash |H|$, choose $\mathbf{a} = (a_0, ..., a_n) \in (\bar{K}^{n+1} \backslash \{\mathbf{0}\})$ such that $P = [a_0, ..., a_n]$, and define Φ and f as above. Choose any sequence of indices $i_0, i_1, i_2, \ldots$ such that (8) holds. Then*

$$\hat{\lambda}_{H, \phi, f}(P, v) = v(l(\mathbf{a})) - \lim_{k \to \infty} \frac{v(X_{i_k}(\Phi^k \mathbf{a}))}{d^k}. \tag{12}$$

Proof. By (7) and (11), for all integers i, j, with $0 \leqslant i, j \leqslant n$, we find

$$s_{ij} = f \cdot \frac{t_i^d}{t_j \circ \phi} = \frac{l(f_0, ..., f_n)}{l^d} \cdot \frac{l^d}{(x_i)^d} \cdot \left(\frac{x_j}{l} \circ \phi\right) = \frac{f_j}{(x_i)^d}.$$

Therefore,

$$s_{i_k i_{k+1}}(\phi^k P) = \frac{f_{i_{k+1}}(\phi^k P)}{x_{i_k}(\phi^k P)^d} = \frac{f_{i_{k+1}}(\Phi^k \mathbf{a})}{X_{i_k}(\Phi^k \mathbf{a})^d} = \frac{X_{i_{k+1}}(\Phi^{k+1} \mathbf{a})}{X_{i_k}(\Phi^k \mathbf{a})^d}.$$

Then, by (9), we compute

$$c_k = -v(s_{i_k i_{k+1}}(\phi^k P)) = d v(X_{i_k}(\Phi^k \mathbf{a})) - v(X_{i_{k+1}}(\Phi^{k+1} \mathbf{a})).$$

Plugging this expression into (10) and letting N go to infinity, we find a telescoping sum which reduces to

$$\hat{\lambda}_{H,\phi,f}(P,v) = v(t_{i_0}(P)) + v(X_{i_0}(\mathbf{a})) - \lim_{k\to\infty} \frac{v(X_{i_{k+1}}(\Phi^{k+1}\mathbf{a}))}{d^{k+1}}$$
$$= v(l(\mathbf{a})) - \lim_{k\to\infty} \frac{v(X_{i_k}(\Phi^k\mathbf{a}))}{d^k},$$

since by definition (7), $t_{i_0} = l/x_{i_0}$. ∎

Remark. As we have shown, a canonical local height is uniquely determined by the choice of a hyperplane H, a morphism ϕ and a rational function $f \in K(\mathbf{P}^n)$ such that

$$\phi^* H = dH + \operatorname{div}(f).$$

It is worth noting that a canonical local height may also be specified by choosing a linear form $l \in K[x_0, ..., x_n]$ and a vector of homogeneous polynomials $\Phi = (f_0, ..., f_n)$ which define a morphism ϕ on $\mathbf{P}^n$ as above. In particular, given such an l and a Φ, there exists a unique function $\hat{\lambda}_{l,\Phi}$ which is a Weil local height associated to $H = \operatorname{div}(l)$ and satisfies

$$\hat{\lambda}_{l,\Phi}(\phi P, v) = d\hat{\lambda}_{l,\Phi}(P, v) + v(f(P)),$$

for all $P \in \mathbf{P}^n \backslash (|H| \cup |\phi^* H|)$, where $f \in K(\mathbf{P}^n)$ is defined by (11). Indeed $\hat{\lambda}_{l,\Phi}$ is just the canonical local height $\hat{\lambda}_{H,\phi,f}$ of Proposition 3.2 and the following corollary. This method of parametrizing canonical local heights is developed in Goldstine [5].

COROLLARY 3.3. *Suppose there exists an increasing sequence of non-negative integers $j_0 < j_1 < j_2 < \cdots$ such that $\phi^{j_k} P \notin |H|$ for all $k \geqslant 0$. Define Φ and* $\mathbf{a}$ *as above. Then*

$$\hat{\lambda}_{H,\phi,f}(P,v) = v(l(\mathbf{a})) + \lim_{k\to\infty} \frac{\lambda_H(\phi^{j_k}P, v) - v(l(\Phi^{j_k}\mathbf{a}))}{d^{j_k}}.$$

Proof. Let $m \geqslant 0$ be any integer such that $\phi^m P \notin |H|$. Then, by the definition of λ_H, we have

$$\lambda_H(\phi^m P, v) = v(l(\phi^m P)) - \min_i \{v(x_i(\phi^m P))\}$$
$$= v(l(\phi^m P)) - v(x_{i_m}(\phi^m P)) \qquad \text{by (8)},$$
$$= v(l(\Phi^m \mathbf{a})) - v(X_{i_m}(\Phi^m \mathbf{a})).$$

Hence, our desired result follows from Proposition 3.2. ∎

For the purpose of making some explicit calculations, we will focus next on a family of morphisms on $\mathbf{P}^1$. To simplify notation we identify $\mathbf{P}^1(K)$ with $K \cup \{\infty\}$; i.e., we identify each point $P = [x, y]$ with its affine coordinate $z = y/x$. Let $e \in K$ and consider the morphism $\tau = \tau_e \colon \mathbf{P}^1 \to \mathbf{P}^1$ defined by

$$\tau[x, y] = [x^2 + y^2, exy] \qquad \text{or, equivalently,} \qquad \tau(z) = \frac{ez}{z^2+1}.$$

Let $l = x$, so $H = \operatorname{div}(x) = \{\infty\}$. Then the rational function $f \in K(\mathbf{P}^1)$ defined by (11) is

$$f(z) = 1 + z^2.$$

For simplicity we write $\hat{\lambda}(z, v)$ for the canonical local height $\hat{\lambda}_{H, \tau, f}(P, v)$ and $\lambda(z, v)$ for the standard local height $\lambda_0(P, v)$ associated to the same divisor H. Note that

$$\lambda(z, v) = \max\left\{0,\, v\left(\frac{x}{y}\right)\right\} = \max\{0, -v(z)\}.$$

Our goal is to compute $\hat{\lambda}(z, v)$ for $z \in K$ and non-archimedean $v \in M_K$.

We first observe that if $v(e) = 0$, then τ has good reduction at v and the polynomials $f_0(x, y) = x^2 + y^2$ and $f_1(x, y) = exy$ satisfy the conditions of Theorem 2.2. Therefore, $\hat{\lambda}(z, v) = \lambda(z, v)$. If $v(e) \neq 0$, then we may have to consider the valuations of the iterates of z. Given $z \in K$, we will write $z_k = \tau^k(z)$ for $k \geqslant 0$. We would like to thank John Tate for his help in obtaining the following two results.

Proposition 3.4. *Let $v \in M_K$ be non-archimedean and suppose $v(e) > 0$.*

(a) *If $v(z) \neq 0$, then $\hat{\lambda}(z, v) = \lambda(z, v)$.*

(b) *If $v(z_k) = 0$ for all k, then $\hat{\lambda}(z, v) = -v(e)$.*

(c) *Otherwise, there is an $m > 0$ such that*

$$v(z_k) = 0 \quad \text{for } 0 \leqslant k < m, \quad v(f(z_k)) = v(e) \quad \text{for } 0 \leqslant k < m-1, \quad \text{and}$$
$$v(z_m) = v(e) - v(f(z_{m-1})) \neq 0.$$

In this case,

$$\hat{\lambda}(z, v) = -\left(1 - \frac{1}{2^{m-1}}\right) v(e) - \frac{1}{2^m} \min\{v(f(z_{m-1})), v(e)\}.$$

In particular,

$$-\left(1-\frac{1}{2^{m-1}}\right)v(e)\geqslant\hat{\lambda}(z,v)\geqslant-\left(1-\frac{1}{2^m}\right)v(e).$$

Proof. By definition of $\hat{\lambda}=\hat{\lambda}_{H,\tau,f}$, we have

$$\hat{\lambda}(\tau z,v)=2\hat{\lambda}(z,v)+v(f(z)). \tag{13}$$

We observe that the region $v(z)>0$ is stable under τ since

$$v(\tau z)=v(e)+v(z)-v(1+z^2)=v(e)+v(z)>0.$$

So suppose $v(z)>0$. Then

$$\lambda(z_k,v)=\max\{0,-v(z_k)\}=0 \qquad \text{for all} \quad k\geqslant 0,$$

and hence the sequence $\hat{\lambda}(z_k,v)$ is bounded for $k\geqslant 0$. But $v(f(z_k))=v(1+z_k^2)=0$ for $k\geqslant 0$, so by (13), $\hat{\lambda}(z_k,v)=2^k\hat{\lambda}(z,v)$. Therefore, $\hat{\lambda}(z,v)=\lambda(z,v)=0$.

Suppose $v(z)<0$. Then $v(f(z))=v(1+z^2)=2v(z)$ and

$$v(\tau z)=v(e)+v(z)-v(1+z^2)=v(e)-v(z)>0.$$

Therefore, $\hat{\lambda}(\tau z,v)=0$, and thus by (13) we conclude

$$\hat{\lambda}(z,v)=-\tfrac{1}{2}v(f(z))=-v(z)=\max\{-v(z),0\}=\lambda(z,v).$$

This completes the proof of part (a).

Now assume $v(z)=0$. If $v(z_k)=0$, then $v(z_{k+1})=v(e)-v(f(z_k))$. Hence, if $v(z_k)=v(z_{k+1})=0$, we have $v(f(z_k))=v(e)$. It follows that either $v(z_k)=0$ for all $k\geqslant 0$ or there is an $m>0$ such that $v(z_k)=0$ for $0\leqslant k<m$, $v(z_m)=v(e)-v(f(z_{m-1}))\neq 0$, and $v(f(z_k))=v(e)$ for $0\leqslant k<m-1$.

From (13) we have $\hat{\lambda}(z,v)=-\frac{1}{2}v(f(z))+\frac{1}{2}\hat{\lambda}(\tau z,v)$, and iterating this equation yields

$$\hat{\lambda}(z,v)=-\frac{1}{2}v(f(z))-\frac{1}{4}v(f(z_1))-\cdots-\frac{1}{2^k}v(f(z_{k-1}))+\frac{1}{2^k}\hat{\lambda}(z_k,v) \tag{14}$$

for all $k\geqslant 1$.

Suppose $v(z_k)=0$ for all $k \geqslant 0$. Then $v(f(z_k))=v(e)$ for all $k \geqslant 0$. Since $\lambda(z_k, v)=0$ for all $k \geqslant 0$, as above it follows that the sequence $\hat{\lambda}(z_k, v)$ is bounded. Hence, letting $k \to \infty$ in (14), we obtain

$$\hat{\lambda}(z, v) = \sum_{i=1}^{\infty} -\frac{1}{2^i} v(e) = -v(e),$$

which proves part (b).

Suppose there is an $m > 0$ such that the conditions of part (c) are satisfied. Applying (14) with $k = m$, we see that

$$\begin{aligned} \hat{\lambda}(z, v) &= -\frac{1}{2} v(e) - \frac{1}{4} v(e) - \cdots - \frac{1}{2^{m-1}} v(e) \\ &\quad - \frac{1}{2^m} v(f(z_{m-1})) + \frac{1}{2^m} \hat{\lambda}(z_m, v) \\ &= -\left(1 - \frac{1}{2^{m-1}}\right) v(e) - \frac{1}{2^m} [v(f(z_{m-1})) - \hat{\lambda}(z_m, v)]. \end{aligned} \tag{15}$$

Since $v(z_m) = v(e) - v(f(z_{m-1})) \neq 0$, by part (a) we have

$$\begin{aligned} \hat{\lambda}(z_m, v) = \lambda(z_m, v) &= \max\{0, -v(z_m)\} \\ &= -\min\{0, v(e) - v(f(z_{m-1}))\}. \end{aligned}$$

Combining this equation with (15) yields our desired formula for $\hat{\lambda}(z, v)$. Finally, we note that $v(f(z_{m-1})) = v(1 + z_{m-1}^2) \geqslant 0$, so $0 \leqslant \min\{v(f(z_{m-1})), v(e)\} \leqslant v(e)$, and we conclude

$$-\left(1 - \frac{1}{2^{m-1}}\right) v(e) \geqslant \hat{\lambda}(z, v) \geqslant -\left(1 - \frac{1}{2^m}\right) v(e). \quad \blacksquare$$

In the remaining case where $v(e) < 0$, we have the following simple formulas when $v(z)$ is not an integer multiple of $v(e)$.

PROPOSITION 3.5. *Let $v \in M_K$ be non-archimedean. Suppose $v(e) < 0$ and $v(z) \notin \mathbf{Z} v(e)$.*

(a) *If $v(e) < v(z) < 0$, then $\hat{\lambda}(z, v) = -\frac{2}{3}(v(z) + v(e))$.*

(b) *If $v(z) < v(e)$ or $v(z) > 0$, let m be the unique positive integer such that $v(e) < |v(z)| + mv(e) < 0$. Then*

$$\hat{\lambda}(z, v) = -\frac{1}{3 \cdot 2^{m-1}} (|v(z)| + (m+1) v(e)).$$

Remark. It will follow from our proof that the region $v(e) < v(z) < 0$ is stable under τ and that m as defined in (b) is the smallest positive integer such that $v(e) < v(z_m) < 0$.

Proof. Recall that $v(f(z)) = 0$ if $v(z) > 0$, and $v(f(z)) = 2v(z)$ if $v(z) < 0$. Hence,

$$v(\tau(z)) = \begin{cases} v(e) + v(z), & \text{if} \quad v(z) > 0; \\ v(e) - v(z), & \text{if} \quad v(z) < 0. \end{cases}$$

Suppose $v(e) < v(z) < 0$. Then

$$v(z_k) = \begin{cases} v(e) - v(z), & \text{if} \quad k \text{ is odd}; \\ v(z), & \text{if} \quad k \text{ is even}. \end{cases}$$

Hence, for all $k \geqslant 0$, we have $v(e) < v(z_k) < 0$ and $v(f(z_k)) = 2v(z_k)$. In addition, $\lambda(z_k, v) = 0$ for all $k \geqslant 0$, so the sequence $\hat{\lambda}(z_k, v)$ is bounded. Therefore, applying (14) and letting $k \to \infty$, we obtain

$$\begin{aligned} \hat{\lambda}(z, v) &= -v(z) - \frac{1}{2}(v(e) - v(z)) - \frac{1}{4} v(z) - \frac{1}{8}(v(e) - v(z)) - \cdots \\ &= -v(z) \sum_{i=0}^{\infty} \left(-\frac{1}{2}\right)^i - \frac{1}{2} v(e) \sum_{i=0}^{\infty} \left(\frac{1}{4}\right)^i \\ &= -\frac{2}{3}(v(z) + v(e)). \end{aligned}$$

Suppose $v(z) > 0$ or $v(z) < v(e)$, and choose m to be the unique positive integer such that $v(e) < |v(z)| + mv(e) < 0$. If $v(z) > 0$, then

$$v(z_1) = v(e) + v(z) < v(z).$$

Hence, by the choice of m, we find

$$v(z_k) = kv(e) + v(z) \qquad \text{for} \quad 0 \leqslant k \leqslant m.$$

On the other hand, if $v(z) < v(e) < 0$, then

$$v(z_1) = v(e) - v(z) > 0.$$

So by the previous case

$$v(z_k) = (k-1)\, v(e) + v(z_1) = kv(e) - v(z) \qquad \text{for} \quad 1 \leqslant k \leqslant m.$$

In either case, we conclude that $v(z_m) = mv(e) + |v(z)|$. Therefore, by part (a), we have

$$\hat{\lambda}(z, v) = \frac{1}{2^m}\,\hat{\lambda}(z_m, v) = \frac{1}{2^m}\left(-\frac{2}{3}\,[v(z_m) + v(e)]\right)$$
$$= -\frac{1}{3 \cdot 2^{m-1}}\,(|v(z)| + (m+1)\,v(e)). \quad \blacksquare$$

4. MORPHISMS WITH HYPERPLANE EIGENDIVISORS

In this section, we will concentrate on a special class of morphisms for which the calculation of canonical local heights is somewhat simpler.

DEFINITION. Let $\phi\colon \mathbf{P}^n \to \mathbf{P}^n$ be a morphism of degree d and let D be a divisor on $\mathbf{P}^n$. D is a *eigendivisor* of ϕ if $\phi^* D = dD$.

In the case where ϕ has an eigendivisor W which is a hyperplane in $\mathbf{P}^n$, we will obtain a simple limit formula for the canonical local height associated to ϕ and W. We begin with a geometric characterization of hyperplane eigendivisors.

LEMMA 4.1. *If W is a hyperplane in $\mathbf{P}^n$, W is an eigendivisor of ϕ if and only if $\phi^{-1}(|W|) = |W|$.*

Proof. Since W is irreducible, the only divisors with support $|W|$ are those of the form nW for n a nonzero integer, and the only one with degree d is dW. $\blacksquare$

When ϕ is a morphism with hyperplane eigendivisor W, we will use the notation that $\hat{\lambda}_{W,\phi} = \hat{\lambda}_{W,\phi,1}$. Thus, by Theorem 2.1, we have

$$\hat{\lambda}_{W,\phi}(\phi P, v) = d\hat{\lambda}_{W,\phi}(P, v) \tag{16}$$

for all $P \in \mathbf{P}^n$, $P \notin |W|$, and all $v \in M$. The following theorem and corollary illustrate the reason for our interest in the hyperplane eigendivisor case.

THEOREM 4.2. *If W is a hyperplane eigendivisor of ϕ and λ_W is any Weil local height function associated to W, then*

$$\hat{\lambda}_{W,\phi}(P, v) = \lim_{m \to \infty} \frac{\lambda_W(\phi^m P, v)}{d^m}.$$

Proof. Iterating (16) shows that

$$\hat{\lambda}_{W,\phi}(P, v) = \frac{\hat{\lambda}_{W,\phi}(\phi^m P, v)}{d^m}$$

for all $m \geqslant 0$. Since the difference $\hat{\lambda}_{W,\phi} - \lambda_W$ is bounded for each $v \in M$, the theorem follows. ∎

Remark. Suppose g is a linear form, $W = \operatorname{div}(g)$, and λ_W is the standard Weil local height

$$\lambda_W(P, v) = \max\left\{ v\left(\frac{g(P)}{x_0(P)}\right), ..., v\left(\frac{g(P)}{x_n(P)}\right)\right\}.$$

Then there is an M-constant γ depending on g such that $\lambda_W(P, v) \geqslant \gamma(v)$ for all $P \in \mathbf{P}^n$. Hence, by Theorem 4.2,

$$\hat{\lambda}_{W,\phi}(P, v) \geqslant 0 \qquad \text{for all} \quad P \in \mathbf{P}^n \text{ and all } v \in M.$$

In addition, since $\hat{\lambda}_{W,\phi}(\phi^n P, v) = d^n \hat{\lambda}_{W,\phi}(P, v)$ and $\hat{\lambda}_{W,\phi} - \lambda_W$ is bounded for each $v \in M$, we conclude that $\hat{\lambda}_{W,\phi}(P, v) = 0$ if and only if the sequence $\lambda_W(\phi^n P, v)$ is bounded as $n \to \infty$.

Corollary 4.3. *Let W be a hyperplane eigendivisor for ϕ and let $P \in \mathbf{P}^n \backslash |W|$. Then P is a pre-periodic point of ϕ if and only if $\hat{\lambda}_{W,\phi}(P, v) = 0$ for all $v \in M$.*

Proof. Suppose P is pre-periodic and let $v \in M$. Since the orbit of P is finite, there exists a $C > 0$ such that $\lambda_W(\phi^m P, v) \leqslant C$ for all m. Thus

$$\lim_{m \to \infty} \frac{\lambda_W(\phi^m P, v)}{d^m} \leqslant \lim_{m \to \infty} \frac{C}{d^m} = 0 \qquad \text{and} \qquad \hat{\lambda}_{W,\phi}(P, v) = 0.$$

Conversely, suppose that $P \in \mathbf{P}^n \backslash |W|$ and $\hat{\lambda}_{W,\phi}(P, v) = 0$ for all $v \in M$. There is some finite extension L of K such that $P \in \mathbf{P}^n(L)$, and by Theorem 2.4,

$$\hat{h}_\phi(P) = \frac{1}{[L : \mathbf{Q}]} \sum_{v \in M_L} [L_v : \mathbf{Q}_v] \, \hat{\lambda}_{W,\phi}(P, v) = 0.$$

By Corollary 1.3, P is pre-periodic. ∎

For computational purposes, it is often convenient to change coordinates.

LEMMA 4.4. *Let ψ be an isomorphism on $\mathbf{P}^n$. Then a morphism $\phi\colon \mathbf{P}^n \to \mathbf{P}^n$ has hyperplane eigendivisor W if and only if $\psi\phi\psi^{-1}$ has hyperplane eigendivisor ψW.*

Proof. Suppose ϕ has a hyperplane eigendivisor $W = \operatorname{div}(g)$. Note that $\psi W = \operatorname{div}(g \circ \psi^{-1}) = (\psi^{-1})^* W$. We find that

$$\begin{aligned}(\psi\phi\psi^{-1})^* (\psi W) &= (\psi^{-1})^* \phi^* \psi^* (\psi W) = (\psi^{-1})^* \phi^* W \\ &= (\psi^{-1})^* (dW) = d(\psi^{-1})^* W = d(\psi W), \qquad (17)\end{aligned}$$

and thus ψW is a hyperplane eigendivisor for $\psi\phi\psi^{-1}$. The converse follows by applying ψ^{-1}. ∎

THEOREM 4.5. *If ϕ has hyperplane eigendivisor W and ψ is an isomorphism on $\mathbf{P}^n$, then*

$$\hat{\lambda}_{W,\phi}(P, v) = \hat{\lambda}_{\psi W,\, \psi\phi\psi^{-1}}(\psi P, v).$$

Proof. Let λ_W and $\lambda_{(\psi^{-1})^* W}$ be Weil local heights associated to W and to $(\psi^{-1})^* W = \psi W$ respectively. By functoriality, there is an M-bounded, M-continuous function $\gamma\colon \mathbf{P}^n \times M \to \mathbf{R}$ such that

$$\lambda_{(\psi^{-1})^* W}(\psi P, v) = \lambda_W(P, v) + \gamma(P, v).$$

We now see that

$$\begin{aligned}\hat{\lambda}_{\psi W,\, \psi\phi\psi^{-1}}(\psi P, v) &= \lim_{m\to\infty} \frac{\lambda_{(\psi^{-1})^* W}((\psi\phi\psi^{-1})^m (\psi P), v)}{d^m} \\ &= \lim_{m\to\infty} \frac{\lambda_{(\psi^{-1})^* W}(\psi(\phi^m P), v)}{d^m} \\ &= \lim_{m\to\infty} \frac{\lambda_W(\phi^m P, v)}{d^m} + \lim_{m\to\infty} \frac{\gamma(\phi^m P, v)}{d^m} \\ &= \hat{\lambda}_{W,\phi}(P, v)\end{aligned}$$

by Theorem 4.2. ∎

Remark. By an argument similar to the preceding proof, we can show that for any morphism ϕ and any isomorphism ψ on $\mathbf{P}^n$ the global canonical heights satisfy

$$\hat{h}_{\psi\phi\psi^{-1}}(\psi P) = \hat{h}_{\phi}(P) \qquad (18)$$

for all $P \in \mathbf{P}^n$. In particular, since this result follows directly from the limit formula (1) for global canonical heights and Theorem 1.1 applied to ψ, (18) holds even if ϕ does not have an eigendivisor.

Given a morphism ϕ with hyperplane eigendivisor W, choose an isomorphism ψ such that $\psi W = H_0$. Then $\psi\phi\psi^{-1}$ is a morphism with hyperplane eigendivisor H_0 and hence it can be written in the form $\psi\phi\psi^{-1}(P) = [x_0(P)^d, f_1(P), ..., f_n(P)]$. A morphism in this form lends itself more readily to calculations than an arbitrary morphism, particularly when $n = 1$. In the next section we will use this remark in conjunction with Theorem 4.2 to compute some canonical local heights on $\mathbf{P}^1$.

We will say that a morphism $\phi: \mathbf{P}^n \to \mathbf{P}^n$ is *diagonalizable* if there is an isomorphism $\psi: \mathbf{P}^n \to \mathbf{P}^n$ such that $\psi\phi\psi^{-1} = \sigma$ where σ has the form

$$\sigma[x_0, x_1, ..., x_n] = [a_0 x_0^d, a_1 x_1^d, ..., a_n x_n^d].$$

These morphisms σ are characterized by the property that each coordinate hyperplane H_i is an eigendivisor for σ. Since any two sets of $n+1$ independent hyperplanes are conjugate, we have the following characterization of diagonalizable morphisms.

LEMMA 4.6. *A morphism $\phi: \mathbf{P}^n \to \mathbf{P}^n$ is diagonalizable if and only if it has $n+1$ hyperplane eigendivisors $W_0, W_1, ..., W_n$ such that*

$$|W_0| \cap |W_1| \cap \cdots \cap |W_n| = \varnothing.$$

The canonical local and global heights for diagonalizable morphisms are expressible in terms of the standard heights via

PROPOSITION 4.7. *Let $\sigma: \mathbf{P}^n \to \mathbf{P}^n$ be the morphism defined by*

$$\sigma[x_0, x_1, ..., x_n] = [a_0 x_0^d, a_1 x_1^d, ..., a_n x_n^d],$$

where $a_0, a_1, ..., a_n \in \bar{K}^*$. Let $b_0, b_1, ..., b_n \in \bar{K}$ be such that $b_i^{d-1} = a_i$.

(a) *For $j = 0, 1, ..., n$, if $P \notin |H_j|$, then*

$$\hat{\lambda}_{H_j, \sigma}(P, v) = \max_{0 \leqslant i \leqslant n} \left\{ v\left(\frac{b_j x_j(P)}{b_i x_i(P)}\right)\right\}.$$

(b) *For all $P = [x_0, x_1, ..., x_n] \in \mathbf{P}^n$,*

$$\hat{h}_\sigma(P) = h([b_0 x_0, b_1 x_1, ..., b_n x_n]),$$

where h is the standard height on $\mathbf{P}^n$ defined in Section 1.

Proof. A straightforward computation shows that the right side of the equation in (a) has the characterizing properties of $\hat{\lambda}_{H_j,\sigma}$. (b) may be deduced directly from (a) or, as in (a), by observing that the map on the right side of the equation in (b) has the defining properties of $\hat{h}_\sigma$. ∎

5. POLYNOMIAL MAPS ON $\mathbf{P}^1$

We now take a closer look at a morphism $\phi\colon \mathbf{P}^1 \to \mathbf{P}^1$ which has a hyperplane eigendivisor. As in Section 3 we identify $\mathbf{P}^1(K)$ with $K \cup \{\infty\}$ by identifying each point $P = [x, y]$ with its affine coordinate $z = y/x$. We choose coordinates so that the point $z = \infty$ is an eigendivisor for ϕ. Then the map $\phi\colon K \to K$ is given by a polynomial of degree $d = \deg(\phi) \geqslant 2$:

$$\phi(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_d z^d, \qquad a_i \in K, \quad a_d \neq 0.$$

We will use the notation $z_k = \phi^k(z)$ for $k \geqslant 0$ and write $\hat{\lambda}(z, v) = \hat{\lambda}_{\infty,\phi}(z, v)$ and $\lambda(z, v) = \lambda_0(z, v) = \max\{-v(z), 0\}$.

To compute $\hat{\lambda}(z, v)$ for non-archimedean v, we will use the following lemma.

Lemma 5.1. *Suppose v is non-archimedean. Let α_v be the minimum of the $d+1$ numbers*

$$\frac{1}{d-i}\, v\left(\frac{a_i}{a_d}\right), \qquad 0 \leqslant i < d, \qquad \text{and} \qquad \frac{1}{d-1}\, v\left(\frac{1}{a_d}\right).$$

Then the region $v(z) < \alpha_v$ is stable under ϕ, and in that region the function $F(z) := -v(z) - (v(a_d)/(d-1))$ satisfies $F(\phi(z)) = dF(z)$.

Proof. If $v(z) < \alpha_v$ then $v(a_i z^i) > v(a_d z^d)$ for $0 \leqslant i \leqslant d-1$, so $v(\phi(z)) = v(a_d z^d)$, and hence $F(\phi(z)) = dF(z)$. Moreover,

$$v(\phi(z)) = v(a_d) + dv(z) < v(a_d) + d\alpha_v \leqslant \alpha_v,$$

because $(d-1)\,\alpha_v \leqslant -v(a_d)$. ∎

Theorem 5.2. *Suppose v is non-archimedean and α_v is defined as in Lemma 5.1. If $v(z_k)$ is bounded below as $k \to \infty$, then $\hat{\lambda}(z, v) = 0$. Otherwise, if m is such that $v(z_m) < \alpha_v$, then*

$$\hat{\lambda}(z, v) = d^{-m}\left(-v(z_m) - \frac{v(a_d)}{d-1}\right) > 0.$$

Proof. If the sequence $v(z_k)$, $k \geqslant 0$, is bounded below, then $\lambda(z_k, v)$ is bounded, and hence by Theorem 4.2,

$$\hat{\lambda}(z, v) = \lim_{k \to \infty} \{d^{-k}\lambda(z_k, v)\} = 0.$$

In the region $v(z) < \alpha_v$ the difference between $\lambda(z, v)$ and $F(z)$ is bounded. Hence

$$\hat{\lambda}(z, v) = \lim_{k \to \infty} \{d^{-k}\lambda(z_k, v)\} = \lim_{k \to \infty} \{d^{-k}F(z_k)\} = F(z),$$

because the lemma shows that $F(z_k) = d^k F(z)$. From this the theorem follows, using $\hat{\lambda}(z, v) = d^{-m}\hat{\lambda}(z_m, v)$. ∎

COROLLARY 5.3. *Suppose v is non-archimedean and α_v is defined as in Lemma* 5.1.

(a) *The following are equivalent for $z \in K$:*

  (i) *$|z_k|_v$ is bounded as $k \to \infty$; i.e., the orbit of z is v-adically bounded.*

  (ii) *$v(z_k) \geqslant \alpha_v$ for all $k \geqslant 0$.*

  (iii) *$\hat{\lambda}(z, v) = 0$.*

(b) *If z is pre-periodic, then $v(z) \geqslant \alpha_v$.*

Proof. (a) is immediate from the theorem, and (b) follows from (a) by Corollary 4.3 (or via (i) implies (ii)). ∎

Let r_i, $1 \leqslant i \leqslant d$, be the roots of $\phi(z)$ in $\bar{K}_v$. Let $R = \{z \in \bar{K}_v \mid v(z) \neq v(r_i)$ for $1 \leqslant i \leqslant d\}$. For $z \in R$, $v(\phi(z))$ depends only on $v(z)$:

$$v(\phi(z)) = v(a_d) + \sum_{i=1}^{d} v(z - r_i) = v(a_d) + \sum_{i=1}^{d} \min\{v(z), v(r_i)\}.$$

Hence we can use Theorem 5.2 to give an explicit expression for $\hat{\lambda}(z, v)$ in terms of $v(z)$ for all $z \in R$ such that $v(\phi(z)) < \alpha_v$.

Recall that the *Newton polygon* of the polynomial ϕ is defined to be the highest convex polygonal line joining $(0, v(a_0))$ and $(d, v(a_d))$ which passes on or below all of the points $(i, v(a_i))$ for $0 \leqslant i \leqslant d$. It is a well-known result (Cf. Koblitz [6], Lemma 4, p. 90) that all of the roots r_i have the same valuation, say $v(r_i) = \beta_v$ for all i, if and only if the Newton polygon of ϕ is a straight line with slope $-\beta_v$. Suppose that, in addition, the Newton polygon line intersects the line $x = 1$ below the x-axis; i.e., $v(a_0) < \beta_v$. Then $\beta_v = \alpha_v$ and we obtain the following explicit expression for $\hat{\lambda}(z, v)$.

THEOREM 5.4. *In the situation of Lemma* 5.1, *suppose that the Newton polygon of* $\phi(z)$ *is a straight line which intersects the line* $x=1$ *below the* x*-axis; i.e., suppose*

$$v(a_i) \geqslant \frac{i}{d} v(a_d) + \frac{d-i}{d} v(a_0) \qquad \textit{for} \quad 0<i<d$$

and

$$v(a_0) < -\frac{v(a_d)}{d-1}.$$

Then the number α_v *defined in Lemma* 5.1 *is the negative of the slope of the Newton polygon line; i.e.,*

$$\alpha_v = \frac{v(a_0) - v(a_d)}{d},$$

and

$$\hat{\lambda}(z, v) = -\min\{v(z), \alpha_v\} - \frac{v(a_d)}{d-1} \qquad \textit{if} \quad v(z) \neq \alpha_v.$$

As a corollary we have

COROLLARY 5.5. *Under the hypotheses of Theorem* 5.4,

$$\hat{\lambda}(z, v) = \begin{cases} -\dfrac{1}{d^m} \min\{v(z_m), \alpha_v\} - \dfrac{v(a_d)}{d-1}, & \textit{if} \quad v(z_m) \neq \alpha_v; \\ 0, & \textit{if} \quad v(z_k) = \alpha_v \textit{ for all } k \geqslant 0 \end{cases}.$$

The proof of Theorem 5.4 is a straightforward calculation using Theorem 5.2 and is left to the reader. The point is that for $v(z) > \alpha_v$, $v(\phi(z)) = v(a_0) < \alpha_v$.

By a change of coordinates of the form $z = aw + b$, $a \neq 0$, one can change the polynomial ϕ representing the map from $\mathbf{P}^1$ to $\mathbf{P}^1$. The equation $z_1 = \phi(z)$ becomes

$$aw_1 + b = \phi(aw+b) = \phi(b) + \phi'(b)\, aw + \cdots + a_d a^d w^d.$$

Hence

$$w_1 = a^{-1}(\phi(b) - b) + \phi'(b)\, w + \cdots + a_d a^{d-1} w^d = \chi(w).$$

In the case $d=2$ we can take $a=a_d^{-1}$ and b the root of the linear polynomial $\phi'(z)$, i.e., $b=-a_1/2a_2$, to achieve $\chi(w)=w^2+c$. Then χ is a polynomial to which Corollary 5.5 applies if $v(c)<0$, and χ has good reduction at v if $v(c)\geqslant 0$. The result is

COROLLARY 5.6. *If $d=2$, there is a unique choice of affine coordinate z such that the polynomial $\phi(z)$ is of the form $\phi_c(z):=z^2+c$. In that case*

(a) *If $v(c)\geqslant 0$, then*

$$\hat{\lambda}(z, v)=\lambda(z, v)=-\min\{v(z), 0\}.$$

(b) *If $v(c)<0$, then*

$$\hat{\lambda}(z, v)=\begin{cases} -\dfrac{1}{2^m}\min\left\{v(z_m), \dfrac{1}{2}v(c)\right\}, & \text{if } \; v(z_m)\neq\dfrac{1}{2}v(c); \\ 0, & \text{if } \; v(z_k)=\dfrac{1}{2}v(c) \text{ for all } k\geqslant 0. \end{cases}$$

Since $\hat{\lambda}(z_0, v)=0$ if z_0 is pre-periodic, we have

COROLLARY 5.7. *Suppose $\phi_c(z)=z^2+c$ has a pre-periodic point z_0.*

(a) *If $v(c)\geqslant 0$, then $v(z_0)\geqslant 0$.*

(b) *If $v(c)<0$, then $v(z_0)=\frac{1}{2}v(c)$.*

6. v-ADIC FILLED JULIA SETS AND PRE-PERIODIC POINTS

We continue with the notational conventions of the preceding section and suppose $\phi\colon K\to K$ is a polynomial map of degree $d\geqslant 2$. Extending Corollary 5.3(a) to the archimedean case, we will show in this section that for all $v\in M$ the canonical local height $\hat{\lambda}(z, v)=0$ if and only if the orbit of z under ϕ is "v-adically bounded." This leads us to define the notion of v-adic filled Julia sets and to show that a point $z\in K$ is pre-periodic for ϕ if and only if for all $v\in M$, z lies in the v-adic filled Julia set of ϕ. By combining these observations with the computations in Section 5 and some archimedean estimates, we will bound the number of rational pre-periodic points of a quadratic **Q**-polynomial map in terms of the number of primes dividing the denominators of its coefficients.

Since ϕ has coefficients in K, given any $v\in M$, we may extend the definition of ϕ to a map from $\bar{K}_v$ to $\bar{K}_v$. Given $z\in\bar{K}_v$ we set $z_m=\phi^m(z)$ for $m\geqslant 0$. As noted in Section 3, if $v\in M$ is fixed, any local height associated to v is defined for all points in $\mathbf{P}^1(\bar{K}_v)$ except those lying on the support of its divisor.

DEFINITION. Let $v \in M$. If $z \in \bar{K}_v$, we say that *the orbit of z under ϕ is bounded with respect to v* if there exists a constant $C > 0$ such that

$$|z_m|_v < C \qquad \text{for all} \quad m \geqslant 0.$$

The *v-adic filled Julia set of ϕ* is the set of all $z \in \bar{K}_v$ such that the orbit of z under ϕ is bounded with respect to v.

Fix an embedding $K \hookrightarrow \mathbf{C}$ and let v_∞ be the standard archimedean absolute value on $\mathbf{C}$. Then the v_∞-adic filled Julia set of ϕ defined above is just the filled Julia set commonly studied in complex dynamical systems (Cf. [4], Section 3.8).

In Theorem 5.2, we showed that for non-archimedean v,

$$\hat{\lambda}(z, v) = -v(z) - \frac{v(a_d)}{d-1} \qquad \text{for} \quad |z|_v \text{ sufficiently large.}$$

For archimedean v we can at least prove the following.

THEOREM 6.1. *Suppose $v \in M$ is archimedean. Then for all $z \in \mathbf{C}$*

$$\hat{\lambda}(z, v) = \log |z|_v + \frac{\log |a_d|_v}{d-1} + o(1) \qquad \textit{as} \quad |z|_v \to \infty. \tag{19}$$

Proof. Put

$$g(z) = \log |z|_v + \frac{\log |a_d|_v}{d-1}.$$

Then

$$g(\phi(z)) - dg(z) = \log \left(\left| \frac{\phi(z)}{a_d z^d} \right|_v \right) \to 0 \qquad \text{as} \quad |z|_v \to \infty.$$

Also, since $d \geqslant 2$, $|\phi(z)|_v / |z|_v \to \infty$ as $|z|_v \to \infty$. Therefore, the regions of the form $|z|_v > R$ are stable under ϕ for large R, and for every $\varepsilon > 0$ there is such an R_ε, which can be given explicitly in terms of the $|a_i|_v$'s and ε, for which $|g(\phi(z)) - dg(z)|_v < \varepsilon$ for $|z|_v > R_\varepsilon$. Hence for $|z|_v > R_\varepsilon$ we have

$$\left| \frac{g(z_m)}{d^m} - \frac{g(z_{m-1})}{d^{m-1}} \right|_v < \frac{\varepsilon}{d^m},$$

and summing we find

$$\left|\frac{g(z_m)}{d^m} - g(z)\right|_v < \frac{\varepsilon}{d} + \frac{\varepsilon}{d^2} + \cdots + \frac{\varepsilon}{d^m} < \frac{\varepsilon/d}{1-1/d} = \frac{\varepsilon}{d-1}.$$

Since we may assume (wlog) that $R_\varepsilon \geqslant 1$, for $|z|_v > R_\varepsilon$ we have

$$\hat{\lambda}(z, v) = \lim_{m\to\infty} \left\{\frac{\log |z_m|_v}{d^m}\right\} = \lim_{m\to\infty} \left\{\frac{g(z_m)}{d^m}\right\},$$

and hence $|\hat{\lambda}(z, v) - g(z)|_v < \varepsilon/(d-1)$. Since ε was arbitrary, this proves (19). ▮

The canonical local height $\hat{\lambda}(\cdot, v)$ associated to v determines the v-adic filled Julia set of ϕ in a natural way.

THEOREM 6.2. *Let $v \in M$. If $z \in \bar{K}_v$, then $\hat{\lambda}(z, v) = 0$ if and only if z is in the v-adic filled Julia set of ϕ.*

Proof. If v is non-archimedean, this result follows directly from Corollary 5.3(a). So assume v is archimedean.

Suppose there is a constant $C > 0$ such that $|z_m|_v \leqslant C$ for all $m \geqslant 0$. Then $\lambda(z_m, v) = \max\{\log |z_m|_v, 0\}$ is bounded, and hence $\hat{\lambda}(z, v) = \lim_{m\to\infty} d^{-m}\lambda(z_m, v) = 0$.

On the other hand, if the sequence $|z_m|_v$ is unbounded, then it follows immediately from Theorem 6.1 that $\hat{\lambda}(z, v) = (1/d^m)\,\hat{\lambda}(z_m, v) > 0$. ▮

Combining the preceding result with Corollary 4.3, we see that an algebraic number $z \in \bar{K}$ has a finite orbit under ϕ if and only if its orbit is bounded with respect to v for all $v \in M$. This remark yields the following corollary.

COROLLARY 6.3. *Suppose $z \in \bar{K}$. Then z is a pre-periodic point of ϕ if and only if z is in the v-adic filled Julia set of ϕ for all $v \in M$.*

We will conclude by proving an upper bound on the number of rational pre-periodic points of a quadratic $\mathbf{Q}$-polynomial map on $\mathbf{P}^1$. As in Corollary 5.6 we let $\phi_c\colon \mathbf{P}^1 \to \mathbf{P}^1$ denote the K-polynomial map

$$\phi_c(z) = z^2 + c, \qquad \text{where} \quad c \in K.$$

As we observed in Section 5, every quadratic K-polynomial map on $\mathbf{P}^1$ is conjugate over K to some ϕ_c with $c \in K$. Since every K-conjugate of a morphism has the same number of K-rational pre-periodic points (P is pre-periodic for ϕ if and only if ψP is pre-periodic for $\psi\phi\psi^{-1}$), we can restrict our focus to the quadratic maps ϕ_c. Corollaries 5.6 and 5.7 provide some information about the v-adic filled Julia set of ϕ_c when v is non-archimedean. To learn about the v_∞-adic filled Julia set of ϕ_c, we begin with the following standard lemma on complex dynamical systems (Cf. [4], p. 270).

LEMMA 6.4. *For any $c \in K$, if $|z| > \max\{|c|, 2\}$ then $|\phi^k(z)| \to \infty$ as $k \to \infty$.*

Proof. Since $|z| > |c|$, we see that $|\phi_c(z)| \geqslant |z|^2 - |c| > |z|^2 - |z| = |z|(|z| - 1)$. We also know that $|z| > 2$, and so $|z| - 1 = 1 + l$ for some positive l. We now have that $|\phi_c(z)| > (1 + l)\, |z|$, and because $|\phi_c(z)| > |z|$, we can repeat this argument to obtain

$$|\phi_c^k(z)| > (1 + l)^k\, |z|.$$

Thus $|\phi_c^k(z)| \to \infty$ as $k \to \infty$. ∎

COROLLARY 6.5. *Suppose $c \in K$ with $|c| \leqslant 2$. If z is in the v_∞-adic filled Julia set of ϕ_c, then $|z| \leqslant 2$.*

For the cases in which $|c| > 2$, the bound on pre-periodic points $z \in K$ yielded by Lemma 6.4, that $|z| \leqslant |c|$, is not quite as satisfying. With a little work, we can get a considerably better bound for these cases.

PROPOSITION 6.6. *Suppose $c \in K$ with $|c| > 2$. Let*

$$\gamma = |c|, \qquad \beta_1 = \frac{1 + \sqrt{1 + 4\gamma}}{2}, \qquad \text{and} \qquad \beta_2 = \sqrt{\gamma - \beta_1}.$$

If z is in the v_∞-adic filled Julia set of ϕ_c, then $\beta_2 \leqslant |z| \leqslant \beta_1$. Furthermore, $\beta_1 - \beta_2 < 2$.

Proof. Note that since $\gamma > 2$, β_1 satisfies $2 < \beta_1 < \gamma$.

Define a sequence $\{b_k\}$ by $b_0 = \gamma$ and $b_k = \sqrt{\gamma + b_{k-1}}$ for $k > 0$. It is not difficult to show that this is a decreasing sequence, and that its limit is β_1. We claim that for $z \in \mathbf{C}$, if $|z| > b_k$ for any k, then its orbit tends to infinity. Lemma 6.4 implies that this is true for $k = 0$. For any $k > 0$, if $|z| > b_k$, then

$$|\phi_c(z)| = |z^2 + c| \geqslant |z|^2 - \gamma > b_k^2 - \gamma = b_{k-1},$$

and thus by mathematical induction, we have established our claim. Therefore, if $|z| > \beta_1$, then it is greater than b_k for some $k > 0$, and thus its orbit is not bounded.

Now suppose that $|z| < \beta_2$. Since $\beta_2 < \sqrt{\gamma}$, $|z|^2 < \gamma$. We compute that

$$|\phi_c(z)| = |z^2 + c| \geqslant \gamma - |z^2| > \gamma - (\beta_2)^2 = \beta_1,$$

and by the above, the orbit of z escapes to infinity. Therefore, for any $z \in \mathbf{C}$, if $|z| > \beta_1$ or $|z| < \beta_2$, then z is not in the v_∞-adic filled Julia set of ϕ_c.

Finally, to check that $\beta_1 - \beta_2 < 2$, note that by the definition of β_1, $\gamma = (\beta_1)^2 - \beta_1$. Since $\beta_1 > 2$, it follows that

$$\beta_2 = \sqrt{\gamma - \beta_1} = \sqrt{(\beta_1)^2 - 2\beta_1} > \beta_1 - 2.$$

Hence, $\beta_1 - \beta_2 < \beta_1 - (\beta_1 - 2) = 2$. ∎

From Corollary 5.7, we deduce that if $c \in \mathbf{Q}$ than any rational pre-periodic point of ϕ_c must have a special form. Using p-adic techniques, this result was obtained independently by Russo and Walde [16].

Proposition 6.7. *Suppose $c \in \mathbf{Q}$ and write $c = a/b$ where a and b are relatively prime integers with $b > 0$. If ϕ_c has a pre-periodic point $z \in \mathbf{Q}$, then*

(i) *$b = e^2$ for some positive integer e, and*

(ii) *$z = w/e$ for some integer w such that* $\gcd(w, e) = 1$.

In particular, if $c \in \mathbf{Z}$, then $z \in \mathbf{Z}$.

Now, using Corollary 6.5 and Propositions 6.6 and 6.7, we can bound the number of rational pre-periodic points of ϕ_c in terms of the number of primes dividing the denominator of c. Our argument begins with a lemma from elementary number theory.

Lemma 6.8. *Let e and u be integers such that $e > 1$ and $\gcd(e, u) = 1$. Let s be the number of distinct odd prime factors of e and define ε by*

$$\varepsilon = \begin{cases} 0, & \textit{if} \quad 4 \nmid e; \\ 1, & \textit{if} \quad 4 \mid e,\ 8 \nmid e; \\ 2, & \textit{if} \quad 8 \mid e. \end{cases}$$

If the congruence

$$X^2 \equiv u \pmod{e} \tag{20}$$

has a solution, then it has exactly $2^{s+\varepsilon}$ distinct solutions modulo e.

Proof. Let p be a prime and r a positive integer such that $p^r \mid e$ and $p^{r+1} \nmid e$. Suppose (20) has a solution. Then $X^2 \equiv u$ must have a solution modulo p^r. If t and v are any two such solutions, then $t^2 \equiv v^2 \equiv u \pmod{p^r}$ which implies $p^r \mid (t+v) \cdot (t-v)$. But $\gcd(e, u) = 1$, so p does not divide t or v. Note that if $p^i \mid (t+v)$ and $p^i \mid (t-v)$, then $p^i \mid 2t$ which implies $p = 2$ and $i = 1$. Therefore, if p is odd, then $t \equiv \pm v \pmod{p^r}$; and if $p = 2$, then $t \equiv \pm v$

(mod 2^{r-1}). Hence, our desired result follows by applying the Chinese Remainder Theorem. ∎

THEOREM 6.9. *Let $c = a/e^2$ where a and e are relatively prime integers with $e > 0$. Define s and ε as in Lemma 6.8. If $c \neq -2$, then the number of rational pre-periodic points of ϕ_c is less than or equal to*

$$2^{s+2+\varepsilon} + 1.$$

The quadratic map ϕ_{-2} has exactly 6 rational pre-periodic points.

Proof. Since ∞ is a fixed point of every map ϕ_c, it suffices to bound the number of rational pre-periodic points of ϕ_c in $\mathbf{Q}$ and add one.

Let $z \in \mathbf{Q}$ be a pre-periodic point of ϕ_c. Suppose first that c is not an integer, so $e > 1$. Since $\phi_c(z)$ is also a rational pre-periodic point of ϕ_c, by Proposition 6.7, there are integers w and w_1 such that

$$z = \frac{w}{e} \qquad \text{and} \qquad \phi_c(z) = z^2 + \frac{a}{e^2} = \frac{w_1}{e},$$

with $\gcd(w, e) = \gcd(w_1, e) = 1$. Hence, $ew_1 = w^2 + a$, so we conclude that

$$w^2 \equiv -a \pmod{e}. \tag{21}$$

By Lemma 6.8, there are exactly $2^{s+\varepsilon}$ distinct w's modulo e which satisfy (21). Hence, if $|c| > 2$, our result follows from Proposition 6.6, since $\beta_1 - \beta_2 < 2$. If $|c| \leqslant 2$, then our result follows from Corollary 6.5.

Now suppose c is an integer. Then, by Proposition 6.7, every rational pre-periodic point of ϕ_c is integral. Thus if $|c| > 2$, our result again follows from Proposition 6.6. It remains only to check the five integer values of c between -2 and 2. Using Lemma 6.4, a few simple calculations show that $\phi_1(z) = z^2 + 1$ and $\phi_2(z) = z^2 + 2$ have no pre-periodic points in $\mathbf{Q}$, $\phi_0(z) = z^2$ and $\phi_{-1}(z) = z^2 - 1$ each have three pre-periodic points in $\mathbf{Q}$ (namely, $0, \pm 1$), and $\phi_{-2}(z) = z^2 - 2$ has exactly five pre-periodic points $(0, \pm 1, \pm 2)$ in $\mathbf{Q}$. This completes the proof. ∎

We remark that Narkiewicz [11] has shown that if c is an integer, then ϕ_c can only have periodic points of orders 1 or 2. It follows from Theorem 2 in Russo and Walde [16] that ϕ_c cannot have both fixed points and periodic points of order 2 if $c \in \mathbf{Z}$. Therefore, if $c \in \mathbf{Z}$, ϕ_c can have at most two periodic points. Recently, Morton and Silverman [10] greatly reduced the bound Narkiewicz [11] had obtained on the maximum period of a K-rational periodic point of a K-polynomial map on $\mathbf{P}^1$. However, for $K = \mathbf{Q}$ and $c \notin \mathbf{Z}$, the bound on the number of rational periodic points of ϕ_c provided by Theorem 6.9 represents a significant improvement over the bounds that can be deduced from the results of [10] and [11].

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