Journal of Pure and Applied Algebra 215 (2011) 1263–1269

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

journal homepage: www.elsevier.com/locate/jpaa

Self-tests for freeness over commutative artinian rings

Liana M. Şega

Department of Mathematics and Statistics, University of Missouri, Kansas City, MO 64110, USA

ARTICLE INFO

Article history:
Received 17 May 2010
Received in revised form 29 July 2010
Available online 9 September 2010
Communicated by S. Iyengar

MSC:
Primary: 13D07
Secondary: 13D02

ABSTRACT

We prove that the Auslander–Reiten conjecture holds for commutative standard graded artinian algebras, in two situations: the first is under the assumption that the modules considered are graded and generated in a single degree. The second is under the assumption that the algebra is *generic* Gorenstein of socle degree 3.

0. Introduction

Conjectures of homological nature play an important role in commutative algebra and representation theory of algebras. We refer to Christensen and Holm [7] and Happel [10] for background material on a cluster of such conjectures. Originally formulated over Artin algebras, many of these statements have meaningful variants for arbitrary noetherian commutative rings. In this paper we are mainly interested in the Auslander–Reiten conjecture, stated in [2] in connection with the Nakayama conjecture.

Let R be a commutative noetherian ring. A commutative version of the Auslander–Reiten conjecture is stated by Auslander et al. [1] as follows: *if a finitely generated R-module M satisfies* $\mathrm{Ext}^i_R(M, M) = 0 = \mathrm{Ext}^i_R(M, R)$ *for all* $i > 0$*, then M is projective.*

It is known that the Auslander–Reiten conjecture follows from a conjecture attributed to Auslander; see [7]. However, Jorgensen and Şega [15] found counterexamples to the Auslander conjecture over commutative artinian rings. In this paper we prove that the Auslander–Reiten conjecture does hold in the cases where the Auslander conjecture was shown to fail.

The rings constructed in [15] are standard graded artinian Koszul algebras and the modules are graded, with a linear resolution; in one of the examples the maximal ideal $\mathfrak{m}$ of R satisfies $\mathfrak{m}^3 = 0$ and in another example the ring is Gorenstein with $\mathfrak{m}^4 = 0$. Local rings $(R, \mathfrak{m}, k)$ with $\mathfrak{m}^3 = 0$ satisfy the Auslander–Reiten conjecture, cf. Huneke et al. [13, Theorem 4.1], but little is known when $\mathfrak{m}^4 = 0$.[1]

When R is a standard graded artinian algebra, we show that the Auslander–Reiten conjecture holds, under an additional assumption (weaker than linearity) on the module.

Theorem 1. *Let R be an artinian standard graded algebra and M a finitely generated graded R-module, generated in a single degree. If* $\mathrm{Ext}^i_R(M, M) = 0 = \mathrm{Ext}^i_R(M, R)$ *for all* $i > 0$*, then M is free.*

Our second result concerns Gorenstein rings. The hypothesis of the following theorem is known to be satisfied by generic standard graded Gorenstein algebras of socle degree 3, as a consequence of a result of Conca et al. [8, Claim 6.5].

E-mail address: segal@umkc.edu.

1 The paper *When are torsionless modules projective?* by R. Luo and Z. Huang, in Journal of Algebra (2008), claims to prove the Auslander–Reiten conjecture for all Gorenstein rings. However, the statement on the second line of page 2163 is not properly justified and it is not clear whether any justification can be given.

0022-4049/$ – see front matter
doi:10.1016/j.jpaa.2010.08.010

Theorem 2. *Let $(R, \mathfrak{m}, k)$ be a local Gorenstein ring with $\mathfrak{m}^4 = 0$ such that there exists a nonzero element $a \in \mathfrak{m}$ with $(0 :_R a)$ principal. Let M be a finitely generated R-module. Then the following are equivalent:*

(1) $\operatorname{Tor}_i^R(M, M) = 0$ *for all* $i \gg 0$*;*
(2) $\operatorname{Ext}_R^i(M, M) = 0$ *for all* $i > 0$*;*
(3) *M is free.*

The paper is organized as follows. Section 1 contains the proof of Theorem 1.

In Section 2 we study absolutely Koszul local rings, as defined by Iyengar and Römer [14], and we prove that asymptotic vanishing of homology over such rings translates into asymptotic vanishing of homology for the associated graded objects (with respect to the maximal ideal). Theorem 2.6 proves the implication (1) $\Longrightarrow$ (3) of Theorem 2 under weaker hypotheses and raises the question whether asymptotic vanishing of Tor provides, in general, a self-test for finite projective dimension.

In Section 3 we use the results of Henriques and Şega [11], which show that the hypothesis of Theorem 2 implies that R is absolutely Koszul, with balanced Hilbert series. We then use the results of Section 2, together with formulas for Hilbert series dictated by vanishing of (co)homology, as found in the work of Avramov et al. [4,5], in order to finish the proof of Theorem 2.

1. Graded algebras

In this section we present general considerations related to the Auslander–Reiten conjecture over commutative rings and then we consider the case of graded algebras.

Let R be a commutative noetherian ring. Given a finitely generated R-module M, we consider the following condition:

$$\mathcal{AR}(M): \quad \operatorname{Ext}_R^i(M, M) = 0 = \operatorname{Ext}_R^i(M, R) \quad \text{for all } i > 0.$$

As recalled in the Introduction, the commutative version of the Auslander–Reiten conjecture can be formulated as follows: if $\mathcal{AR}(M)$ holds, then M is projective.

Remark 1.1. Let M be a finitely generated R-module and N a syzygy of M in a free resolution of M over R. Note that $\operatorname{pd}_R(M) < \infty$ if and only if $\operatorname{pd}_R(N) < \infty$. Usual homological considerations show:

(1) If $\mathcal{AR}(M)$ holds, then so does $\mathcal{AR}(N)$.
Indeed, one may assume that N is a first syzygy and consider the exact sequence

$$0 \to N \to R^n \to M \to 0.$$

Applying $\operatorname{Hom}_R(-, R)$ and considering the induced long exact sequence, one obtains $\operatorname{Ext}_R^i(N, R) = 0$ for all $i > 0$. The long exact sequence induced by $\operatorname{Hom}_R(M, -)$ yields $\operatorname{Ext}_R^i(M, N) = 0$ for all $i > 1$ and then the long exact sequence induced by $\operatorname{Hom}_R(-, N)$ gives $\operatorname{Ext}_R^i(N, N) = 0$ for all $i > 0$.
(2) If R is self-injective and $\mathcal{AR}(N)$ holds, then so does $\mathcal{AR}(M)$.
To see this, reverse the argument above; the condition $\operatorname{Ext}_R^1(M, R) = 0$ is granted by the hypothesis on R.
(3) If R is self-injective, then $\mathcal{AR}(M)$ holds if and only if $\mathcal{AR}(M^*)$ holds, where M^* denotes the R-module $\operatorname{Hom}_R(M, R)$.
This statement follows from the sequence of isomorphisms below, obtained as a consequence of Hom-tensor adjointness and the hypothesis on R.

$$(\operatorname{Ext}_R^i(M, M))^* \cong \operatorname{Tor}_i^R(M, M^*) \cong \operatorname{Tor}_i^R(M^*, M) \cong (\operatorname{Ext}_R^i(M^*, M^*))^*.$$

We say that R is a *standard graded algebra* if it is graded, $k = R_0$ is a field, and R is generated (as an algebra) over k by finitely many elements of R_1. The Hilbert series of a finitely generated graded R-module $M = \oplus_{i \geqslant r} M_i$ is the formal Laurent series:

$$\operatorname{Hilb}_M(t) = \sum_{i=r}^{\infty} (\operatorname{rank}_k M_i) t^i \in \mathbb{Z}((t)).$$

For every $d \in \mathbb{Z}$, let $M(d)$ denote the graded R-module with $M(d)_j = M_{j+d}$ for each j.

1.2. *Ext vanishing and Hilbert series.* We recall a result of Avramov et al. [5, Theorem 1]: if R is a standard graded algebra and M, N are finitely generated graded R-modules such that $\operatorname{Ext}_R^i(M, N) = 0$ for $i \gg 0$, then there is an equality of rational functions:

$$\sum_i (-1)^i \operatorname{Hilb}_{\operatorname{Ext}_R^i(M,N)}(t) = \frac{\operatorname{Hilb}_M(t^{-1}) \operatorname{Hilb}_N(t)}{\operatorname{Hilb}_R(t^{-1})}. \tag{1.2.1}$$

Proof of Theorem 1. We have $R = \oplus_{i=0}^{s} R_i$, with $R_s \neq 0$ and $R_0 = k$, a field. Assume that M is generated in degree r. By replacing M with $M(r)$ we may assume that M is generated by M_0. We then have $M_i = 0$ for all $i > s$. Let N be the first syzygy in a minimal graded free resolution of M over R:

$$0 \to N \to R^p \to M \to 0,$$

where $p = \operatorname{rank}_k M_0$. Note that $N_i = 0$ for $i \leq 0$ and $i > s$. By Remark 1.1(1), $\mathcal{AR}(N)$ holds. Assume that $N \neq 0$, hence $s > 0$.

If $s = 1$, then N is a k-vector space concentrated in degree 1. The fact that $\operatorname{Ext}_R^i(N, N) = 0$ for all $i > 0$ implies that R is regular, hence $s = 0$, a contradiction.

Now assume that $s > 1$. Set

$$\operatorname{Hilb}_N(t) = n_1 t + n_2 t^2 + \cdots + n_s t^s \quad \text{and} \quad \operatorname{Hilb}_R(t) = 1 + r_1 t + \cdots + r_s t^s.$$

Then formula (1.2.1) gives:

$$\operatorname{Hilb}_{\operatorname{Hom}_R(N,N)}(t) = \frac{(n_1 t^{-1} + n_2 t^{-2} + \cdots + n_s t^{-s})(n_1 t + n_2 t^2 + \cdots + n_s t^s)}{1 + r_1 t^{-1} + \cdots + r_s t^{-s}}.$$

Clearing denominators, we have:

$$(t^s + r_1 t^{s-1} + \cdots + r_s) \operatorname{Hilb}_{\operatorname{Hom}_R(N,N)}(t) = (n_1 t^{s-1} + n_2 t^{s-2} + \cdots + n_s)(n_1 t + n_2 t^2 + \cdots + n_s t^s). \tag{1.2.2}$$

Let x denote the coefficient in degree 0 of $\operatorname{Hilb}_{\operatorname{Hom}_R(N,N)}(t)$. The coefficient in degree 0 of the left hand side of (1.2.2) is at least $r_s x$, while the right hand side of the equation has no term in degree zero. Since $r_s \neq 0$, it follows that $x = 0$. Noting that the identity is a degree zero element in $\operatorname{Hom}_R(N, N)$, we obtain a contradiction.

We conclude that $N = 0$, hence M is free. □

2. Absolutely Koszul local rings

In the remainder of this paper we assume that $(R, \mathfrak{m}, k)$ is a local ring; the notation identifies $\mathfrak{m}$ as the maximal ideal and k as the residue field. If M is a finitely generated R-module, then the ith *Betti number* of M is defined to be

$$\beta_i^R(M) = \operatorname{rank}_k \operatorname{Tor}_i^R(M, k).$$

The *Poincaré series* of M over R is the formal power series

$$\mathrm{P}_M^R(t) = \sum_{i \in \mathbb{N}} \beta_i^R(M) t^i \in \mathbb{Z}[\![t]\!].$$

We adopt the following notation for the associated graded objects over R: we let R^{g} denote the associated graded ring and M^{g} denote the associated graded module of an R-module M; that is,

$$R^{\mathrm{g}} = \bigoplus_{i \geq 0} \mathfrak{m}^i/\mathfrak{m}^{i+1} \quad \text{and} \quad M^{\mathrm{g}} = \bigoplus_{i \geq 0} \mathfrak{m}^i M/\mathfrak{m}^{i+1} M.$$

Remark 2.1. If M is a finitely generated R-module and N is a syzygy in a minimal free resolution of M over R, then $N \subseteq \mathfrak{m} R^n$ for some n. In particular, $(0 :_R \mathfrak{m})N = 0$. If R is artinian, we conclude that $\deg \operatorname{Hilb}_{N^{\mathrm{g}}}(t) < \deg \operatorname{Hilb}_{R^{\mathrm{g}}}(t)$.

2.2. *Linearity defect and Koszul modules.* Given a minimal free resolution F of a finitely generated R-module M

$$F = \cdots \to F_{n+1} \to F_n \to \cdots \to F_0 \to M \to 0$$

one constructs the complex

$$\operatorname{lin}^R(F) = \cdots \to {F_{n+1}}^{\mathrm{g}}(-n-1) \to {F_n}^{\mathrm{g}}(-n) \to \cdots \to {F_0}^{\mathrm{g}} \to M^{\mathrm{g}} \to 0$$

with differentials induced from F. Herzog and Iyengar [12, 1.7] defined the *linearity defect* of M to be the number:

$$\operatorname{ld}_R(M) = \sup\{i \in \mathbb{Z} \mid \mathrm{H}_i(\operatorname{lin}^R(F)) \neq 0\}.$$

A module M is said to be *Koszul* if $\operatorname{ld}_R(M) = 0$. A local ring R is said to be *Koszul* if its residue field k is Koszul as a module over R.

From [12, 1.5] the following statements are equivalent for a finitely generated R-module M:

(1) M is a Koszul R-module;
(2) $\operatorname{lin}^R(F)$ is a minimal graded free resolution of M^{g} over R^{g}.

As a consequence of this equivalence, we have: If M is Koszul, and N is a syzygy of M in a minimal free resolution F, then N is Koszul and N^{g} is a syzygy of M^{g} in the minimal graded free resolution $\operatorname{lin}^R(F)$.

As proved in [12, Prop. 1.8], the Poincaré series of a Koszul R-module M satisfies

$$\mathrm{P}_M^R(t) = \mathrm{P}_{M^{\mathrm{g}}}^{R^{\mathrm{g}}}(t) = \operatorname{Hilb}_M(-t) \operatorname{Hilb}_R(-t)^{-1}. \tag{2.2.1}$$

Following Iyengar and Römer [14, 2.10], we say that the ring R is *absolutely Koszul* if it satisfies the property that every finitely generated R-module has finite linearity defect; equivalently, has a Koszul syzygy.

Proposition 2.3. *Let $(R, \mathfrak{m}, k)$ be a local ring and M, N be finitely generated Koszul R-modules. Assume that one of the following hypotheses is satisfied:*

(1) $\mathrm{Tor}_i^R(M, N) = 0$ *for all* $i > 0$ *and* $\mathrm{ld}_R(M \otimes_R N) < \infty$*;*
(2) $\mathrm{Tor}_i^R(M, N) = 0$ *for all* $i \gg 0$ *and R is absolutely Koszul.*

Then $\mathrm{Tor}_i^{R^{\mathrm{g}}}(M^{\mathrm{g}}, N^{\mathrm{g}}) = 0$ *for all* $i \gg 0$.

Proof. To simplify notation, given a minimal free resolution P, we set $P^{\mathrm{g}} = \mathrm{lin}^R(P)$. Let F, respectively G, be a minimal free resolution of M, respectively N. Then F^{g} is a minimal graded free resolution of M^{g} over R^{g} and G^{g} is a minimal graded free resolution of N^{g} over R^{g}.

Assuming (1), we have $\mathrm{Tor}_i^R(M, N) = 0$ for all $i > 0$, hence $F \otimes_R G$ is a minimal free resolution of $M \otimes_R N$. Since $\mathrm{ld}_R(M \otimes_R N) < \infty$, we conclude that $\mathrm{H}_i((F \otimes_R G)^{\mathrm{g}}) = 0$ for $i \gg 0$. By [14, Lemma 2.7], we know that $F^{\mathrm{g}} \otimes_{R^{\mathrm{g}}} G^{\mathrm{g}} \cong (F \otimes_R G)^{\mathrm{g}}$, hence we have $\mathrm{H}_i(F^{\mathrm{g}} \otimes_{R^{\mathrm{g}}} G^{\mathrm{g}}) = 0$ for $i \gg 0$, implying the desired conclusion.

Assuming (2), let S denote a syzygy in a minimal free resolution of M over R such that $\mathrm{Tor}_i^R(S, N) = 0$ for all $i > 0$. Note that the pair of modules S, N satisfies hypothesis (1), and thus $\mathrm{Tor}_i^{R^{\mathrm{g}}}(S^{\mathrm{g}}, N^{\mathrm{g}}) = 0$ for $i \gg 0$. Since M is Koszul, we have that S^{g} is a syzygy of M^{g} (see 2.2) and hence $\mathrm{Tor}_i^{R^{\mathrm{g}}}(M^{\mathrm{g}}, N^{\mathrm{g}}) = 0$ for $i \gg 0$. □

Remark 2.4. Let A be a standard graded k-algebra and let S, T be finitely generated graded A-modules. By Avramov and Buchweitz [4, Lemma 7(ii)], one has the following equality of formal Laurent series:

$$\frac{\mathrm{Hilb}_S(t) \cdot \mathrm{Hilb}_T(t)}{\mathrm{Hilb}_A(t)} = \sum_i (-1)^i \, \mathrm{Hilb}_{\mathrm{Tor}_i^A(S,T)}(t). \tag{2.4.1}$$

When A is artinian and $\mathrm{Tor}_i^A(S, T) = 0$ for all $i > 0$ we obtain the formula:

$$\mathrm{rank}_k(S \otimes_A T) = \frac{\mathrm{rank}_k(S)\,\mathrm{rank}_k(T)}{\mathrm{rank}_k(A)}. \tag{2.4.2}$$

If M is an R-module, $\lambda(M)$ denotes its length.

Proposition 2.5. *Let R be an absolutely Koszul artinian local ring and M, N be two finitely generated R-modules such that* $\mathrm{Tor}_i^R(M, N) = 0$ *for all* $i > 0$*. The following formula then holds:*

$$\lambda(M \otimes_R N) = \frac{\lambda(M)\lambda(N)}{\lambda(R)}. \tag{2.5.1}$$

Proof. Let Z be a first syzygy of N, so that there is an exact sequence

$$0 \to Z \to R^n \to N \to 0.$$

Then $\mathrm{Tor}_i^R(M, Z) = 0$ for all $i > 0$ and there is an exact sequence:

$$0 \to M \otimes_R Z \to M^n \to M \otimes_R N \to 0.$$

A length count in these sequences gives:

$$\lambda(Z) = n\lambda(R) - \lambda(N)$$
$$\lambda(M \otimes_R Z) = n\lambda(M) - \lambda(M \otimes_R N).$$

Based on these two formulas, note that (2.5.1) holds if and only if

$$\lambda(M \otimes_R Z) = \frac{\lambda(M)\lambda(Z)}{\lambda(R)}.$$

We may thus replace both M and N with a syzygy. Since R is absolutely Koszul, we may assume that both M and N are Koszul. Then Proposition 2.3 gives $\mathrm{Tor}_i^{R^{\mathrm{g}}}(M^{\mathrm{g}}, N^{\mathrm{g}}) = 0$ for $i \gg 0$. Recall that if S is a syzygy of M, then S^{g} is a syzygy of M^{g}. Further replacing M with a syzygy, we may assume thus that $\mathrm{Tor}_i^{R^{\mathrm{g}}}(M^{\mathrm{g}}, N^{\mathrm{g}}) = 0$ for all $i > 0$ and then the desired formula is given by (2.4.2). □

The next theorem indicates a possible counterpart of the Auslander–Reiten conjecture, in terms of the vanishing of Tor. The author does not know any examples of finitely generated R-modules M for which $\mathrm{Tor}_i^R(M, M) = 0$ for all $i \gg 0$ and M does not have finite projective dimension.

Theorem 2.6. *Let $(R, \mathfrak{m}, k)$ be an absolutely Koszul artinian local ring such that the polynomial* $\mathrm{Hilb}_{R^{\mathrm{g}}}(t)$ *is square free in* $\mathbb{Z}[t]$*.*

If M is a finitely generated R-module such that $\mathrm{Tor}_i^R(M, M) = 0$ *for all* $i \gg 0$*, then M is free.*

In particular, under the hypotheses of the theorem, we obtain a proof for the commutative version of a conjecture of Tachikawa considered by Avramov et al. [5]:

Corollary 2.7. *If D is a dualizing module for R and* $\mathrm{Ext}_R^i(D, R) = 0$ *for all* $i \gg 0$*, then R is Gorenstein.*

Proof. Note that $\mathrm{Ext}_R^i(D, R) = 0$ for all $i \gg 0$ implies $\mathrm{Tor}_i^R(D, D) = 0$ for all $i \gg 0$. Also, D is free if and only if R is Gorenstein. □

Proof of Theorem 2.6. Since R is absolutely Koszul, M has a Koszul syzygy. Using an argument similar to that of Remark 1.1(1), we replace M with a Koszul syzygy in a minimal free resolution.

Proposition 2.3(2) yields $\mathrm{Tor}_i^{R^{\mathrm{g}}}(M^{\mathrm{g}}, M^{\mathrm{g}}) = 0$ for $i \gg 0$. Then Remark 2.4 shows that the expression

$$\frac{\mathrm{Hilb}_{M^{\mathrm{g}}}(t) \cdot \mathrm{Hilb}_{M^{\mathrm{g}}}(t)}{\mathrm{Hilb}_{R^{\mathrm{g}}}(t)}$$

is a polynomial. In consequence, $\mathrm{Hilb}_{R^{\mathrm{g}}}(t)$ divides $(\mathrm{Hilb}_{M^{\mathrm{g}}}(t))^2$. Since $\mathrm{Hilb}_{R^{\mathrm{g}}}(t)$ is square free, it follows that $\mathrm{Hilb}_{R^{\mathrm{g}}}(t)$ divides $\mathrm{Hilb}_{M^{\mathrm{g}}}(t)$. If $M \neq 0$, we conclude that $\deg \mathrm{Hilb}_{M^{\mathrm{g}}}(t) \geq \deg \mathrm{Hilb}_{R^{\mathrm{g}}}(t)$. This inequality contradicts Remark 2.1, hence $M = 0$. □

3. Small Gorenstein local rings

In this section we prove the Auslander–Reiten conjecture for a large class of Gorenstein local rings $(R, \mathfrak{m}, k)$ with $\mathfrak{m}^4 = 0$.

3.1. *Exact zero divisors.* In [11], Henriques and Şega defined an *exact zero divisor* on R to be an element $a \in R$ such that $0 \neq (0 \colon a) \cong R/aR \neq 0$. Elias and Rossi [9, Theorem 3.3] show that, if R is Gorenstein, k is algebraically closed of characteristic zero and $\mathrm{Hilb}_{R^{\mathrm{g}}}(t) = 1 + et + et^2 + t^3$, then R is *canonically graded*, that is, $R \cong R^{\mathrm{g}}$. Furthermore, as described in [11, 3.5, 4.3], results of Conca et al. [8, Claim 6.5] show that generic artinian Gorenstein standard graded algebras of socle degree 3 admit an exact zero divisor.

When R is Gorenstein, note that a nonzero element $a \in \mathfrak{m}$ is an exact zero divisor if and only if the ideal $(0 :_R a)$ is principal; see [11, 4.1].

Remark 3.2. In [16, Theorem 2.3], the author shows that if R is a local Gorenstein ring such that $\mathrm{rank}_k\, \mathfrak{m}/\mathfrak{m}^2 \leq 3$ and M, N are finitely generated R-modules with $\mathrm{Tor}_i^R(M, N) = 0$ or $\mathrm{Ext}_R^i(M, N) = 0$ for $i \gg 0$, then M or N has finite projective dimension.

We now consider Theorem 2 in the introduction. For convenience, we rewrite its statement below.

Theorem 3.3. *Let* $(R, \mathfrak{m}, k)$ *be a local Gorenstein ring with* $\mathfrak{m}^4 = 0$ *such that there exists a nonzero element* $a \in \mathfrak{m}$ *with* $(0 :_R a)$ *principal. Let M be a finitely generated R-module. Then the following are equivalent:*

(1) $\mathrm{Tor}_i^R(M, M) = 0$ *for all* $i \gg 0$*;*
(2) $\mathrm{Ext}_R^i(M, M) = 0$ *for all* $i > 0$*;*
(3) *M is free.*

Remark 3.4. A result of Avramov and Buchweitz [3, Theorem III] gives that the asymptotic vanishing of $\mathrm{Ext}_R^i(M, N)$ is equivalent to the asymptotic vanishing of $\mathrm{Tor}_i^R(M, N)$ whenever R is a complete intersection and M, N are finitely generated R-modules. This remarkable property does not extend to all Gorenstein local rings; counterexamples (with $M \ncong N$) have been found among rings satisfying the hypothesis of Theorem 3.3, cf. [15, Prop. 3.9].

Proof of Theorem 3.3. Obviously, (3) implies both (1) and (2).

By [11, Theorem 4.2, Corollary 4.4], one has $\mathrm{Hilb}_{R^{\mathrm{g}}}(t) = 1 + et + et^2 + t^3$ and R is absolutely Koszul. Using Remark 3.2, we may assume that $e \geq 4$. Note that

$$1 + et + et^2 + t^3 = (1 + t)(1 + (e - 1)t + t^2)$$

and both factors are irreducible. In particular, $\mathrm{Hilb}_{R^{\mathrm{g}}}(t)$ is square free.

(1) $\Longrightarrow$ (3): The statement follows immediately from Theorem 2.6.

(2) $\Longrightarrow$ (3): Since R is artinian and Gorenstein, thus self-injective, condition (2) holds if and only if $\mathcal{AR}(M)$ holds. Remark 1.1 gives that $\mathcal{AR}(M)$ holds if and only if $\mathcal{AR}(N)$ holds, where N is a syzygy of M or M^*.

Assume that M satisfies (2). Since M is free if and only if N (as above) is free, we may replace M with a syzygy of M or M^* whenever needed for the proof.

We first replace M with a Koszul syzygy in a minimal free resolution of M. Also, let N denote a Koszul syzygy in a minimal free resolution of M^*. Note that condition (2) implies that $\mathrm{Tor}_i^R(M, M^*) = 0$ for all $i > 0$ (see Remark 1.1(3)), and consequently $\mathrm{Tor}_i^R(M, N) = 0$ for all $i > 0$.

Since both M and N are Koszul, Proposition 2.3(2) yields $\mathrm{Tor}_i^{R^{\mathrm{g}}}(M^{\mathrm{g}}, N^{\mathrm{g}}) = 0$ for $i \gg 0$. Thus Remark 2.4 shows that the expression

$$\frac{\mathrm{Hilb}_{M^{\mathrm{g}}}(t) \cdot \mathrm{Hilb}_{N^{\mathrm{g}}}(t)}{\mathrm{Hilb}_{R^{\mathrm{g}}}(t)}$$

is a polynomial. In consequence, the polynomial $1 + (e-1)t + t^2$ divides the product $\mathrm{Hilb}_{M^{\mathrm{g}}}(t) \cdot \mathrm{Hilb}_{N^{\mathrm{g}}}(t)$. Since both M and N are syzygies in a minimal free resolution, their Hilbert series have degree at most 2 by Remark 2.1. One of $\mathrm{Hilb}_{M^{\mathrm{g}}}(t)$ or $\mathrm{Hilb}_{N^{\mathrm{g}}}(t)$ is thus divisible by the polynomial $1 + (e-1)t + t^2$, and hence it is a scalar multiple of it.

Recalling that we may replace M with N, we may assume that

$$\mathrm{Hilb}_{M^{\mathrm{g}}}(t) = c(1 + (e-1)t + t^2) \tag{3.4.1}$$

for some c, hence the length of M is $\lambda(M) = c(e+1)$. Since $\mathrm{Tor}_i^R(M, M^*) = 0$ for all $i > 0$, we apply formula (2.5.1):

$$\lambda(M \otimes_R M^*) = \frac{\lambda(M)\lambda(M^*)}{\lambda(R)} = \frac{\lambda(M)^2}{\lambda(R)} = \frac{c^2(e+1)^2}{2(e+1)} = \frac{c^2(e+1)}{2}.$$

Since $M \otimes_R M^*$ and $\mathrm{Hom}_R(M, M)$ are Matlis dual, they have equal length, hence

$$\lambda(\mathrm{Hom}_R(M, M)) = \frac{c^2(e+1)}{2}. \tag{3.4.2}$$

In view of (3.4.1), formula (2.2.1) gives that M has Betti numbers equal to c, hence M has a minimal free resolution:

$$\cdots \to R^c \to R^c \to M \to 0.$$

Applying $\mathrm{Hom}_R(-, M)$ to the sequence above, we obtain an exact sequence

$$0 \to \mathrm{Hom}_R(M, M) \to M^c \xrightarrow{\varphi} M^c.$$

A length count, using (3.4.2), gives:

$$\lambda(\mathrm{Im}\,\varphi) = \lambda(M^c) - \lambda(\mathrm{Hom}_R(M, M)) = c^2(e+1) - \frac{c^2(e+1)}{2} = \frac{c^2(e+1)}{2}. \tag{3.4.3}$$

Note that $\varphi(M^c) \subseteq \mathfrak{m}M^c$ (because the resolution of M above is minimal), hence, restricting the target of φ to $\mathfrak{m}M^c$, Lemma 3.5 below gives:

$$\lambda(\mathrm{Im}\,\varphi) \le \lambda(\mathfrak{m}^2 M^c) + \lambda(M^c/\mathfrak{m}M^c).$$

Using (3.4.1), we obtain $\lambda(\mathfrak{m}^2 M^c) = c^2 = \lambda(M^c/\mathfrak{m}M^c)$, thus:

$$\lambda(\mathrm{Im}\,\varphi) \le 2c^2. \tag{3.4.4}$$

Recall that $e \ge 4$. Formulas (3.4.3) and (3.4.4) then yield

$$\frac{5}{2}c^2 \le \frac{c^2(e+1)}{2} \le 2c^2$$

hence $c = 0$ and thus $M = 0$. □

Lemma 3.5. *Let $(R, \mathfrak{m}, k)$ be a local ring. If $\varphi\colon M \to N$ is a homomorphism of R-modules of finite length, then:*

$$\lambda(\mathrm{Im}\varphi) \le \lambda(\mathfrak{m}N) + \lambda(M/\mathfrak{m}M).$$

Proof. Set $W = \mathrm{Im}\,\varphi$. We have:

$$\begin{aligned} \lambda(W) &= \lambda(\mathfrak{m}W) + \lambda(W/\mathfrak{m}W) \\ &\le \lambda(\mathfrak{m}N) + \lambda(M/\mathfrak{m}M) \end{aligned}$$

where the inequality is due to the inclusion $\mathfrak{m}W \subseteq \mathfrak{m}N$ and the fact that the minimal number of generators of W is less than or equal to the minimal number of generators of M. □

Acknowledgements

The author would like to thank Dan Zacharia for the encouragement to pursue this topic, and Luchezar Avramov, Srikanth Iyengar and the referee for suggestions regarding the exposition.

References

[1] M. Auslander, S. Ding, Ø. Solberg, Liftings and weak liftings of modules, J. Algebra 156 (1993) 273–317.
[2] M. Auslander, I. Reiten, On a generalized version of the Nakayama conjecture, Proc. Amer. Math. Soc. 52 (1975) 69–74.
[3] L.L. Avramov, R.-O. Buchweitz, Support varieties and cohomology over complete intersections, Invent. Math. 142 (2000) 285–318.
[4] L.L. Avramov, R.-O. Buchweitz, Lower bounds for Betti numbers, Compos. Math. 86 (1993) 147–158.
[5] L.L. Avramov, R.-O. Buchweitz, J.D. Sally, Laurent coefficients and Ext of finite graded modules, Math. Ann. 307 (1997) 401–415.
[6] L.L. Avramov, S. Iyengar, L.M. Şega, Free resolutions over short local rings, J. Lond. Math. Soc. 78 (2008) 459–476.
[7] L.W. Christensen, H. Holm, Algebras that satisfy Auslander's condition on vanishing of cohomology, Math. Z. 265 (2010) 21–40.
[8] A. Conca, M.E. Rossi, G. Valla, Gröbner flags and Gorenstein algebras, Compos. Math. 129 (2001) 95–121.
[9] J. Elias, M.E. Rossi, Isomorphism classes of short Gorenstein local rings via Macaulay's inverse system, Preprint, arXiv:0911.3565.
[10] D. Happel, Homological conjectures in representation theory of finite-dimensional algebras, Sherbrook Lecture Notes Series (1991), available from http://www.math.ntnu.no/~oyvinso/Nordfjordeid/Program/references.html.
[11] I.B. Henriques, L.M. Şega, Free resolutions over short Gorenstein local rings, Math. Z., in press (doi:10.1007/s00209-009-0639-z).
[12] J. Herzog, S. Iyengar, Koszul modules, J. Pure Appl. Algebra 201 (2005) 154–188.
[13] C. Huneke, L.M. Şega, A.N. Vraciu, Vanishing of Ext and Tor over some Cohen–Macaulay local rings, Illinois J. Math. 48 (2004) 295–317.
[14] S. Iyengar, T. Römer, Linearity defects of modules over commutative rings, J. Algebra 322 (2009) 3212–3237.
[15] D.A. Jorgensen, L.M. Şega, Nonvanishing cohomology and classes of Gorenstein rings, Adv. Math. 188 (2004) 470–490.
[16] L.M. Şega, Homological properties of powers of the maximal ideal of a local ring, J. Algebra 241 (2001) 827–858.