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Third homology of general linear groups over rings with many units

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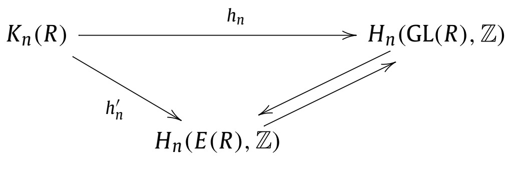
ABSTRACT

For a commutative ring R with many units, we describe the kernel of $H_3(\text{inc}) : H_3(\text{GL}_2(R), \mathbb{Z}) \to H_3(\text{GL}_3(R), \mathbb{Z})$. Moreover we show that the elements of this kernel are of order at most two. As an application we study the indecomposable part of $K_3(R)$.

Introduction

Interest in the study of the homology of general linear groups has arose mostly because of their close connection with the K-theory of rings. For any ring R and any positive integer n, there are natural homomorphisms



where $E(R)$ is the elementary subgroup of the stable general linear group $\text{GL}(R)$ and h_n and h'_n ($n \geqslant 2$ for h'_n) are the Hurewicz maps coming from algebraic topology [10, Chap. 2].

It is known that $K_1(R) \overset{h_1}{\simeq} H_1(\text{GL}(R), \mathbb{Z})$, $K_2(R) \overset{h'_2}{\simeq} H_2(E(R), \mathbb{Z})$ [10, Chap. 2]. The homomorphism $h'_3 : K_3(R) \to H_3(E(R), \mathbb{Z})$ is surjective with 2-torsion kernel [12, Corollary 5.2], [9, Proposition 2.5].

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Homological stability type theorems, are very powerful tools for the study of K-theory of rings. Suslin has proved that for an infinite field F, we have the homological stability

$$H_n\big(\mathrm{GL}_n(F),\mathbb{Z}\big) \xrightarrow{\sim} H_n\big(\mathrm{GL}_{n+1}(F),\mathbb{Z}\big) \xrightarrow{\sim} H_n\big(\mathrm{GL}_{n+2}(F),\mathbb{Z}\big) \xrightarrow{\sim} \cdots,$$

and used this to prove many interesting results [11]. For example he showed that we have an exact sequence

$$H_n\big(\mathrm{GL}_{n-1}(F),\mathbb{Z}\big) \xrightarrow{H_n(\mathrm{inc})} H_n\big(\mathrm{GL}_n(F),\mathbb{Z}\big) \longrightarrow K_n^M(F) \longrightarrow 0.$$

Suslin has conjectured that the kernel of

$$H_n\big(\mathrm{GL}_{n-1}(F),\mathbb{Z}\big) \longrightarrow H_n\big(\mathrm{GL}_n(F),\mathbb{Z}\big)$$

is a torsion group [9, Problem 4.13]. These results can be generalized over rings with many units [4], e.g. semilocal rings with infinite residue fields. Also Suslin's conjecture can be asked in this more general setting [7]. A positive answer to this conjecture is known only for $n \leqslant 4$ [3,8,7].

It was known that when F is an infinite field, the kernel of the homomorphism $H_3(\mathrm{GL}_2(F),\mathbb{Z}) \to H_3(\mathrm{GL}_3(F),\mathbb{Z})$ is a 2-power torsion group [8]. In this article we generalize this to all commutative rings with many units. In fact we do more. Here we describe the kernel of

$$H_3(\mathrm{inc}) : H_3\big(\mathrm{GL}_2(R),\mathbb{Z}\big) \longrightarrow H_3\big(\mathrm{GL}_3(R),\mathbb{Z}\big),$$

where R is a commutative ring with many units. Our main theorem claims that the elements of $\ker(H_3(\mathrm{inc}))$ are of the form

$$\sum \mathbf{c}\big(\mathrm{diag}(a,1),\mathrm{diag}(1,b),\mathrm{diag}\big(c,c^{-1}\big)\big)$$

provided that

$$\sum a\otimes\{b,c\} + b\otimes\{a,c\} = 0 \in R^*\otimes_{\mathbb{Z}} K_2^M(R).$$

Moreover by an easy argument we will show that $\ker(H_3(\mathrm{inc}))$ is a 2-torsion group. It is highly expected that this kernel should be trivial, at least when R is a field [5, Section 5].

It is known that, the map $H_3(\mathrm{inc})$ is closely related to the indecomposable part of $K_3(R)$, i.e. $K_3(R)^{\mathrm{ind}} := K_3(R)/K_3^M(R)$ [8,5]. As an application of our main theorem we show that

$$K_3(R)^{\mathrm{ind}} \otimes_{\mathbb{Z}} \mathbb{Z}[1/2] \simeq H_0\big(R^*, H_3\big(\mathrm{SL}_2(R),\mathbb{Z}[1/2]\big)\big).$$

If $R^* = R^{*2}$, then we get the isomorphism

$$K_3(R)^{\mathrm{ind}} \simeq H_3\big(\mathrm{SL}_2(R),\mathbb{Z}\big).$$

Previously these results were known only for infinite fields [8].

Notation

In this article by $H_i(G)$ we mean the homology of group G with integral coefficients, namely $H_i(G,\mathbb{Z})$. By GL_n (resp. SL_n) we mean the general (resp. special) linear group $\mathrm{GL}_n(R)$ (resp. $\mathrm{SL}_n(R)$), where R is a commutative ring with 1. If $A \to A'$ is a homomorphism of abelian groups, by A'/A we mean $\mathrm{coker}(A \to A')$ and we take other liberties of this kind. For a group A, by $A_{\mathbb{Z}[1/2]}$ we mean $A \otimes_{\mathbb{Z}} \mathbb{Z}[1/2]$.

1. Third homology of product of two abelian groups

In this section we will study the homology group $H_3(A \times B)$, where A and B are abelian groups.

First we assume $A = B = \mathbb{Z}/n$. By applying the Künneth formula [13, Proposition 6.1.13] to $H_3(\mathbb{Z}/n \times \mathbb{Z}/n)$ and using the calculation of the homology of finite cyclic groups [13, Theorem 6.2.2, Example 6.2.3], we obtain the exact sequence

$$0 \longrightarrow H_3(\mathbb{Z}/n) \oplus H_3(\mathbb{Z}/n) \longrightarrow H_3(\mathbb{Z}/n \times \mathbb{Z}/n) \longrightarrow \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Z}/n) \longrightarrow 0.$$

If $p_i : \mathbb{Z}/n \times \mathbb{Z}/n \to \mathbb{Z}/n$, $i = 1, 2$, is projection on the i-th factor, then

$$(p_{1*}, p_{2*}) : H_3(\mathbb{Z}/n \times \mathbb{Z}/n) \longrightarrow H_3(\mathbb{Z}/n) \oplus H_3(\mathbb{Z}/n)$$

splits the above exact sequence. Thus we obtain a canonical splitting map

$$\theta_{n,n} : \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Z}/n) \longrightarrow H_3(\mathbb{Z}/n \times \mathbb{Z}/n).$$

If $\langle \bar{1}, n, \bar{1} \rangle$ is the image of $\bar{1} \in \mathbb{Z}/n$ under the isomorphism

$$\mathbb{Z}/n \xrightarrow{\simeq} \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Z}/n),$$

then one can show that $\theta_{n,n}(\langle \bar{1}, n, \bar{1} \rangle) = \chi_{n,n}$, where

$$\chi_{n,n} := \sum_{i=1}^{n} \Big(\big[(\bar{1},0)|(0,\bar{1})|(0,\bar{i})\big] - \big[(0,\bar{1})|(\bar{1},0)|(0,\bar{i})\big] + \big[(0,\bar{1})|(0,\bar{i})|(\bar{1},0)\big] + \big[(\bar{1},0)|(\bar{i},0)|(0,\bar{1})\big] - \big[(\bar{1},0)|(0,\bar{1})|(\bar{i},0)\big] + \big[(0,\bar{1})|(\bar{1},0)|(\bar{i},0)\big] \Big)$$

[6, Chap. V, Proposition 10.6], [8, Proposition 4.1]. If $A = \mathbb{Z}/m$ and $B = \mathbb{Z}/n$, then the same approach shows that the exact sequence

$$0 \longrightarrow H_3(\mathbb{Z}/m) \oplus H_3(\mathbb{Z}/n) \longrightarrow H_3(\mathbb{Z}/m \times \mathbb{Z}/n) \longrightarrow \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/m, \mathbb{Z}/n) \longrightarrow 0$$

splits canonically. The splitting map

$$\theta_{m,n} : \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/m, \mathbb{Z}/n) \longrightarrow H_3(\mathbb{Z}/m \times \mathbb{Z}/n)$$

can be computed similar to $\theta_{n,n}$. In fact if $\langle \overline{m/d}, d, \overline{n/d} \rangle$ is the image of $\bar{1} \in \mathbb{Z}/(m,n)$ under the isomorphism $\mathbb{Z}/(m,n) \simeq \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/m, \mathbb{Z}/n)$, then $\theta_{m,n}(\langle \overline{m/d}, d, \overline{n/d} \rangle) = \chi_{m,n}$, where

$$\begin{aligned}\chi_{n,m} := \sum_{i=1}^{n} \Bigg(&\left[\left(\frac{\overline{m}}{d},0\right)\middle|\left(0,\frac{\overline{n}}{d}\right)\middle|\left(0,\frac{\overline{in}}{d}\right)\right] - \left[\left(0,\frac{\overline{n}}{d}\right)\middle|\left(\frac{\overline{m}}{d},0\right)\middle|\left(0,\frac{\overline{in}}{d}\right)\right] \\ &+ \left[\left(0,\frac{\overline{n}}{d}\right)\middle|\left(0,\frac{\overline{in}}{d}\right)\middle|\left(\frac{\overline{m}}{d},0\right)\right] + \left[\left(\frac{\overline{m}}{d},0\right)\middle|\left(\frac{\overline{im}}{d},0\right)\middle|\left(0,\frac{\overline{n}}{d}\right)\right] \\ &- \left[\left(\frac{\overline{m}}{d},0\right)\middle|\left(0,\frac{\overline{n}}{d}\right)\middle|\left(\frac{\overline{im}}{d},0\right)\right] + \left[\left(0,\frac{\overline{n}}{d}\right)\middle|\left(\frac{\overline{m}}{d},0\right)\middle|\left(\frac{\overline{im}}{d},0\right)\right] \Bigg).\end{aligned}$$

In the next proposition we extend these results to all abelian groups.

Proposition 1.1. *Let A and B be abelian groups. Then we have the canonical decomposition*

$$H_3(A \times B) = \bigoplus_{i+j=3} H_i(A) \otimes H_j(B) \oplus \mathrm{Tor}_1^{\mathbb{Z}}(A, B).$$

Proof. By the Künneth formula we have the exact sequence

$$0 \longrightarrow \bigoplus_{i+j=3} H_i(A) \otimes H_j(B) \longrightarrow H_3(A \times B) \longrightarrow \mathrm{Tor}_1^{\mathbb{Z}}(A, B) \longrightarrow 0.$$

We will construct a canonical splitting map

$$\mathrm{Tor}_1^{\mathbb{Z}}(A, B) \longrightarrow H_3(A \times B).$$

It is known that direct limit with directed set index, is an exact functor and it commutes with the homology group [2, Chap. V, Section 5, Exercise 3] and the functor Tor [13, Corollary 2.6.17]. Since any abelian group can be written as direct limit of its finitely generated subgroups, we may assume that A and B are finitely generated abelian groups. On the other hand,

$$\mathrm{Tor}_1^{\mathbb{Z}}(A, B) \simeq \mathrm{Tor}_1^{\mathbb{Z}}(A_{tor}, B_{tor}),$$

where A_{tor} and B_{tor} are the torsion subgroups of A and B respectively. So we may even assume that A and B are finite abelian groups. Let

$$A = \mathbb{Z}/m_1 \times \cdots \times \mathbb{Z}/m_r, \qquad B = \mathbb{Z}/n_1 \times \cdots \times \mathbb{Z}/n_s.$$

Now consider the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H_3(\mathbb{Z}/m_i) \oplus H_3(\mathbb{Z}/n_j) & \longrightarrow & H_3(\mathbb{Z}/m_i \times \mathbb{Z}/n_j) & \longrightarrow & \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/m_i, \mathbb{Z}/n_j) & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow \scriptstyle{\mathrm{inc}_{m_i,n_j}} & & \big\downarrow \scriptstyle{\mathrm{inc}} & & \\
0 & \longrightarrow & \bigoplus_{i+j=3} H_i(A) \otimes H_j(B) & \longrightarrow & H_3(A \times B) & \longrightarrow & \mathrm{Tor}_1^{\mathbb{Z}}(A, B) & \longrightarrow & 0
\end{array}$$

We have seen that the first row of this diagram splits by the canonical map θ_{m_i,n_j}. Thus the composition

$$\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/m_i, \mathbb{Z}/n_j) \xrightarrow{\mathrm{inc}_{m_i,n_j} \circ \theta_{m_i,n_j}} H_3(A \times B) \longrightarrow \mathrm{Tor}_1^{\mathbb{Z}}(A, B)$$

is the natural inclusion map. Since

$$\mathrm{Tor}_1^{\mathbb{Z}}(A, B) = \bigoplus_{\substack{1 \leqslant j \leqslant s \\ 1 \leqslant i \leqslant r}} \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/m_i, \mathbb{Z}/n_j),$$

we obtain a map $\theta_{A,B} : \mathrm{Tor}_1^{\mathbb{Z}}(A, B) \to H_3(A \times B)$ that decomposes our exact sequence canonically. In fact $\theta_{A,B} = \sum_{i,j} \mathrm{inc}_{m_i,n_j} \circ \theta_{m_i,n_j}$. $\square$

2. The third homology of GL_2

A commutative ring R with 1 is called a *ring with many units* if for any $n \geqslant 2$ and for any finite number of surjective linear forms $f_i : R^n \to R$, there exists a $v \in R^n$ such that, for all i, $f_i(v) \in R^*$. Important examples of rings with many units are semilocal rings with infinite residue fields. In particular for an infinite field F, any commutative finite dimensional F-algebra is a semilocal ring and so is a ring with many units. In this article we always assume that R is a commutative ring with many units.

Let

$$R^{*3} \times \mathrm{GL}_0 \overset{\mathrm{inc}}{\hookrightarrow} R^{*2} \times \mathrm{GL}_1 \overset{\mathrm{inc}}{\hookrightarrow} R^* \times \mathrm{GL}_2 \overset{\mathrm{inc}}{\hookrightarrow} \mathrm{GL}_3$$

be the natural diagonal inclusions. Here by R^{*n} we mean $R^* \times \cdots \times R^*$ (n-times). Let

$$\begin{aligned}
&\sigma_2^1 := \mathrm{inc} : R^* \times \mathrm{GL}_2 \longrightarrow \mathrm{GL}_3, \\
&\sigma_1^1 : R^{*2} \times \mathrm{GL}_1 \longrightarrow R^* \times \mathrm{GL}_2, && (a,b,c) \mapsto (b,a,c), \\
&\sigma_1^2 = \mathrm{inc} : R^{*2} \times \mathrm{GL}_1 \longrightarrow R^* \times \mathrm{GL}_2, && (a,b,c) \mapsto (a,b,c), \\
&\sigma_0^1 : R^{*3} \times \mathrm{GL}_0 \longrightarrow R^{*2} \times \mathrm{GL}_1, && (a,b,c) \mapsto (b,c,a), \\
&\sigma_0^2 : R^{*3} \times \mathrm{GL}_0 \longrightarrow R^{*2} \times \mathrm{GL}_1, && (a,b,c) \mapsto (a,c,b), \\
&\sigma_0^3 = \mathrm{inc} : R^{*3} \times \mathrm{GL}_0 \longrightarrow R^{*2} \times \mathrm{GL}_1, && (a,b,c) \mapsto (a,b,c).
\end{aligned}$$

It is easy to see that the chain of maps

$$\begin{aligned}
H_3\big(R^{*3} \times \mathrm{GL}_0\big) &\xrightarrow{\sigma_{0*}^1 - \sigma_{0*}^2 + \sigma_{0*}^3} H_3\big(R^{*2} \times \mathrm{GL}_1\big) \\
&\xrightarrow{\sigma_{1*}^1 - \sigma_{1*}^2} H_3\big(R^* \times \mathrm{GL}_2\big) \xrightarrow{\sigma_{2*}^1} H_3(\mathrm{GL}_3) \longrightarrow 0
\end{aligned}$$

is a chain complex. The following result has been proved in [8, Corollary 3.5].

Theorem 2.1. *The sequence*

$$H_3\big(R^{*2} \times \mathrm{GL}_1\big) \xrightarrow{\sigma_{1*}^1 - \sigma_{1*}^2} H_3\big(R^* \times \mathrm{GL}_2\big) \xrightarrow{\sigma_{2*}^1} H_3(\mathrm{GL}_3) \longrightarrow 0$$

is exact.

Using the Künneth formula [13, Proposition 6.1.13], we have the decomposition $H_3(R^* \times \mathrm{GL}_2) = \bigoplus_{i=0}^4 S_i$, where

$$\begin{aligned}
&S_0 = H_3(\mathrm{GL}_2), \\
&S_i = H_i\big(R^*\big) \otimes H_{3-i}(\mathrm{GL}_2), \quad 1 \leqslant i \leqslant 3, \\
&S_4 = \mathrm{Tor}_1^{\mathbb{Z}}\big(R^*, H_1(\mathrm{GL}_2)\big) \simeq \mathrm{Tor}_1^{\mathbb{Z}}\big(\mu(R), \mu(R)\big).
\end{aligned}$$

Note that by the homological stability, $R^* \simeq H_1(\mathrm{GL}_1) \simeq H_1(\mathrm{GL}_2)$ [4, Theorem 1]. This decomposition is canonical. The splitting map

$$S_4 \simeq \mathrm{Tor}_1^{\mathbb{Z}}\big(\mu(R), \mu(R)\big) \longrightarrow H_3\big(R^* \times \mathrm{GL}_2\big)$$

is given by the composition

$$S_4 \simeq \mathrm{Tor}_1^{\mathbb{Z}}\big(\mu(R), \mu(R)\big) \xrightarrow{\theta_{R,R}} H_3\big(R^* \times R^*\big) \xrightarrow{q_*} H_3\big(R^* \times \mathrm{GL}_2\big),$$

where

$$q : R^* \times R^* \longrightarrow R^* \times \mathrm{GL}_2, \quad (a, b) \mapsto (a, b, 1),$$

and $\theta_{R,R}$ is obtained from Proposition 1.1. Using the decomposition

$$H_2(\mathrm{GL}_2) = H_2(\mathrm{GL}_1) \oplus K_2^M(R)$$

[4, Theorem 2], we have $S_1 = S_1' \oplus S_1''$, where

$$S_1' = R^* \otimes H_2(\mathrm{GL}_1), \qquad S_1'' = R^* \otimes K_2^M(R).$$

We should remark that the inclusion $K_2^M(R) \to H_2(\mathrm{GL}_2)$, in the decomposition of $H_2(\mathrm{GL}_2)$, is given by the formula

$$\{a, b\} \mapsto \mathbf{c}\big(\mathrm{diag}(a, 1), \mathrm{diag}\big(b, b^{-1}\big)\big)$$

[3, Proposition A.11]. For the definition of Milnor's K-groups, $K_n^M(R)$, over commutative rings and their study over rings with many units, we refer the interested readers to Subsection 3.2 of [4].

Let us introduce the notation $\mathbf{c}(-,-)$ in a more general setting and state some of its main properties. These will be used frequently in this article. Let G be a group and set

$$\mathbf{c}(g_1, g_2, \ldots, g_n) := \sum_{\sigma \in \Sigma_n} \mathrm{sign}(\sigma)[g_{\sigma(1)}|g_{\sigma(2)}|\ldots|g_{\sigma(n)}] \in H_n(G),$$

where $g_1, \ldots, g_n \in G$ pairwise commute and Σ_n is the symmetric group of degree n. Here we use the bar resolution of G [2, Chapter I, Section 5] to define the homology of G.

Lemma 2.2. *Let G and G' be two groups.*

(i) *If $h_1 \in G$ commutes with all the elements $g_1, \ldots, g_n \in G$, then*

$$\mathbf{c}(g_1 h_1, g_2, \ldots, g_n) = \mathbf{c}(g_1, g_2, \ldots, g_n) + \mathbf{c}(h_1, g_2, \ldots, g_n).$$

(ii) *For every $\sigma \in \Sigma_n$, $\mathbf{c}(g_{\sigma(1)}, \ldots, g_{\sigma(n)}) = \mathrm{sign}(\sigma)\mathbf{c}(g_1, \ldots, g_n)$.*

(iii) *The cup product of $\mathbf{c}(g_1, \ldots, g_p) \in H_p(G)$ and $\mathbf{c}(g_1', \ldots, g_q') \in H_q(G')$ is $\mathbf{c}((g_1, 1), \ldots, (g_p, 1), (1, g_1'), \ldots, (1, g_q')) \in H_{p+q}(G \times G')$.*

Proof. The proofs follow from direct computations, so we leave it to the interested readers. □

Again using the Künneth formula and Proposition 1.1, we obtain the canonical decomposition $H_3(R^{*2} \times \mathrm{GL}_1) = \bigoplus_{i=0}^{8} T_i$, where

$$
\begin{aligned}
T_0 &= H_3(\mathrm{GL}_1),\\
T_1 &= \bigoplus_{i=1}^{3} T_{1,i} = \bigoplus_{i=1}^{3} H_i\big(R_1^*\big) \otimes H_{3-i}(\mathrm{GL}_1),\\
T_2 &= \bigoplus_{i=1}^{3} T_{2,i} = \bigoplus_{i=1}^{3} H_i\big(R_2^*\big) \otimes H_{3-i}(\mathrm{GL}_1),\\
T_3 &= R_1^* \otimes R_2^* \otimes H_1(\mathrm{GL}_1),\\
T_4 &= \mathrm{Tor}_1^{\mathbb{Z}}\big(R_1^*, R_2^*\big) \simeq \mathrm{Tor}_1^{\mathbb{Z}}\big(\mu(R), \mu(R)\big),\\
T_5 &= \mathrm{Tor}_1^{\mathbb{Z}}\big(R_1^*, H_1(\mathrm{GL}_1)\big) \simeq \mathrm{Tor}_1^{\mathbb{Z}}\big(\mu(R), \mu(R)\big),\\
T_6 &= \mathrm{Tor}_1^{\mathbb{Z}}\big(R_2^*, H_1(\mathrm{GL}_1)\big) \simeq \mathrm{Tor}_1^{\mathbb{Z}}\big(\mu(R), \mu(R)\big),\\
T_7 &= R_1^* \otimes H_2\big(R_2^*\big),\\
T_8 &= H_2\big(R_1^*\big) \otimes R_2^*.
\end{aligned}
$$

Here by R_i^* we mean the i-th component of $R^* \times \cdots \times R^*$. Now we give an explicit description of restriction of the map $\alpha := \sigma_{1*}^1 - \sigma_{1*}^2$ on all T_i's. By direct computations one sees that

$$
\begin{aligned}
&\alpha|_{T_0} : T_0 \longrightarrow S_0, && x \mapsto 0,\\
&\alpha|_{T_{1,i}} : T_{1,i} \longrightarrow S_0 \oplus S_i, && x_i \otimes x_i' \mapsto \big(x_i \cup x_i', -x_i \otimes x_i'\big),\ 1 \leqslant i \leqslant 3,\\
&\alpha|_{T_{2,i}} : T_{2,i} \longrightarrow S_0 \oplus S_i, && y_i \otimes y_i' \mapsto \big(-y_i \cup y_i', y_i \otimes y_i'\big),\ 1 \leqslant i \leqslant 3,\\
&\alpha|_{T_3} : T_3 \longrightarrow S_1, && a \otimes b \otimes c \mapsto -b \otimes (a \cup c) - a \otimes (b \cup c),\\
&\alpha|_{T_4} : T_4 \longrightarrow S_4, && z \mapsto 0,\\
&\alpha|_{T_5} : T_5 \longrightarrow S_0 \oplus S_4, && u \mapsto \big(\sigma_{1*}^1(u), -u\big),\\
&\alpha|_{T_6} : T_6 \longrightarrow S_0 \oplus S_4, && v \mapsto \big(-\sigma_{1*}^2(v), v\big),\\
&\alpha|_{T_7} : T_7 \longrightarrow S_1 \oplus S_2, && d \otimes u' \mapsto \big(-d \otimes u', u' \otimes d\big),\\
&\alpha|_{T_8} : T_8 \longrightarrow S_1 \oplus S_2, && v' \otimes e \mapsto \big(e \otimes v', -v' \otimes e\big),
\end{aligned}
$$

where $x \cup y$ is the cup product of x and y.

3. The kernel of $H_3(\mathrm{GL}_2) \to H_3(\mathrm{GL}_3)$

Our goal in this article is to study the kernel of the map $\mathrm{inc}_* : H_3(\mathrm{GL}_2) \to H_3(\mathrm{GL}_3)$. So let $x \in \ker(\mathrm{inc}_*)$. Then

$$(x, 0, 0, 0, 0) \in \ker\big(\sigma_{2*}^1\big) \subseteq \bigoplus_{i=0}^{4} S_i = H_3\big(R^* \times \mathrm{GL}_2\big).$$

By Theorem 2.1 and by the explicit description of $\alpha = \sigma_{1*}^1 - \sigma_{1*}^2$ given in the previous section, there exists an element

$$l = \Big(0, \big(x_i \otimes x_i'\big)_{1 \leqslant i \leqslant 3}, \big(y_i \otimes y_i'\big)_{1 \leqslant i \leqslant 3}, \sum a \otimes b \otimes c, 0, u, v, d \otimes u', v' \otimes e\Big)$$

in $H_3(R^{*2} \times \mathrm{GL}_1)$ such that $\alpha(l) = (x, 0, 0, 0, 0)$.

Set $\beta := \sigma_{0*}^1 - \sigma_{0*}^2 + \sigma_{0*}^3$, and consider the following summands of $H_3(R^{*3} \times \mathrm{GL}_0)$,

$$T_1' := R_1^* \otimes H_2\big(R_2^*\big), \qquad T_2' := H_2\big(R_1^*\big) \otimes R_2^*.$$

By easy computations one sees that

$$\beta|_{T_1'} : T_1' \longrightarrow T_{1,1} \oplus T_{1,2} \oplus T_7, \quad f \otimes w \mapsto (-f \otimes w, w \otimes f, f \otimes w),$$
$$\beta|_{T_2'} : T_2' \longrightarrow T_{1,1} \oplus T_{1,2} \oplus T_8, \quad w' \otimes f' \mapsto \big(f' \otimes w', -w' \otimes f', w' \otimes f'\big).$$

So we may assume $d \otimes u' = 0$, $v' \otimes e = 0$. Therefore we have

$$\sum_{i=1}^{3} x_i \cup x_i' - \sum_{i=1}^{3} y_i \cup y_i' + \sigma_{1*}^1(u) - \sigma_{1*}^2(v) = x,$$
$$-x_1 \otimes x_1' + y_1 \otimes y_1' - \sum \big[b \otimes (a \cup c) + a \otimes (b \cup c)\big] = 0,$$
$$-x_2 \otimes x_2' + y_2 \otimes y_2' = 0,$$
$$-x_3 \otimes x_3' + y_3 \otimes y_3' = 0,$$
$$-u + v = 0.$$

Therefore we obtain the following relations

$$x = x_1 \cup x_1' - y_1 \cup y_1' \in S_0 = H_3(\mathrm{GL}_2),$$
$$x_1 \otimes x_1' - y_1 \otimes y_1' = -\sum b \otimes (a \cup c) + a \otimes (b \cup c) \in S_1.$$

Under the decomposition $H_2(\mathrm{GL}_2) = H_2(\mathrm{GL}_1) \oplus K_2^M(R)$, we have

$$a \cup b = \mathbf{c}\big(\mathrm{diag}(a, 1), \mathrm{diag}(1, b)\big) = \big(\mathbf{c}(a, b), \{a, b\}\big).$$

Thus under the decomposition $S_1 = S_1' \oplus S_1''$, we have

$$\Big(x_1 \otimes x_1' - y_1 \otimes y_1' + \sum b \otimes \mathbf{c}(a, c) + a \otimes \mathbf{c}(b, c), \sum b \otimes \{a, c\} + a \otimes \{b, c\}\Big) = 0,$$

and hence

$$x_1 \otimes x_1' - y_1 \otimes y_1' = -\sum b \otimes \mathbf{c}(a, c) + a \otimes \mathbf{c}(b, c),$$
$$\sum b \otimes \{a, c\} + a \otimes \{b, c\} = 0.$$

Therefore

$$x = -\sum \mathbf{c}\big(\mathrm{diag}(a, 1), \mathrm{diag}(1, b), \mathrm{diag}(1, c)\big) + \mathbf{c}\big(\mathrm{diag}(b, 1), \mathrm{diag}(1, a), \mathrm{diag}(1, c)\big)$$
$$= \sum \mathbf{c}\big(\mathrm{diag}(a, 1), \mathrm{diag}(1, b), \mathrm{diag}\big(c, c^{-1}\big)\big),$$

such that $\sum a \otimes \{b, c\} + b \otimes \{a, c\} = 0$. From now on, we will use the following notation:

$$l_{a,b,c} = \mathbf{c}\big(\mathrm{diag}(a, 1), \mathrm{diag}(1, b), \mathrm{diag}\big(c, c^{-1}\big)\big).$$

Hence we have proved most parts of the following theorem.

Theorem 3.1. *Let R be a commutative ring with many units. Then the kernel of* $\mathrm{inc}_* : H_3(\mathrm{GL}_2) \to H_3(\mathrm{GL}_3)$ *consists of elements of the form* $\sum \mathbf{c}(\mathrm{diag}(a, 1), \mathrm{diag}(1, b), \mathrm{diag}(c, c^{-1}))$ *provided that*

$$\sum a \otimes \{b, c\} + b \otimes \{a, c\} = 0 \in R^* \otimes K_2^M(R).$$

In particular $\ker(\mathrm{inc}_*) \subseteq R^* \cup H_2(\mathrm{GL}_1) \subseteq H_3(\mathrm{GL}_2)$*, where the cup product is induced by the diagonal inclusion* $\mathrm{inc} : R^* \times \mathrm{GL}_1 \to \mathrm{GL}_2$*. Moreover* $\ker(\mathrm{inc}_*)$ *is a* 2*-torsion group.*

Proof. The only part that remains to be proved is that $\ker(\mathrm{inc}_*)$ is a 2-torsion group. Let $x \in \ker(\mathrm{inc}_*)$. For simplicity we may assume that $x = l_{a,b,c} = \mathbf{c}(\mathrm{diag}(a, 1), \mathrm{diag}(1, b), \mathrm{diag}(c, c^{-1}))$, such that $a \otimes \{b, c\} + b \otimes \{a, c\} = 0$. Let Φ be the following composition

$$R^* \otimes K_2^M(R) \xrightarrow{\mathrm{id}_{R^*} \otimes \iota} R^* \otimes H_2(\mathrm{GL}_2) \xrightarrow{\cup} H_3\big(R^* \times \mathrm{GL}_2\big) \xrightarrow{\alpha_*} H_3(\mathrm{GL}_2),$$

where $\iota : K_2^M(R) \to H_2(\mathrm{GL}_2)$ is described in the previous section, $\cup$ is the cup product and $\alpha : R^* \times \mathrm{GL}_2 \to \mathrm{GL}_2$ is given by $(a, A) \mapsto aA$. It is easy to see that

$$\Phi\big(a \otimes \{b, c\}\big) = \mathbf{c}\big(\mathrm{diag}(a, a), \mathrm{diag}(b, 1), \mathrm{diag}\big(c, c^{-1}\big)\big).$$

Now with easy computations, one sees that

$$\begin{aligned} 0 &= \Phi(0) \\ &= \Phi\big(a \otimes \{b, c\} + b \otimes \{a, c\}\big) \\ &= \mathbf{c}\big(\mathrm{diag}(a, a), \mathrm{diag}(b, 1), \mathrm{diag}\big(c, c^{-1}\big)\big) + \mathbf{c}\big(\mathrm{diag}(b, b), \mathrm{diag}(a, 1), \mathrm{diag}\big(c, c^{-1}\big)\big) \\ &= -2l_{a,b,c}. \end{aligned}$$

This completes the proof of the theorem. □

Remark 3.2. One can show directly that if $a \otimes \{b, c\} + b \otimes \{a, c\} = 0$, then $l_{a,b,c} \in \ker(\mathrm{inc}_* : H_3(\mathrm{GL}_2) \to H_3(\mathrm{GL}_3))$. To see this, let Ψ be the following composition

$$R^* \otimes K_2^M(R) \xrightarrow{\mathrm{id}_{R^*} \otimes \iota} R^* \otimes H_2(\mathrm{GL}_2) \xrightarrow{\cup} H_3\big(R^* \times \mathrm{GL}_2\big) \longrightarrow H_3(\mathrm{GL}_3).$$

Then it is easy to see that

$$\Psi\big(a \otimes \{b, c\}\big) = \mathbf{c}\big(\mathrm{diag}(a, 1, 1), \mathrm{diag}(1, b, 1), \mathrm{diag}\big(1, c, c^{-1}\big)\big).$$

Now we have

$$\begin{aligned}
\mathrm{inc}_*(l_{a,b,c}) &= +\mathbf{c}\big(\mathrm{diag}(1,a,1), \mathrm{diag}(1,1,b), \mathrm{diag}\big(1,c,c^{-1}\big)\big) \\
&= +\mathbf{c}\big(\mathrm{diag}(a,1,1), \mathrm{diag}(1,b,1), \mathrm{diag}\big(c,c^{-1},1\big)\big) \\
&= -\mathbf{c}\big(\mathrm{diag}(a,1,1), \mathrm{diag}(1,b,1), \mathrm{diag}(1,c,1)\big) \\
&\quad - \mathbf{c}\big(\mathrm{diag}(b,1,1), \mathrm{diag}(1,a,1), \mathrm{diag}(1,c,1)\big) \\
&= -\mathbf{c}\big(\mathrm{diag}(a,1,1), \mathrm{diag}(1,b,1), \mathrm{diag}(1,1,c,)\big) \\
&\quad - \mathbf{c}\big(\mathrm{diag}(a,1,1), \mathrm{diag}(1,b,1), \mathrm{diag}\big(1,c,c^{-1}\big)\big) \\
&\quad - \mathbf{c}\big(\mathrm{diag}(b,1,1), \mathrm{diag}(1,a,1), \mathrm{diag}(1,1,c)\big) \\
&\quad - \mathbf{c}\big(\mathrm{diag}(b,1,1), \mathrm{diag}(1,a,1), \mathrm{diag}\big(1,c,c^{-1}\big)\big) \\
&= -\Psi\big(a\otimes\{b,c\} + b\otimes\{a,c\}\big) \\
&= 0.
\end{aligned}$$

Corollary 3.3. *Let R be a ring with many units.*

(i) *The natural map* $\mathrm{inc}_*: H_3(\mathrm{GL}_2, \mathbb{Z}[1/2]) \to H_3(\mathrm{GL}_3, \mathbb{Z}[1/2])$ *is injective.*
(ii) *If $R^* = R^{*2} = \{a^2 \mid a \in R^*\}$, then* $\mathrm{inc}_*: H_3(\mathrm{GL}_2) \to H_3(\mathrm{GL}_3)$ *is injective.*

Proof. The part (i) immediately follows from Theorem 3.1. Let $R^* = R^{*2}$. By Theorem 3.1, we may assume that $x \in \ker(\mathrm{inc}_*)$ is of the form $l_{a,b,c} \in H_3(\mathrm{GL}_2)$ such that $a\otimes\{b,c\}+b\otimes\{a,c\} = 0$. Let $c = c'^2$ for some $c' \in R^*$. Then $l_{a,b,c} = 2l_{a,b,c'}$ and $2(a\otimes\{b,c'\} + b\otimes\{a,c'\}) = 0$. Since $K_2^M(R)$ is uniquely 2-divisible [1, Proposition 1.2], $R^*\otimes K_2^M(R)$ is uniquely 2-divisible too. Hence $a\otimes\{b,c'\}+b\otimes\{a,c'\} = 0$. Now from Theorem 3.1, it follows that $2l_{a,b,c'} = 0$. Therefore $l_{a,b,c} = 0$ and hence $\mathrm{inc}_*: H_3(\mathrm{GL}_2) \to H_3(\mathrm{GL}_3)$ is injective. □

Example 3.4. Let $R = \mathbb{R}$. It is well know that $K_2^M(\mathbb{R}) \simeq \langle\{-1,-1\}\rangle \oplus V$, where V is uniquely divisible and is generated by elements $\{a,b\}$ with $a,b > 0$. Let $l_{a,b,c} \in H_3(\mathrm{GL}_2(\mathbb{R}))$ be such that $a\otimes\{b,c\} + b\otimes\{a,c\} = 0$. If $a > 0$, then $a\otimes\{b,c\} = a\otimes\{-b,c\} = a\otimes\{b,-c\} = a\otimes\{-b,-c\}$, so we may assume that $b,c > 0$. Now with an argument as in the proof of the previous corollary, one sees that $l_{a,b,c} = 0$. A similar argument works if $b > 0$ or if $c > 0$. If $a,b,c < 0$, then one can easily reduce the problem to the case that $a = b = c = -1$, and it is trivial to see that $l_{-1,-1,-1} = 0$. Therefore $\mathrm{inc}_*: H_3(\mathrm{GL}_2(\mathbb{R})) \to H_3(\mathrm{GL}_3(\mathbb{R}))$ is injective.

Remark 3.5. Consider the following chain of maps

$$R^{*\otimes 3} \otimes K_0^M(R) \xrightarrow{\delta_0^{(3)}} R^{*\otimes 2} \otimes K_1^M(R) \xrightarrow{\delta_1^{(3)}} R^* \otimes K_2^M(R) \xrightarrow{\delta_2^{(3)}} K_3^M(R) \longrightarrow 0,$$

where

$$\delta_2^{(3)}: a\otimes\{b,c\} \mapsto \{a,b,c\},$$

$$\delta_1^{(3)}: a\otimes b\otimes\{c\} \mapsto a\otimes\{b,c\} + b\otimes\{a,c\},$$

$$\delta_0^{(3)}: a\otimes b\otimes c \mapsto b\otimes c\otimes\{a\} + a\otimes c\otimes\{b\} + a\otimes b\otimes\{c\}.$$

It is easy to see that this is, in fact, a chain complex. It is not difficult to see that $\ker(\delta_2^{(3)}) = \mathrm{im}(\delta_1^{(3)})$ (see the proof of Theorem 3.2 in [5]). Under the composition

$$R^{*\otimes 3} \longrightarrow R^* \otimes H_2(R^*) \longrightarrow H_3(\mathrm{GL}_2),$$

defined by

$$a \otimes b \otimes c \mapsto a \otimes \mathbf{c}(b, c) \mapsto \mathbf{c}(\mathrm{diag}(a, 1), \mathrm{diag}(1, b), \mathrm{diag}(1, c)),$$

one can see that $\mathrm{im}(\delta_0^{(3)})$ maps to zero. Thus we obtain a surjective map

$$\ker(\delta_1^{(3)})/\mathrm{im}(\delta_0^{(3)}) \longrightarrow \ker(H_3(\mathrm{GL}_2) \to H_3(\mathrm{GL}_3)),$$
$$\sum a \otimes b \otimes c + \mathrm{im}(\delta_0^{(3)}) \mapsto \sum l_{a,b,c}.$$

Lemma 3.6. *Let R be a ring with many units.*

(i) *We have the exact sequence*

$$0 \longrightarrow H_3(\mathrm{SL}_2, \mathbb{Z}[1/2])_{R^*} \longrightarrow H_3(\mathrm{SL}, \mathbb{Z}[1/2]) \longrightarrow K_3^M(R)_{\mathbb{Z}[1/2]} \longrightarrow 0.$$

(ii) *If $R^* = R^{*2} = \{a^2 \mid a \in R^*\}$, then we have the exact sequence*

$$0 \longrightarrow H_3(\mathrm{SL}_2) \longrightarrow H_3(\mathrm{SL}) \longrightarrow K_3^M(R) \longrightarrow 0.$$

Proof. The proof is similar to the proof of Theorem 6.1 and Corollary 6.2 in [8]. □

Theorem 3.7. *Let R be a ring with many units.*

(i) *We have the isomorphism*

$$K_3(R)^{\mathrm{ind}} \otimes \mathbb{Z}[1/2] \simeq H_3(\mathrm{SL}_2, \mathbb{Z}[1/2])_{R^*}.$$

(ii) *If $R^* = R^{*2} = \{a^2 \mid a \in R^*\}$, then*

$$K_3(R)^{\mathrm{ind}} \simeq H_3(\mathrm{SL}_2).$$

Proof. The proof is similar to the proof of Theorem 6.4 in [8]. □

Remark 3.8. Previously Lemma 3.6 and Theorem 3.7 were known only for infinite fields [8, Corollary 6.2, Proposition 6.4].

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