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The D'Yakonov Fully Explicit Variant of the Iterative Decomposition Method

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Abstract—In this paper, a new iterative alternating decomposition (IADE) scheme of (4, 2) order of accuracy is developed to solve the one-dimensional parabolic problem. It is based on the two-stage fractional splitting strategy suggested by D'Yakonov and found to be generally more accurate than the recently developed (2, 2) accurate alternating group explicit (AGE) method of Peaceman-Rachford variant. As the method is fully explicit, its feature can be fully utilized for parallelization by means of a domain decomposition strategy. © 2001 Elsevier Science Ltd. All rights reserved.

Keywords—Alternating group explicit (AGE) method, Iterative alternating decomposition explicit (IADE) method, D'Yakonov fractional splitting.

1. INTRODUCTION

Consider the following heat equation:

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}, \qquad 0 \le x \le 1, \quad t > 0, \tag{1}$$

subject to the initial-boundary conditions

$$\begin{aligned} U(x,0) &= f(x), \qquad 0 < x < 1, \\ U(0,t) &= g(t), \\ U(1,t) &= h(t), \qquad 0 < t \le T. \end{aligned} \tag{2}$$

The interval $0 < x < 1$ is divided into a grid of points of spacing $\Delta x = 1/(m+1)$. Similarly, the T interval is divided into steps of Δt. Then, on the proposed lattice, the different operators in (1) are approximated by central differences.

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A generalized finite difference approximation to the differential equation (1) at the point $(x_i, t_{j+1/2})$ is given by

$$\begin{aligned} &-\lambda\theta u_{i-1,j+1} + (1+2\lambda\theta)u_{i,j+1} - \lambda\theta u_{i+1,j+1} \\ &= \lambda(1-\theta)u_{i-1,j} + [1-2\lambda(1-\theta)]u_{i,j} + \lambda(1-\theta)u_{i+1,j}, \qquad i = 1,2,\dots,m, \end{aligned} \tag{3}$$

which is displayed in matrix form as

$$\begin{bmatrix} a & b & & & & & \\ c & a & b & & O & & \\ & c & a & b & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \ddots & \ddots & \ddots & \\ & & O & & c & a & b \\ & & & & & c & a \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{m-1} \\ u_m \end{bmatrix}_{j+1} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_{m-1} \\ f_m \end{bmatrix}, \tag{4}$$

i.e.,

$$A\underset{\sim}{u} = \underset{\sim}{f}, \tag{5}$$

where

$$\begin{aligned} c = b &= -\lambda\theta, & a &= 1 + 2\lambda\theta, \\ f_1 &= \lambda(1-\theta)\,(u_{0,j} + u_{2,j}) + [1-2\lambda(1-\theta)] + \lambda\theta u_{0,j}, \\ f_i &= \lambda(1-\theta)\,(u_{i-1,j} + u_{i+1,j}) + [1-2\lambda(1-\theta)]u_{i,j}, & i &= 2,3,\dots,m-2,m-1, \\ f_m &= \lambda(1-\theta)\,(u_{m-1,j} + u_{m+1,j}) + [1-2\lambda(1-\theta)] + \lambda\theta u_{m+1,j+1}, \\ \underline{u} &= (u_{1,j+1}, u_{2,j+1}, \dots, u_{m,j+1})^\top, & \underline{f} &= (f_1, f_2, \dots, f_m)^\top, \end{aligned} \tag{6}$$

and $\lambda = \Delta t/(\Delta x)^2$ with the increments Δx, Δt given by $\Delta x = 1/(m+1)$ and $\Delta t = T/(n+1)$. We recall that equation (3) corresponds to the fully implicit, the Crank-Nicolson, and the classical explicit methods when θ takes the values of 1, 1/2, and 0.

2. FORMULATION AND COMPUTATION OF THE IADE SCHEME

In the development of the ADI scheme to solve the two-dimensional heat equation, D'Yakonov [1] proposed a high order $(4,2)$ accurate unconditionally stable two-step method involving the solution of tridiagonal sets of equations along lines parallel to the x- and y-axes at the first and second steps. Using the well-known fact of the parabolic-elliptic correspondence, we shall now employ this fractional splitting of D'Yakonov to obtain the following $(4,2)$ accurate, stable, and convergent two-stage iterative procedure for a fixed acceleration parameter $r > 0$.

Consider the iterative formulae

$$\begin{aligned} (rI + L)\underset{\sim}{u}^{(p+1/2)} &= (rI - gL)(rI - gR)\underset{\sim}{u}^{(p)} + h\underset{\sim}{f}, \\ (rI + R)\underset{\sim}{u}^{(p+1)} &= \underset{\sim}{u}^{(p+1/2)}, \end{aligned} \tag{7}$$

and

$$g = \frac{6+r}{6}, \qquad h = \frac{r(12+r)}{6}.$$

Note that by combining the two equations in (7) and eliminating $\underset{\sim}{u}^{(p+1/2)}$, we find that as $p \to \infty$, we have

$$\left(L + R - \frac{1}{6}LR\right)\underset{\sim}{u} = \underset{\sim}{f}. \tag{8}$$

This suggests that the coefficient matrix A in (5) can be decomposed into

$$A = L + R - \frac{1}{6}LR. \tag{9}$$

To retain the tridiagonal structure of A as in (4), the constituent matrices L and R take the bidiagonal forms (lower and upper, respectively)

$$L = \begin{bmatrix} 1 & & & & & \\ l_1 & 1 & & O & & \\ & l_2 & 1 & & & \\ & & \ddots & \ddots & & \\ & O & & l_{m-2} & 1 & \\ & & & & l_{m-1} & 1 \end{bmatrix}_{(m\times m)},$$

and

$$R = \begin{bmatrix} e_1 & u_1 & & & & \\ & e_2 & u_2 & & O & \\ & & e_3 & u_3 & & \\ & & & \ddots & \ddots & \\ & & O & & e_{m-1} & u_{m-1} \\ & & & & & e_m \end{bmatrix}_{(m\times m)}. \tag{10}$$

Equating the entries of the matrices in (9) leads to the determination of e_i, u_i, $i = 1, 2, \dots, m$, in the recursive form

$$\begin{aligned} e_1 &= \frac{6(a-1)}{5}, \\ u_i &= \frac{6b}{5}, \qquad l_i = \frac{6c}{6-e_i}, \qquad e_i \neq 6, \\ e_{i+1} &= \frac{6(a + l_i u_i/6 - 1)}{5}, \qquad i = 1, 2, \dots, m-1. \end{aligned} \tag{11}$$

The explicit form of (7) is given by

$$\begin{aligned} \underset{\sim}{u}^{(p+1/2)} &= (rI + L)^{-1}\left\{(rI - gL)(rI - gR)\underset{\sim}{u}^{(p)} + h\underset{\sim}{f}\right\}, \qquad \text{and} \\ \underset{\sim}{u}^{(p+1)} &= (rI + R)^{-1}\underset{\sim}{u}^{(p+1/2)}. \end{aligned} \tag{12}$$

Since L and R are bidiagonal, the inverse of $(rI + L)$ and $(rI + R)$ take a full lower and upper triangular form given by

$$(rI + L)^{-1} = \begin{bmatrix} \frac{1}{d} & & & & & & \\ \frac{-l_1}{d^2} & \frac{1}{d} & & & & & \\ \frac{l_1 l_2}{d^3} & \frac{-l_2}{d^2} & \frac{1}{d} & & & & \\ \vdots & \vdots & \ddots & \ddots & & & \\ \alpha_{i,1} & \alpha_{i,2} & \cdots & \alpha_{i,i-1} & \alpha_{i,i} & & \\ \vdots & \vdots & & & \ddots & \ddots & \\ \alpha_{m,1} & \alpha_{m,2} & \alpha_{m,3} & \cdots & \cdots & \alpha_{m,m-1} & \alpha_{m,m} \end{bmatrix}_{m\times m},$$

where

$$\alpha_{i,k} = \frac{(-1)^{i-k+2}}{d^{i-(k-1)}} \prod_{j=k}^{i-1} l_j, \qquad i = 1,2,\dots,m, \quad k = 1,2,\dots,i, \tag{13}$$

with

$$\prod_{j=k}^{i-1} l_j = \begin{cases} l_k l_{k+1} \cdots l_{i-1}, & k < i-1, \\ l_{i-1}, & k = i-1, \\ 1, & k = i, \end{cases} \tag{14}$$

$$d = 1 + r, \tag{15}$$

and

$$(rI + R)^{-1} = \begin{bmatrix} \frac{1}{d_1} & \frac{-u_1}{d_1 d_2} & \frac{u_1 u_2}{d_1 d_2 d_3} & \cdots & \beta_{1,j} & \cdots & \beta_{1,m} \\ & \frac{1}{d_2} & \frac{-u_2}{d_2 d_3} & \cdots & \beta_{2,j} & \cdots & \beta_{2,m} \\ & & & \ddots & \vdots & & \vdots \\ & & & & \beta_{j,j} & & \beta_{j,m} \\ & & O & & & \ddots & \vdots \\ & & & & & & \beta_{m-1,m} \\ & & & & & & \beta_{m,m} \end{bmatrix}_{m \times m},$$

where

$$\beta_{k,j} = (-1)^{j-k} \frac{\prod_{i=k}^{j-1} u_i}{\prod_{i=k}^{j} d_i}, \qquad j = 1,2,\dots,m, \quad k = 1,2,\dots,j, \tag{16}$$

with

$$\prod_{i=k}^{j-1} u_i = \begin{cases} u_k u_{k+1} \cdots u_{j-1}, & k < j-1, \\ u_{j-1}, & k = j-1, \\ 1, & k = j, \end{cases} \tag{17}$$

and

$$d_i = r + e_i. \tag{18}$$

By carrying out the relevant multiplications in (12), we obtain the following equations for computation at each of the intermediate levels:

(i) at the $(p+1/2)^{\text{th}}$ iterate,

$$\begin{aligned} u_1^{(p+1/2)} &= \frac{s\left(s_1 u_1^{(p)} + w_1 u_2^{(p)} + h f_1\right)}{d}, \\ u_i^{(p+1/2)} &= \frac{-l_{i-1} u_{i-1}^{(p+1/2)} + v_{i-1} s_{i-1} u_{i-1}^{(p)} + (v_{i-1} w_{i-1} + s s_i) u_i^{(p)} + s w_i u_{i+1}^{(p)} + h f_i}{d}, \\ &\quad i = 2,3,4,\dots,m-1, \\ u_m^{(p+1/2)} &= \frac{-l_{m-1} u_{m-1}^{(p+1/2)} + v_{m-1} s_{m-1} u_{m-1}^{(p)} + (v_{m-1} w_{m-1} + s s_m) u_m^p + h f_m}{d}, \end{aligned} \tag{19}$$

where

$$\begin{aligned} s &= r - g, & q &= (1+g) r, \\ s_i &= r - g e_i, & i &= 1,2,\dots,m, \\ w_i &= -g u_i, & i &= 1,2,\dots,m-1, \\ v_i &= -g l_i, & i &= 1,2,\dots,m-1. \end{aligned} \tag{20}$$

Equations (19) may be written fully in their explicit form as

$$
\begin{aligned}
u_1^{(p+1/2)} &= \frac{s\left(s_1 u_1^{(p)} + w_1 u_2^{(p)} + hf_1\right)}{d}, \\
u_1^{(p+1/2)} &= \frac{(-1)^{i+1} q \left[\prod_{j=1}^{i-1} l_j\right] s_1 u_1^{(p)}}{d^i} + \sum_{j=2}^{i-1} \frac{(-1)^{i+j+1} q \left[\prod_{k=j}^{i-1} l_j\right] \left(l_{j-1} w_{j-1} - s_j d\right) u_j^{(p)}}{d^{i-j+2}} \\
&\quad - \frac{(l_{i-1} q w_{i-1} - s s_i d) u_i^{(p)}}{d^2} + \frac{s w_i u_{i+1}^{(p)}}{d} \\
&\quad + h \frac{\sum_{j=1}^{i} \left[(-1)^{j+i} f_j \prod_{k=j}^{i-1} l_k\right]}{d^{i-j+1}}, \qquad \text{for } i = 2, 3, \dots, m-1, m;
\end{aligned} \tag{21}
$$

(ii) at the $(p+1)^{\text{th}}$ iterate,

$$
\begin{aligned}
u_m^{(p+1)} &= \frac{u_m^{(p+1/2)}}{d_m}, \\
u_i^{(p+1)} &= \frac{u_i^{(p+1/2)} - u_i u_{i+1}^{(p+1)}}{d_i}, \qquad i = m-1, m-2, \dots, 2, 1.
\end{aligned} \tag{22}
$$

The fully explicit form of (22) is given by

$$
u_i^{(p+1)} = \frac{\sum_{k=i}^{m} (-1)^z \left[\prod_{j=k+1}^{m} (r + e_j)\right] u_k^{(p+1/2)} \prod_{l=i}^{k-1} u_l}{\prod_{n=i}^{m} (r + e_n)}, \tag{23}
$$

where

$$
z = \begin{cases} k+1, & i \text{ odd}, \\ k, & i \text{ even}, \end{cases} \qquad \text{for } i = 1, 2, \dots, m.
$$

The IADE algorithm is executed by using the equations (21) and (23) in alternate sweeps along the points in the interval $(0, 1)$ until a specified convergence criterion is satisfied.

3. NUMERICAL EXPERIMENTS

The application of the AGE and IADE algorithms is now demonstrated on the following problems. The convergence criterion is taken as eps $= 10^{-4}$, and the accuracy of the solution at the grid points is determined by computing its root mean square (rms) error.

Experiment 1 (Problem 1)

The following problem is taken from [2]:

$$
\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}, \qquad 0 \le x \le 1, \tag{24}
$$

with the initial condition

$$
U(x, 0) = 4x(1 - x), \qquad 0 \le x \le 1, \tag{25}
$$

and the boundary conditions

$$
U(0, t) = U(1, t) = 0, \qquad t \ge 0. \tag{26}
$$

The exact solution is given by

$$
U(x, t) = \frac{32}{\pi^2} \sum_{k=1,(2)}^{\infty} \frac{1}{k^3} e^{-\pi^2 k^2 t} \sin(k\pi x). \tag{27}
$$

Table 1. Absolute errors of the numerical solution to Problem 1.

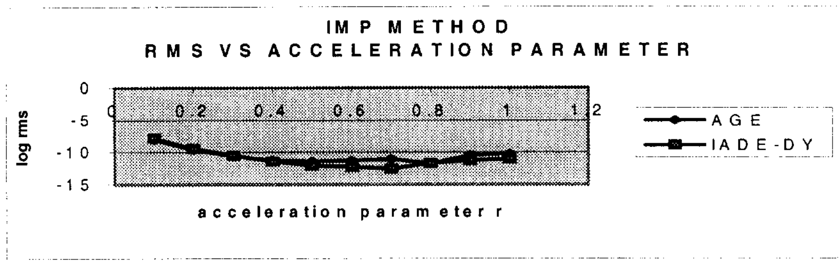
Method \ x	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	Root Mean Square Error (RMS)	No. of Iterations
$\lambda = 0.5,\ t = 0.25,\ \Delta t = 0.005,\ \Delta x = 0.1,\ \text{eps} = 10^{-4}$											
AGE											
IMP	2.18×10^{-3}	4.15×10^{-3}	5.69×10^{-3}	6.70×10^{-3}	7.04×10^{-3}	6.69×10^{-3}	5.71×10^{-3}	4.14×10^{-3}	2.19×10^{-3}	2.76×10^{-5}	3
CN	5.32×10^{-4}	1.02×10^{-3}	1.39×10^{-3}	1.64×10^{-3}	1.72×10^{-3}	1.64×10^{-3}	1.40×10^{-3}	1.01×10^{-4}	5.43×10^{-4}	1.65×10^{-6}	2
IADE-DY											
IMP	7.53×10^{-4}	1.36×10^{-3}	8.50×10^{-5}	9.74×10^{-4}	1.71×10^{-3}	2.05×10^{-3}	1.97×10^{-3}	1.52×10^{-3}	8.00×10^{-4}	1.92×10^{-6}	3
CN	2.05×10^{-4}	4.05×10^{-4}	7.53×10^{-4}	1.01×10^{-4}	1.15×10^{-3}	1.16×10^{-3}	1.03×10^{-3}	7.85×10^{-4}	4.38×10^{-4}	7.05×10^{-7}	3
Exact Solution	0.0270461	0.0514447	0.0708075	0.0832392	0.0875229	0.0832392	0.0708075	0.0514447	0.0270461	–	–
$\lambda = 1.0,\ t = 0.5,\ \Delta t = 0.01,\ \Delta x = 0.1,\ \text{eps} = 10^{-4}$											
AGE											
IMP	6.28×10^{-4}	1.26×10^{-3}	1.64×10^{-3}	1.96×10^{-3}	2.04×10^{-3}	1.93×10^{-3}	1.69×10^{-3}	1.20×10^{-3}	6.98×10^{-4}	2.36×10^{-6}	2
CN	7.28×10^{-5}	1.46×10^{-4}	1.90×10^{-4}	2.27×10^{-4}	2.36×10^{-4}	2.24×10^{-4}	1.97×10^{-4}	1.39×10^{-4}	8.12×10^{-5}	3.16×10^{-8}	2
IADE-DY											
IMP	7.11×10^{-4}	1.37×10^{-3}	1.47×10^{-3}	1.42×10^{-3}	1.28×10^{-3}	1.07×10^{-3}	8.03×10^{-4}	5.17×10^{-4}	2.41×10^{-4}	1.15×10^{-7}	3
CN	3.34×10^{-5}	6.31×10^{-5}	3.81×10^{-5}	1.64×10^{-5}	3.90×10^{-7}	8.96×10^{-6}	1.18×10^{-5}	9.67×10^{-6}	4.86×10^{-6}	7.96×10^{-10}	2
Exact Solution	0.0022936	0.0043628	0.0060048	0.0070591	0.0074224	0.0070590	0.0060048	0.0043628	0.0022936	–	–

Table 2. Absolute errors of the numerical solution to Problem 2.

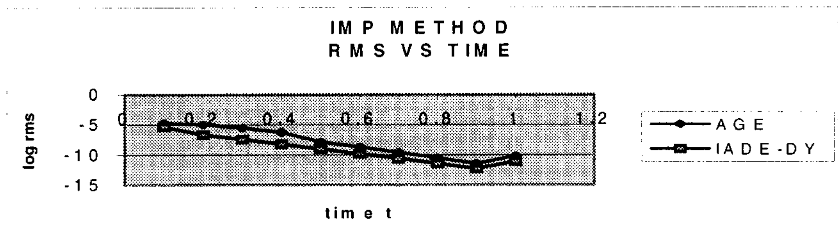
$\lambda = 0.5,\ t = 0.25,\ \Delta t = 0.005,\ \Delta x = 0.1,\ \text{eps} = 10^{-4}$											
Method \ x	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	Root Mean Square Error (RMS)	No. of Iterations
AGE											
IMP	2.06×10^{-3}	4.02×10^{-3}	5.40×10^{-3}	6.40×10^{-3}	6.70×10^{-3}	6.36×10^{-3}	5.48×10^{-3}	3.92×10^{-3}	2.17×10^{-3}	2.51×10^{-6}	3
CN	5.16×10^{-4}	9.91×10^{-4}	1.35×10^{-3}	1.59×10^{-3}	1.67×10^{-3}	1.59×10^{-3}	1.36×10^{-3}	9.82×10^{-4}	5.27×10^{-4}	1.56×10^{-6}	2
IADE-DY											
IMP	7.19×10^{-4}	1.29×10^{-3}	5.31×10^{-5}	9.78×10^{-4}	1.69×10^{-3}	2.02×10^{-3}	1.94×10^{-3}	1.49×10^{-3}	7.87×10^{-4}	1.85×10^{-6}	3
CN	2.14×10^{-4}	4.13×10^{-4}	7.50×10^{-4}	9.97×10^{-4}	1.13×10^{-3}	1.12×10^{-3}	9.94×10^{-4}	7.49×10^{-4}	4.14×10^{-4}	6.69×10^{-7}	3
Exact Solution	0.1950648	0.3707771	0.5098716	0.5989617	0.6296137	0.5989617	0.5098716	0.3707705	0.1950648	–	–
AGE											
IMP	5.36×10^{-4}	1.17×10^{-3}	1.42×10^{-3}	1.74×10^{-3}	1.79×10^{-3}	1.68×10^{-3}	1.53×10^{-3}	1.02×10^{-3}	7.03×10^{-4}	1.84×10^{-6}	2
CN	6.51×10^{-5}	1.37×10^{-4}	1.71×10^{-4}	2.07×10^{-4}	2.14×10^{-3}	2.02×10^{-4}	1.81×10^{-4}	1.24×10^{-4}	7.95×10^{-5}	2.62×10^{-8}	2
IADE-DY											
IMP	6.86×10^{-4}	1.32×10^{-3}	1.41×10^{-3}	1.37×10^{-3}	1.23×10^{-3}	1.03×10^{-3}	7.70×10^{-4}	4.95×10^{-4}	2.30×10^{-4}	1.06×10^{-6}	3
CN	3.11×10^{-5}	5.86×10^{-5}	3.35×10^{-5}	1.18×10^{-5}	4.04×10^{-6}	1.30×10^{-5}	1.53×10^{-5}	1.22×10^{-5}	6.24×10^{-6}	6.96×10^{-10}	2
Exact Solution	0.0022224	0.0042273	0.0058184	0.0068399	0.0071919	0.0068399	0.0058184	0.0042273	0.0022224	–	–

Table 3. Absolute errors of the numerical solution to Problem 3.

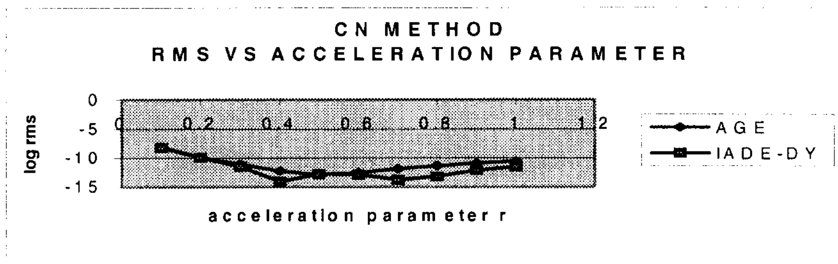
Method \ x	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	Root Mean Square Error (RMS)	No. of Iterations
$\lambda = 0.5,\ t = 0.25,\ \Delta t = 0.005,\ \Delta x = 0.1,\ \text{eps} = 10^{-4}$											
AGE											
IMP	2.25×10^{-3}	4.34×10^{-3}	5.72×10^{-3}	6.60×10^{-3}	6.72×10^{-3}	6.19×10^{-3}	5.20×10^{-3}	3.63×10^{-3}	1.99×10^{-3}	2.53×10^{-5}	3
CN	5.47×10^{-4}	1.04×10^{-3}	1.40×10^{-3}	1.62×10^{-3}	1.67×10^{-3}	1.56×10^{-3}	1.31×10^{-3}	9.33×10^{-4}	4.97×10^{-4}	1.56×10^{-6}	2
IADE-DY											
IMP	6.16×10^{-4}	1.26×10^{-3}	6.40×10^{-4}	1.54×10^{-4}	1.50×10^{-4}	2.81×10^{-4}	2.85×10^{-4}	2.14×10^{-4}	1.11×10^{-4}	2.94×10^{-7}	3
CN	1.15×10^{-4}	2.17×10^{-4}	4.63×10^{-4}	6.34×10^{-4}	7.15×10^{-4}	7.05×10^{-4}	6.14×10^{-4}	4.60×10^{-4}	2.56×10^{-4}	2.60×10^{-7}	3
Exact Solution	0.0262974	0.0499947	0.0687562	0.0807455	0.0848050	0.0805631	0.0684611	0.0496995	0.0261150	–	–
$\lambda = 1.0,\ t = 0.5,\ \Delta t = 0.01,\ \Delta x = 0.1,\ \text{eps} = 10^{-4}$											
AGE											
IMP	5.37×10^{-4}	1.17×10^{-3}	1.42×10^{-3}	1.74×10^{-3}	1.79×10^{-3}	1.68×10^{-3}	1.53×10^{-3}	1.02×10^{-3}	7.04×10^{-4}	1.84×10^{-6}	2
CN	6.53×10^{-5}	1.37×10^{-4}	1.71×10^{-4}	2.08×10^{-4}	2.15×10^{-4}	2.02×10^{-4}	1.82×10^{-4}	1.25×10^{-4}	7.97×10^{-5}	2.64×10^{-8}	2
IADE-DY											
IMP	6.10×10^{-4}	1.18×10^{-3}	1.31×10^{-3}	1.34×10^{-3}	1.29×10^{-3}	1.16×10^{-3}	9.56×10^{-4}	6.83×10^{-4}	3.57×10^{-4}	1.09×10^{-6}	3
CN	4.80×10^{-5}	9.04×10^{-5}	7.96×10^{-5}	6.72×10^{-5}	5.47×10^{-5}	4.29×10^{-5}	3.21×10^{-5}	2.21×10^{-5}	1.18×10^{-5}	3.09×10^{-9}	2
Exact Solution	0.0022224	0.0042273	0.0058184	0.0068399	0.0071919	0.0068399	0.0058184	0.0042273	0.0022224	–	–



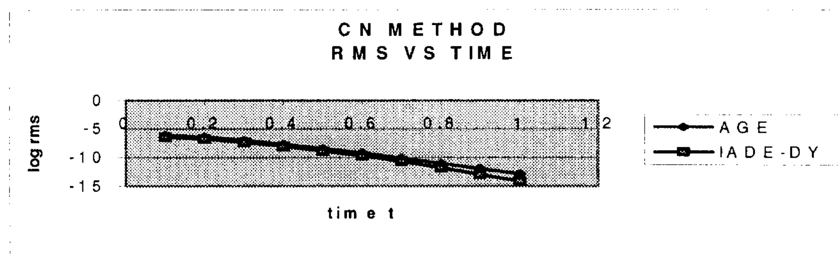
$\Delta x = 0.1$, $r = 0.1(0.1)1.0$, $t = 1.0$



$\Delta x = 0.1$, r (AGE) $= 0.5$, r(IADE) $= 0.6$, $t = 0.1(0.1)1.0$



$\Delta x = 0.1$, $r = 0.1(0.1)1.0$, $t = 1.0$



$\Delta x = 0.1$, r (AGE) $= 0.5$, r(IADE) $= 0.4$, $t = 0.1(0.1)1.0$

Figure 1. For Problem 1 ($\lambda = 0.5$).

Experiment 2 (Problem 2)

This experiment deals with the following problem:

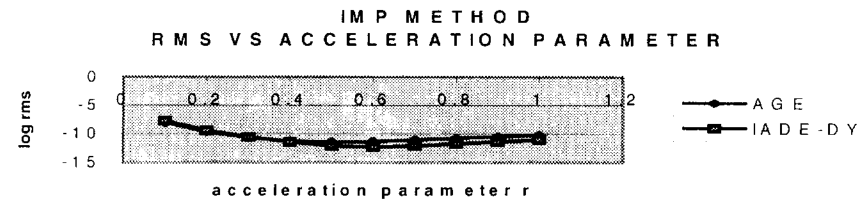
$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}, \qquad 0 \le x \le 1, \tag{28}$$

subject to the initial condition

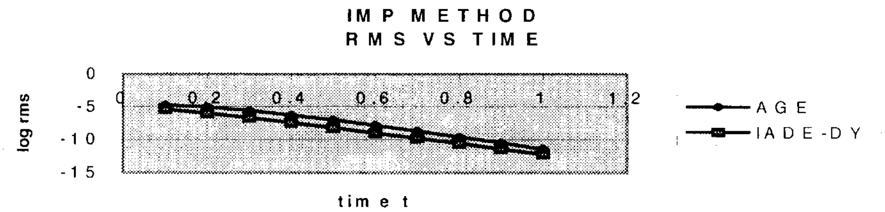
$$U(x,0) = \sin(\pi x), \qquad 0 \le x \le 1, \tag{29}$$

and the boundary conditions

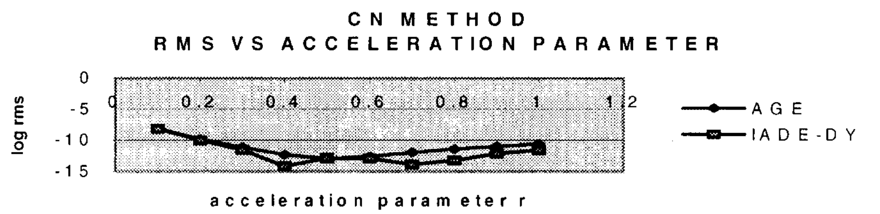
$$U(0,t) = U(1,t) = 0, \qquad t \ge 0. \tag{30}$$



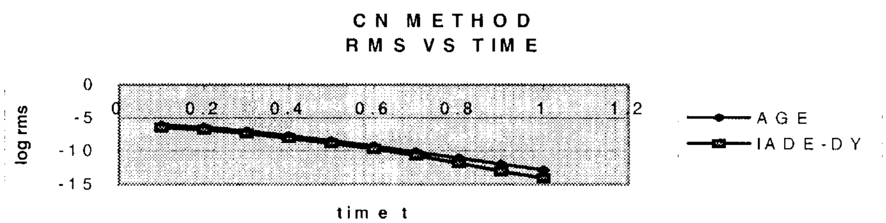
$\Delta x = 0.1$, $r = 0.1(0.1)1.0$, $t = 1.0$



$\Delta x = 0.1$, r (AGE) $= 0.5$, r(IADE) $= 0.6$, $t = 0.1(0.1)1.0$



$\Delta x = 0.1$, $r = 0.1(0.1)1.0$, $t = 1.0$



$\Delta x = 0.1$, r (AGE) $= 0.5$, r(IADE) $= 0.4$, $t = 0.1(0.1)1.0$

Figure 2. For Problem 2 ($\lambda = 0.5$).

The exact solution is given by

$$U(x,t) = e^{-\pi^2 t}\sin(\pi x). \tag{31}$$

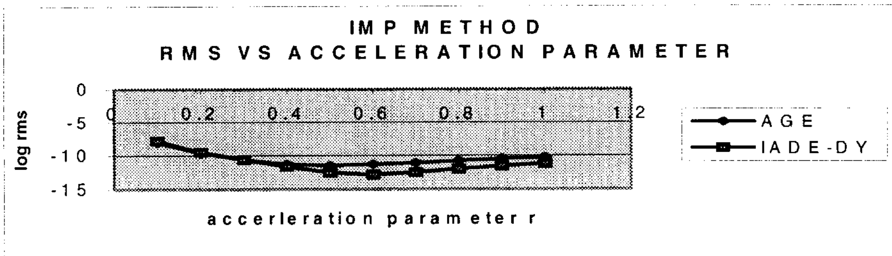
Experiment 3 (Problem 3)

This experiment deals with the following problem taken from [3]:

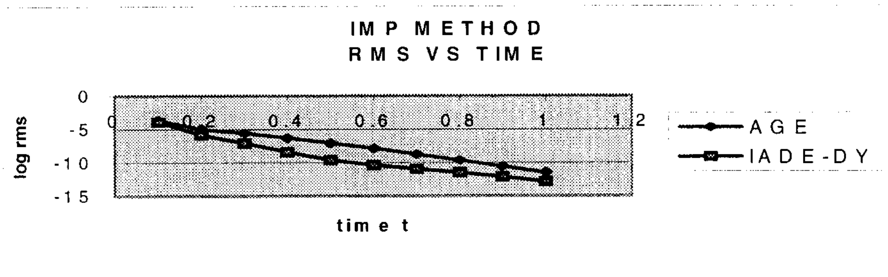
$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}, \qquad 0 \le x \le 1, \tag{32}$$

subject to the initial condition

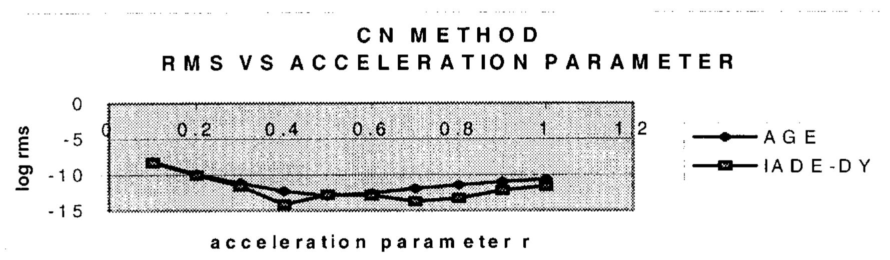
$$U(x,0) = \sin(\pi x)(1 + 6\cos(\pi x)), \tag{33}$$



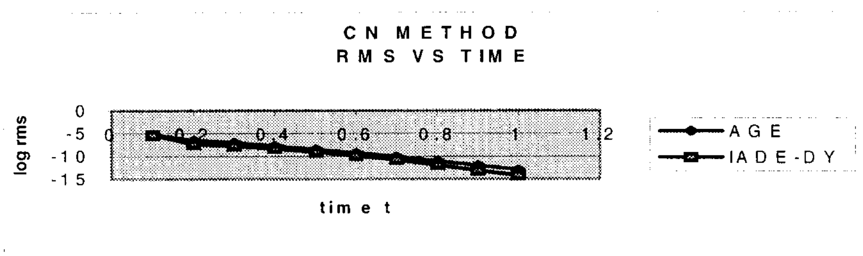
$\Delta x = 0.1$, $r = 0.1(0.1)1.0$, $t = 1.0$



$\Delta x = 0.1$, r (AGE) = 0.5, r(IADE) = 0.6, t= 0.1(0.1)1.0



$\Delta x = 0.1$, $r = 0.1(0.1)1.0$, $t = 1.0$



$\Delta x = 0.1$, r (AGE) = 0.5, r(IADE) = 0.4, $t = 0.1(0.1)1.0$

Figure 3. For Problem 3 ($\lambda = 0.5$).

and the periodic boundary conditions

$$U(0,t) = U(1,t) = 0, \qquad t \geq 0. \tag{34}$$

The exact solution is given by

$$U(x,t) = \sin(\pi x)e^{-\pi^2 t} + 3\sin(2\pi x)e^{-4\pi^2 t}. \tag{35}$$

Table 1 provides the absolute errors of the numerical solution for Problem 1 at each of the grid points at time levels of $t = 0.25$ and $t = 0.5$ using a mesh ratio of $\lambda = 0.5$ and $\lambda = 1.0$, respectively. The higher accuracy of the IADE scheme when compared with the AGE method is evidenced from these errors and is also reflected from the lower magnitude of the root mean square error (rms). The same observation is also featured in Tables 2 and 3 for Problems 2 and 3. Obviously, the higher accuracy is more prominent when the IADE scheme employs the $(2,2)$ accurate method of C-N as opposed to the $(2,1)$ accurate fully implicit formula. These are achieved through an iterative process which requires only two to three iterations.

Figures 1–3 show a plot of the logarithm of the root mean square (rms) of the numerical solutions vs. the acceleration parameter r for values of r in the range 0.1 to 1 in steps of 0.1 for the three problems. In the same graphs, we also construct plots of $\log \text{rms}$ vs. time for $t = 0.1(0.1)1$ with different grid sizes, but with r chosen to provide the most rapid convergence. The relatively higher accuracy of the IADE scheme is amply displayed from these graphs.

The IADE scheme using the D'Yakonov variant, therefore, affords its users many advantages as an alternative iterative method with respect to stability, accuracy, and rate of convergence. And, as the method is fully explicit, its feature can be fully utilized for parallelization.

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