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Coefficient bounds for biholomorphic mappings which have a parametric representation ☆

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ABSTRACT

Let B be the unit ball in $\mathbb{C}^n$ with respect to an arbitrary norm $\|\cdot\|$ and let $f(z,t)$ be a g-Loewner chain such that $z=0$ is a zero of order $k+1$ of $e^{-t}f(z,t)-z$ for each $t\geqslant 0$. In this paper, the authors obtain coefficient bounds of mappings in $S^0_{g,k+1}(B)$. These results generalize the related works of Hamada, Honda and Kohr.

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1. Introduction

Let $\mathbb{C}^n$ denote the space of n complex variables $z=(z_1,\dots,z_n)$ with respect to an arbitrary norm $\|\cdot\|$. Let $B=\{z\in\mathbb{C}^n\colon \|z\|<1\}$. Let B^n be the Euclidean unit ball in $\mathbb{C}^n$, D be the unit disc in $\mathbb{C}$. Let $L(\mathbb{C}^n,\mathbb{C}^m)$ be the space of all continuous linear operators from $\mathbb{C}^n$ into $\mathbb{C}^m$. For each $z\in\mathbb{C}^n\setminus\{0\}$, we define $T(z)=\{l_z\in L(\mathbb{C}^n,\mathbb{C})\colon \|l_z\|=1,\ l_z(z)=\|z\|\}$. According to the Hahn–Banach theorem, $T(z)$ is nonempty. Let $H(B)$ be the set of all holomorphic mappings from B into $\mathbb{C}^n$. Notice that for fixed $z\in\mathbb{C}^n$, $\forall\alpha(\neq 0)\in\mathbb{C}$, when l_z is chosen and fixed, then $\|\frac{|\alpha|}{\alpha}l_z\|=\|l_z\|\leqslant 1$, and $\frac{|\alpha|}{\alpha}l_z(\alpha z)=\frac{|\alpha|}{\alpha}\alpha l_z(z)=|\alpha|\|z\|=\|\alpha z\|$, so we can assume $l_{\alpha z}=\frac{|\alpha|}{\alpha}l_z$. A holomorphic mapping $f:B\to\mathbb{C}^n$ is said to be biholomorphic if the inverse f^{-1} exists and is holomorphic on the open set $f(B)$. A mapping $f\in H(B)$ is said to be locally biholomorphic if the Fréchet derivative $Df(z)$ has a bounded inverse for each $z\in B$. We say that f is normalized if $f(0)=0$ and $Df(0)=I$, where I represents the identity in $L(\mathbb{C}^n,\mathbb{C}^n)$. Let $S(B)$ be the set of all normalized biholomorphic mappings.

If $f,g\in H(B)$, we say that f is subordinate to g ($f\prec g$) if there exists a Schwarz mapping v (i.e. $v\in H(B)$ and $\|v(z)\|\leqslant\|z\|$, $z\in B$) such that $f=g\circ v$. A mapping $F:B\times[0,\infty]\to\mathbb{C}^n$ is called a Loewner chain if $F(\cdot,t)$ is biholomorphic on B, $F(0,t)$ is biholomorphic on B, $F(0,t)=0$, $DF(0,t)=e^tI$ for $t\geqslant 0$ and

$$F(z,s)\prec F(z,t),\quad z\in B,\ 0\leqslant s\leqslant t<\infty.$$

The following set play a key role in our discussion:

$$\mathcal{M}=\big\{h\in H(B)\colon\ h(0)=0,\ Dh(0)=I,\ \Re e\big[l_z\big(h(z)\big)\big]\geqslant 0,\ z\in B,\ l_z\in T(z)\big\}.$$

In [7] (see also [2,6]), the following result is proved:

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Lemma 1. *Let $f(z,t)$ be a Loewner chain and $v = v(z,s,t)$ be the transition mapping of $f(z,t)$. Then $f(z,\cdot)$ is locally Lipschitz continuous on $[0,\infty)$, locally uniformly with respect to $z \in B$, and there exists a mapping $h = h(z,t)$ such that $h(\cdot,t) \in \mathcal{M}$, $t \geqslant 0$, $h(z,\cdot)$ is measurable on $[0,\infty)$, and*

$$\frac{\partial f}{\partial t}(z,t) = Df(z,t)h(z,t), \quad \text{a.e. } t \geqslant 0$$

and for all $z \in B$. Also $v(z,s,t)$ satisfies the initial value problem

$$\frac{\partial v}{\partial t} = -h(v,t), \quad \text{a.e. } t \geqslant s, \qquad v(z,s,s) = z,$$

and for all $z \in B$ and $s \geqslant 0$. Moreover, if $\{e^{-t} f(z,t)\}_{t \geqslant 0}$ is a normal family on B, then for every $s \geqslant 0$,

$$\lim_{t\to\infty} e^t v(z,s,t) = f(z,s)$$

and the above limit holds locally uniformly on B.

Definition 2. (See [2].) Let $f : B \to \mathbb{C}^n$ be a normalized holomorphic mapping. We say that f has parametric representation if there exists a mapping $h = h(z,t)$ which satisfies the condition in Lemma 1 such that $f(z) = \lim_{t\to\infty} e^t v(z,t)$ locally uniformly on B, where $v = v(z,t)$ is the unique solution of the initial value problem

$$\frac{\partial v}{\partial t} = -h(v,t), \quad \text{a.e. } t \geqslant 0, \qquad v(z,0) = z,$$

for all $z \in B$.

Let $S^0(B)$ be the set of all mappings which have parametric representation on B. Then $S^0(B) \subset S(B)$ [2]. It is well known that in the case of one variable $S^0(D) = S(D)$; however, in $\mathbb{C}^n$, $n \geqslant 2$, $S^0(B) \subsetneqq S(B)$ [14].

Definition 3. Let $g \in H(D)$ be a biholomorphic function such that $g(0) = 1$, $g(\bar{\xi}) = \overline{g(\xi)}$, for $\xi \in D$, $\Re e\, g(\xi) > 0$ on $\xi \in D$, and assume g satisfies the following conditions for $r \in (0,1)$:

$$\begin{cases} \min\limits_{|\xi|=r} \Re e\, g(\xi) = \min\bigl\{g(r), g(-r)\bigr\}, \\ \max\limits_{|\xi|=r} \Re e\, g(\xi) = \max\bigl\{g(r), g(-r)\bigr\}. \end{cases}$$

We define $\mathcal{M}_g$ to be the class of mappings given by

$$\mathcal{M}_g = \left\{ p \in H(B)\colon\ p(0) = 0,\ Dp(0) = I,\ \frac{1}{\|z\|} l_z\bigl(p(z)\bigr) \in g(D),\ z \in B \backslash \{0\},\ l_z \in T(z) \right\}.$$

The class $\mathcal{M}_g$ has been introduced by Kohr [13] on B^n and by Graham, Hamada and Kohr [2] on the unit ball with respect to an arbitrary norm in $\mathbb{C}^n$.

Definition 4. (See [6].) Let $g : D \to \mathbb{C}$ be a biholomorphic function satisfying the assumptions of Definition 1. Also let $f \in H(B)$. We say that $f \in S_g^0(B)$ if there exists a mapping $h : B \times [0,\infty] \to \mathbb{C}^n$ which satisfies the conditions

(i) for each $t \geqslant 0$, $h(\cdot,t) \in \mathcal{M}_g$;
(ii) for each $z \in B$, $h(z,t)$ is a measurable function of $t \in [0,\infty]$;
(iii) $\lim_{t\to\infty} e^t v(z,t) = f(z)$ locally uniformly on B, where $v = v(z,t)$ is the solution of the initial value problem

$$\frac{\partial v}{\partial t} = -h(v,t), \quad \text{a.e. } t \geqslant 0, \qquad v(z,0) = z,$$

for all $z \in B$.

The class $S_g^0(B)$ is called the class of mappings which have g-parametric representation on B. We say that a mapping $f : B \times [0,\infty] \to \mathbb{C}^n$ is a g-Loewner chain if and only if $f(z,t)$ is a Loewner chain such that $\{e^{-t} f(z,t)\}_{t \geqslant 0}$ is a normal family on B and the mapping $h(z,t)$ which occurs in the Loewner differential equation

$$\frac{\partial f}{\partial t}(z,t) = Df(z,t)h(z,t), \quad \text{a.e. } t \geqslant 0,$$

satisfies $h(\cdot,t) \in \mathcal{M}_g$ for a.e. $t \geqslant 0$ (see [2,3,12]). Obviously, if $g(\xi) = \frac{(1+\xi)}{1-\xi}$, $\xi \in D$, then $S_g^0(B)$ reduces to the set $S^0(B)$.

We denote by S_{k+1}^0 (respectively $S_{g,k+1}^0(B)$) the subset of $S^0(B)$ (respectively $S_g^0(B)$) consisting of mappings f for which there exists a Loewner chain (respectively a g-Loewner chain) $f(z,t)$ such that $\{e^{-t} f(z,t)\}_{t \geqslant 0}$ is a normal family on B, $f = f(\cdot,0)$ and $z = 0$ is a zero of order $k+1$ of $e^{-t} f(z,t) - z$ for each $t \geqslant 0$ (see [4,5,7,8,10,11,13]).

The aim of this paper is to give coefficient bounds in $S_{g,k+1}^0(B)$. These results generalize the corresponding results of [10].

2. Preliminaries

In order to prove the desired results, we first give some lemmas.

Lemma 5. *(See [1].) If $f(z) = a_0 + \sum_{n=1}^{\infty} a_n z^n \in H(D)$, and $f(D) \subset D$, then*

$$|a_n| \leqslant 1 - |a_0|^2, \quad n = 1, 2, \ldots.$$

The following formula is that of Faà di Bruno to deal with the higher derivatives of compound functions.

Lemma 6. *(See [15].) Let G, Ω be domains in $\mathbb{C}$, $f \in H(G)$, $g \in H(\Omega)$. If $f(G) \subset \Omega$, then*

$$(g \circ f)^{(n)}(z) = \sum \frac{n!}{l_1! \cdots l_n!} g^{(l)}\big(f(z)\big)\left(\frac{f'(z)}{1!}\right)^{l_1} \cdots \left(\frac{f^{(n)}(z)}{n!}\right)^{l_n}, \quad z \in G,$$

where $l = l_1 + \cdots + l_n$ and the sum is over all $l_1, \ldots, l_n$ for which $l_1 + 2l_2 + \cdots + nl_n = n$.

Lemma 7. *If $f \in H(D)$, g is a biholomorphic function on D, $f(0) = g(0)$, $f'(0) = \cdots = f^{(k-1)}(0) = 0$, and $f \prec g$, then*

$$\frac{|f^{(n)}(0)|}{n!} \leqslant \left|g'(0)\right|, \quad n = k, \ldots, 2k-1.$$

Proof. Since $f \prec g$, there exists a function $\varphi \in H(D, D)$, $\varphi(0) = 0$ such that $\varphi = g^{-1} \circ f$. By Lemma 6, we have

$$\varphi^{(n)}(0) = \big(g^{-1} \circ f\big)^{(n)}(0) = \sum \frac{n!}{l_1! \cdots l_n!}\big[g^{-1}\big]^{(l)}\big(f(0)\big)\left(\frac{f'(0)}{1!}\right)^{l_1} \cdots \left(\frac{f^{(n)}(0)}{n!}\right)^{l_n},$$

where $l = l_1 + \cdots + l_n$ and the sum is over all $l_1, \ldots, l_n$ for which $l_1 + 2l_2 + \cdots + nl_n = n$. In view of the assumption of Lemma 7 and the above equality, we easily deduce that

$$\varphi^{(n)}(0) = \frac{f^{(n)}(0)}{g'(0)}, \quad n = k, \ldots, 2k-1.$$

Therefore, according to Lemma 5, we obtain

$$\frac{|f^{(n)}(0)|}{n!} \leqslant \left|g'(0)\right|, \quad n = k, \ldots, 2k-1.$$

This completes the proof. □

3. Main results

Theorem 8. *Let g satisfy the assumptions of Definition 3 and $f \in S^0_{g,k+1}(B)$. Then*

$$\frac{|l_z(D^m f(0)(z^m))|}{m!} \leqslant \frac{1}{m-1}\left|g'(0)\right|\|z\|^m, \quad z \in B,\ l_z \in T(z),\ m = k+1, \ldots, 2k.$$

Proof. Since $f \in S^0_{g,k+1}(B)$, there is a g-Loewner chain $f(z,t)$ such that $f(z) = f(z,0)$, $z \in B$. Also there exist a mapping $h_t(z) = h(z,t) \in \mathcal{M}_g$ for each $t \geqslant 0$, measurable in t for each $z \in B$, such that for almost all $t \geqslant 0$,

$$\frac{\partial f}{\partial t}(z,t) = Df(z,t)h(z,t), \quad \forall z \in B. \tag{1}$$

Fix $z \in B \setminus \{0\}$, $l_z \in T(z)$, $t_0 \geqslant 0$ and denote $z_0 = \frac{z}{\|z\|}$. Let $p_{t_0} : D \to \mathbb{C}$ be given by

$$p_{t_0}(\xi) = \begin{cases} \frac{1}{\xi} l_z(h_{t_0}(\xi z_0)), & \xi \neq 0, \\ 1, & \xi = 0. \end{cases} \tag{2}$$

Then $p_{t_0} \in H(D)$, $p_{t_0}(0) = g(0) = 1$, and since $h_{t_0}(z) \in \mathcal{M}_g$, we deduce that

$$p_{t_0}(\xi) = \frac{1}{\xi} l_z\big(h_{t_0}(\xi z_0)\big) = \frac{1}{\xi} l_{z_0}\big(h_{t_0}(\xi z_0)\big) = \frac{1}{\|\xi z_0\|} l_{\xi z_0}\big(h_{t_0}(\xi z_0)\big) \in g(D), \quad \xi \in D \setminus \{0\}.$$

Therefore $p_{t_0} \prec g$. Using the fact that $z = 0$ is a zero of order $k+1$ of $e^{-t} f(z,t) - z$, we have

$$f(\xi z,t)=e^t z\xi+\sum_{m=k+1}^{\infty}\frac{D^m f(0,t)(z^m)}{m!}\xi^m$$

and

$$\frac{\partial f}{\partial t}(z\xi,t)=e^t z\xi+\sum_{m=k+1}^{\infty}\frac{\partial}{\partial t}\left[\frac{D^m f(0,t)(z^m)}{m!}\right]\xi^m.$$

After simple computations, in view of (1), we obtain for almost all $t\geqslant 0$ that

$$h(\xi z,t)=z\xi+\sum_{m=k+1}^{\infty}\frac{D^m h(0,t)(z^m)}{m!}\xi^m \tag{3}$$

and

$$\frac{\partial}{\partial t}\left[\frac{D^m f(0,t)(z^m)}{m!}\right]=\frac{D^m f(0,t)(z^m)}{(m-1)!}+\frac{e^t D^m h(0,t)(z^m)}{m!},\quad m=k+1,\dots,2k, \tag{4}$$

where $z\in B$, and $\xi\in D$. Taking into account (2) and (3), for $z=z_0$, and $t=t_0\geqslant 0$ such that (1) holds, we have

$$p_{t_0}(\xi)=1+\sum_{m=k+1}^{\infty}\frac{l_z(D^m h(0,t_0)(z_0^m))}{m!}\xi^{m-1}. \tag{5}$$

It is clear that $p_{t_0}(\xi)$ satisfies the hypothesis of Lemma 7, thus we have

$$\frac{|p_{t_0}^{(n)}(0)|}{n!}\leqslant\left|g'(0)\right|,\quad n=k,\dots,2k-1. \tag{6}$$

Combining the relations (5) and (6), we obtain

$$\frac{|l_z(D^m h_{t_0}(0)(z^m))|}{m!}\leqslant\left|g'(0)\right|\|z\|^m,\quad z\in B\setminus 0,\ l_z\in T(z),\ m=k+1,\dots,2k. \tag{7}$$

Let

$$q_{m,z}(T)=e^{-mT}D^m f(0,T)\bigl(z^m\bigr)-D^m f(0,0)\bigl(z^m\bigr)-\int_0^T e^{-(m-1)t}D^m h(0,t)\bigl(z^m\bigr)\,dt,\quad m=k+1,\dots,2k,$$

for fixed $z\in B$ and $T\geqslant 0$. Since $q_z'(T)=0$ for almost all $T\geqslant 0$ by (4), we have $q_z(T)=q_z(0)=0$. From this we have the equality

$$e^{-mT}l_z\bigl(D^m f(0,T)\bigl(z^m\bigr)\bigr)-l_z\bigl(D^m f(0,0)\bigl(z^m\bigr)\bigr)=\int_0^T l_z\bigl(e^{-(m-1)t}D^m h(0,t)\bigl(z^m\bigr)\bigr)\,dt,\quad m=k+1,\dots,2k. \tag{8}$$

Next, in view of Corollary 11 in [10], we have

$$\bigl\|f(z,T)\bigr\|\leqslant e^T\|z\|\exp\int_0^{\|z\|}\left[\frac{1}{\min\{g(x^k),g(-x^k)\}}-1\right]\frac{dx}{x},\quad z\in B. \tag{9}$$

Using the Cauchy formula

$$\frac{1}{m!}D^m f(0,T)\bigl(u^m\bigr)=\frac{1}{2\pi i}\int_{|\zeta|=r}\frac{f(\zeta u,T)}{\zeta^{m+1}}d\zeta,\quad r<1,$$

for $u\in\mathbb{C}^n$, $\|u\|=1$, and taking into account (9), we easily obtain that

$$\lim_{T\to+\infty}e^{-mT}D^m f(0,T)\bigl(z^m\bigr)=0,\quad m\geqslant k+1.$$

Letting $T\to+\infty$ in (8) and using the above equality and (6), we deduce that

$$\frac{|l_z(D^m f(0)(z^m))|}{m!}\leqslant\frac{1}{m-1}\left|g'(0)\right|\|z\|^m,\quad z\in B,\ l_z\in T(z),\ m=k+1,\dots,2k.$$

This completes the proof. □

Remark 9. Theorem 8 generalizes the corresponding result of [10], when $m = k + 1$, Theorem 8 was obtained by Hamada, Honda and Kohr [10]. Moreover, Theorem 1 improves some results of Hamada and Honda [9] by omitting the convexity assumption on $g(z)$.

Using Theorem 8, we obtain the following corollary by an argument similar to that in the proof of [10, Corollary 25].

Corollary 10. *Let g satisfy the assumptions of Definition* 3 *and* $f \in S^0_{g,k+1}(B)$. *Then*

$$\left\| \frac{1}{m!} D^m f(0)\bigl(w^m\bigr) \right\| \leqslant b_{m-1} \bigl|g'(0)\bigr|, \quad m = k+1, \dots, 2k, \ \|w\| = 1,$$

where $b_{m-1} = \frac{m^{\frac{m}{m-1}}}{m-1}$.

For $g(\zeta) = \frac{1+\zeta}{1-\zeta}$, $\zeta \in D$, we obtain the following corollary.

Corollary 11. *If* $f \in S^0_{k+1}(B)$, *then*

$$\frac{|l_w(D^m f(0)(w^m))|}{m!} \leqslant \frac{2}{m-1}, \quad m = k+1, \dots, 2k, \ \|w\| = 1, \ l_w \in T(w).$$

Moreover, for $\|w\| = 1$, *we have*

$$\left\| \frac{1}{m!} D^m f(0)\bigl(w^m\bigr) \right\| \leqslant 2b_{m-1}, \quad m = k+1, \dots, 2k, \ \|w\| = 1,$$

where $b_{m-1} = \frac{m^{\frac{m}{m-1}}}{m-1}$.

Remark 12. Corollaries 10, 11 generalize the corresponding results of [10], when $m = k + 1$, Corollaries 10, 11 were obtained by Hamada, Honda and Kohr [10].

At present, we do not know whether the following conjecture is true for the class $f \in S^0_{k+1}(B)$. This is a version of the Bieberbach conjecture in several complex variables.

Conjecture 13. *If* $f \in S^0_{k+1}(B)$, *then*

$$\frac{|l_w(D^m f(0)(w^m))|}{m!} \leqslant \frac{2}{m-1}, \quad m \geqslant k+1, \ \|w\| = 1, \ l_w \in T(w).$$

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