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Shared Values and Normal Families of Meromorphic Functions[1]

Huaihui Chen

E-mail: hhchen@pine.njnu.edu.cn

and

Mingliang Fang

Department of Mathematics, Nanjing Normal University, Nanjing 210097
E-mail: mlfang@pine.njnu.edu.cn

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In this paper, we study the normality of a family of meromorphic functions concerning shared values and prove the following theorem: Let $\mathscr{F}$ be a family of meromorphic functions in a domain D, let $k \geq 2$ be a positive integer, and let a, b, c be complex numbers such that $a \neq b$. If, for each $f \in \mathscr{F}$, f and $f^{(k)}$ share a and b in D, and the zeros of $f(z) - c$ are of multiplicity $\geq k+1$, then $\mathscr{F}$ is normal in D.

Key Words: meromorphic function; normality; shared value.

1. INTRODUCTION

Let D be a domain in $\mathbb{C}$, f a meromorphic function, and $a \in \mathbb{C}$. Set

$$\overline{E}(a, f) = \{z : z \in D, f(z) = a\}.$$

Two meromorphic functions f and g are said to share the value a in D if $\overline{E}(a, f) = \overline{E}(a, g)$. If two meromorphic functions f and g share the value a in $\mathbb{C}$, then we say that f and g share a.

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Mues and Steinmetz [10] proved

THEOREM A. *Let f be a non-constant meromorphic function and let a_1, a_2, and a_3 be distinct complex numbers. If f and f' share a_1, a_2, a_3, then $f \equiv f'$.*

Schwick [16] discovered a connection between normality criteria and shared values. He proved

THEOREM B. *Let $\mathscr{F}$ be a family of meromorphic functions in a domain D and let a_1, a_2, and a_3 be distinct complex numbers. If f and f' share a_1, a_2, and a_3 in D for each $f \in \mathscr{F}$, then $\mathscr{F}$ is normal in D.*

This result has undergone various extensions [12, 19, 20], culminating in the following result of Pang and Zalcman [13].

THEOREM C. *Let $\mathscr{F}$ be a family of meromorphic functions in a domain D and let a, b be two non-zero distinct complex numbers. If f and f' share a and b in D for each $f \in \mathscr{F}$, then $\mathscr{F}$ is normal in D.*

Frank and Schwick [6] generalized Theorem A as follows

THEOREM D. *Let f be a non-constant meromorphic function, k a positive integer, and let a_1, a_2, and a_3 be distinct complex numbers. If f and $f^{(k)}$ share a_1, a_2, a_3, then $f \equiv f^{(k)}$.*

Naturally, we ask what can be stated if f' is replaced by $f^{(k)}$ for $k \geq 2$ in Theorems B–C. Frank and Schwick [7] observed that Theorem B does not admit the obvious extension obtained by replacing f' by $f^{(k)}$. In this paper, we prove

THEOREM 1. *Let $\mathscr{F}$ be a family of meromorphic functions in a domain D, let $k \geq 2$ be a positive integer, and let a, b, c be complex numbers such that $a \neq b$. If, for each $f \in \mathscr{F}$, f and $f^{(k)}$ share a and b in D, and the zeros of $f(z) - c$ are of multiplicity $\geq k+1$, then $\mathscr{F}$ is normal in D.*

For a family of holomorphic functions we have

THEOREM 2. *Let $\mathscr{F}$ be a family of holomorphic functions in a domain D, let $k \geq 2$ be a positive integer, and let a, b, c be complex numbers such that $a \neq b$. If, for each $f \in \mathscr{F}$, f and $f^{(k)}$ share a and b in D, and the zeros of $f(z) - c$ are of multiplicity $\geq k$, then $\mathscr{F}$ is normal in D.*

Remark 1. The following example shows that some assumption on the zeros of $f(z) - c$ is required for Theorems 1 and 2 to hold.

Let $\mathscr{F} = \{f_n(z): f_n(z) = n(e^z - e^{\lambda z}), n = 1, 2, 3, \ldots\}$, where $\lambda^k = 1$, $\lambda \neq 1$, $k \geq 2$, $D = \{z : |z| < 1\}$. Then $\mathscr{F}$ is a family of holomorphic functions in a domain D. Obviously, for each $f \in \mathscr{F}$, $f \equiv f^{(k)}$, f and $f^{(k)}$ share any number b in D. But $\mathscr{F}$ is not normal in D.

2. SOME LEMMAS

For the proof of our results, we need the following lemmas.

LEMMA 1 [2, 3, 9, 15, 20]. *Let $\mathcal{F}$ possess the property that every function $f \in \mathcal{F}$ has only zeros of multiplicity at least k. If $\mathcal{F}$ is not normal at a point* 0, *then for* $0 \leq \alpha < k$, *there exist*

(a) *a number* r, $0 < r < 1$;

(b) *a sequence of complex numbers* $z_n \to 0$, $|z_n| < r < 1$;

(c) *a sequence of functions* $f_n \in \mathcal{F}$; *and*

(d) *a sequence of positive numbers* $\rho_n \to 0$

such that $g_n(\xi) = \rho_n^{-\alpha} f_n(z_n + \rho_n \xi)$ *converges locally uniformly with respect to the spherical metric to a non-constant meromorphic function* $g(\xi)$ *on* $\mathbb{C}$, *and moreover*, g *is of order at most two*.

Remark 2. In Lemma 1, if $\mathcal{F}$ is a family of holomorphic functions, then $g(\xi)$ is of order at most one (see [3]). If $\mathcal{F}$ satisfies the additional assumption that there exists $M > 0$ such that $|f^{(k)}(z)| \leq M$ whenever $f(z) = 0$ for any $f \in \mathcal{F}$, then we can take $\alpha = k$ (see [12]).

LEMMA 2 [17, p. 22]. *Let* $R(z)$ *be a non-constant rational function, let* k *be a positive integer, and let* b *be a non-zero complex number. If the zeros of* $R(z)$ *are of multiplicity at least* $k + 1$, *and* $R^{(k)}(z) \neq b$, *then* $R(z) = (\gamma z + \delta)^{k+1}/(\alpha z + \beta)$, *where* α, β, γ, δ *are constants such that* $\alpha\gamma \neq 0$, $|\beta| + |\delta| \neq 0$.

LEMMA 3 [1, p. 360; 17, p. 21; 18, p. 34]. *Let* $f(z)$ *be a transcendental meromorphic function of finite order*, k *a positive integer. If the zeros of* $f(z)$ *are of multiplicity at least* $k + 1$, *then* $f^{(k)} - b$ *has infinitely many zeros for any non-zero complex number* b.

LEMMA 4. *Let* $f(z)$ *be a meromorphic function of finite order, let* b *be a non-zero complex number, and let* k *be a positive integer. If the zeros of* $f(z)$ *are of multiplicity at least* k, $\overline{E}(0, f) = \overline{E}(0, f^{(k)})$ *and* $f^{(k)}(z) \neq b$, *then* $f(z)$ *is a constant*.

Proof. Obviously, the zeros of $f(z)$ are of multiplicity at least $k + 1$ by the assumption, and f cannot be a polynomial of degree $k + 1$. If $f(z)$ is a transcendental meromorphic function with finite order, then by Lemma 3, $f^{(k)} = b$ has infinitely many solutions, a contradiction. Hence $f(z)$ is a rational function. Suppose that $f(z)$ is a non-constant rational function. Then it follows from Lemma 2 that $f(z) = (\gamma z + \delta)^{k+1}/(\alpha z + \beta)$, where α, β, γ, δ are constants such that $\alpha\gamma \neq 0$, $|\beta| + |\delta| \neq 0$. Hence $f^{(k)}(z) = b + A/(\alpha z + \beta)^{k+1}$, where A is a non-zero constant, $\overline{E}(0, f^{(k)}) = \{z : b$

$+ A/(\alpha z + \beta)^{k+1} = 0\}$, and $\overline{E}(0, f) = \{-\delta/\gamma\}$, $\overline{E}(0, f) \neq \overline{E}(0, f^{(k)})$. We arrive at a contradiction. This completes the proof of the lemma.

LEMMA 5 [8, p. 14; 9, p. 60]. *Let $f(z)$ be a meromorphic function, let a be a non-zero complex number, and let k be a positive integer. If $f(z) \neq 0$, $f^{(k)}(z) \neq a$, then $f(z)$ is a constant.*

LEMMA 6 [21, p. 38]. *Let $f(z)$ be a transcendental meromorphic function, and let $a_1(z)$, $a_2(z)$ be distinct meromorphic functions satisfying $T(r, a_i) = S(r, f)$, $i = 1, 2$. Then*

$$T(r, f) \leq \overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f - a_1}\right) + \overline{N}\left(r, \frac{1}{f - a_2}\right) + S(r, f).$$

3. PROOFS OF THEOREMS 1 AND 2

Proof of Theorem 1. We may assume that $D = \{|z| < 1\}$. Suppose that $\mathscr{F}$ is not normal in D; without loss of generality, we assume that $\mathscr{F}$ is not normal at $z_0 = 0$. Then by Lemma 1, there exist

(a) a number r, $0 < r < 1$;

(b) a sequence of complex numbers $z_n \to 0$, $|z_n| < r < 1$;

(c) a sequence of functions $f_n \in \mathscr{F}$; and

(d) a sequence of positive numbers $\rho_n \to 0$

such that $g_n(\xi) = \rho_n^{-k}(f_n(z_n + \rho_n \xi) - c)$ converges locally uniformly with respect to the spherical metric to a non-constant meromorphic function $g(\xi)$. Moreover, $g(\xi)$ is of order at most two.

By Hurwitz's theorem, the zeros of $g(\xi)$ are of multiplicity $\geq k + 1$. Now, we distinguish three cases.

Case 1. $c = a = 0$, $b \neq 0$. Then we know that the zeros of $f(z)$ are of multiplicity $\geq k + 1$, and f and $f^{(k)}$ share 0 and b, for each $f \in \mathscr{F}$. Since the zeros of $g(\xi)$ are of multiplicity $\geq k + 1$, we have $\overline{E}(0, g) \subset \overline{E}(0, g^{(k)})$.

Obviously, $g^{(k)} \not\equiv 0$. Suppose that $g^{(k)}(\xi_0) = 0$. Then there exist ξ_n, $\xi_n \to \xi_0$, such that

$$g_n^{(k)}(\xi_n) = f_n^{(k)}(z_n + \rho_n \xi_n) = 0, \qquad n = 1, 2, \dots .$$

Hence $f_n(z_n + \rho_n \xi_n) = 0$ and $g_n(\xi_n) = 0$ for $n = 1, 2, \dots$, since f_n and $f_n^{(k)}$ share 0. Thus we get $g(\xi_0) = \lim_{n \to \infty} g_n(\xi_n) = 0$. This shows that $\overline{E}(0, g^{(k)}) \subset \overline{E}(0, g)$. Hence we have proved that $\overline{E}(0, g) = \overline{E}(0, g^{(k)})$. It is clearly that $g^{(k)} \not\equiv b$.

Next, suppose that there exists ξ_0 satisfying $g^{(k)}(\xi_0) = b$. Then, by Hurwitz's theorem, there exists a sequence ξ_n such that $\xi_n \to \xi_0$ and $g_n^{(k)}(\xi_n) = f_n^{(k)}(z_n + \rho_n \xi_n) = b$ for $n = 1, 2, \ldots$. Since f_n and $f_n^{(k)}$ share b, we have $f_n(z_n + \rho_n \xi_n) = b$ and $g_n(\xi_n) = \rho_n^{-k} f_n(z_n + \rho_n \xi_n) = \rho_n^{-k} b \to \infty$. This contradicts that $\lim_{n\to\infty} g_n(\xi_n) = g(\xi_0) \neq \infty$. So, $g^{(k)}(\xi) \neq b$. Now by Lemma 4 we conclude that $g(\xi)$ is a constant, a contradiction.

Case 2. $c = a \neq 0$. Then we have

$$\overline{E}(0, g) = \overline{E}(a, g^{(k)}). \tag{3.1}$$

Indeed, suppose that $g(\xi_0) = 0$. Then by Hurwitz's theorem there exist ξ_n, $\xi_n \to \xi_0$ and

$$g_n(\xi_n) = \rho_n^{-k}\left(f_n(z_n + \rho_n \xi_n) - c\right) = 0.$$

Thus $f_n(z_n + \rho_n \xi_n) = c = a$. Since f_n and $f_n^{(k)}$ share a, we have

$$g_n^{(k)}(\xi_n) = f_n^{(k)}(z_n + \rho_n \xi_n) = a.$$

Hence $g^{(k)}(\xi_0) = \lim_{n\to\infty} g_n^{(k)}(\xi_n) = a$, and we have $\overline{E}(0, g) \subset \overline{E}(a, g^{(k)})$.

Obviously, $g^{(k)} \not\equiv a$. If $g^{(k)}(\xi_0) = a$, then by Hurwitz's theorem, there exist ξ_n, $\xi_n \to \xi_0$ and

$$g_n^{(k)}(\xi_n) = f_n^{(k)}(z_n + \rho_n \xi_n) = a.$$

Since f_n and $f_n^{(k)}$ share a, we have $f_n(z_n + \rho_n \xi_n) = a$. Thus $g(\xi_0) = \lim_{n\to\infty} g_n(\xi_n) = \lim_{n\to\infty}(f_n(z_n + \rho_n \xi_n) - c) = 0$, and we have $\overline{E}(a, g^{(k)}) \subset \overline{E}(0, g)$. Hence (3.1) is proved. Since the zeros of $g(\xi)$ are of multiplicity $\geq k + 1$, we get by (3.1) that $g \neq 0$ and $g^{(k)} \neq a$. By Lemma 5, $g(\xi)$ is a constant, a contradiction.

Case 3. $c \neq a$, $c \neq b$. Then using the same argument as we do in Case 1, we deduce that $g^{(k)}(\xi) \neq a, b$. Since $g(\xi)$ is of order at most two, we have

$$\frac{g^{(k)}(\xi) - a}{g^{(k)}(\xi) - b} = e^{a_1\xi^2 + a_2\xi + a_3},$$

where a_1, a_2, and a_3 are constants. Thus we have

$$g^{(k)}(\xi) = \frac{a - be^{a_1\xi^2 + a_2\xi + a_3}}{1 - e^{a_1\xi^2 + a_2\xi + a_3}}. \tag{3.2}$$

Assume that $|a_1| + |a_2| \neq 0$. Then $g^{(k)}(\xi)$ has infinitely many poles of multiplicity 1 or 2. However, a pole of $g^{(k)}(\xi)$ has multiplicity $\geq k + 1$.

We arrive at a contradiction, since $k \geq 2$. This shows that $g^{(k)}(\xi)$ is a constant and $g(\xi)$ is a polynomial of degree $\leq k$, which contradicts the assumption that the zeros of $g(\xi)$ are of multiplicity $\geq k+1$ and $g(\xi)$ is a non-constant function.

If $c \neq a$, $c = b$, then as in Cases 1–2, we get a contradiction. Thus we have proved that $\mathscr{F}$ is normal in D. The theorem is proved.

Proof of Theorem 2. We may assume that $D = \{|z| < 1\}$. Suppose that $\mathscr{F}$ is not normal in D; without loss of generality we assume that $\mathscr{F}$ is not normal at $z_0 = 0$.

In the following, we consider two cases:

Case 1. $c = a$. Then, by Lemma 1 (Remark 2), there exist

(a) a number r, $0 < r < 1$;

(b) a sequence of complex numbers $z_n \to 0$, $|z_n| < r < 1$;

(c) a sequence of functions $f_n \in \mathscr{F}$; and

(d) a sequence of positive numbers $\rho_n \to 0$

such that $g_n(\xi) = \rho_n^{-k}(f_n(z_n + \rho_n \xi) - c)$ converges locally uniformly to a non-constant entire function $g(\xi)$. Moreover, $g(\xi)$ is of order at most one.

Now, we distinguish two subcases.

Case 1.1. $c = a = 0$, $b \neq 0$. In this case, we get a contradiction as in the proof of Theorem 1.

Case 1.2. $c = a \neq 0$. Then the zeros of $g(\xi)$ are of multiplicity $\geq k$, $g^{(k)}(\xi) \neq b$, and

$$\overline{E}(0, g) = \overline{E}\big(a, g^{(k)}\big). \tag{3.3}$$

Suppose that $g(\xi)$ is a polynomial; then $g(\xi)$ is of degree $\leq k$, since $g^{(k)}(\xi) \neq b$. Hence $\overline{E}(a, g^{(k)}) = \phi$ or $\mathbb{C}$. However, $\overline{E}(0, g)$ contains only finitely many points, a contradiction.

Now we assume that $g(\xi)$ is a transcendental entire function. Then

$$g^{(k)}(\xi) = b + e^{a_1 \xi + a_2},$$

$$g(\xi) = p(\xi) + \frac{1}{a_1^k} e^{a_1 \xi + a_2},$$

where a_1, a_2 are constants such that $a_1 \neq 0$, $p(\xi)$ is a polynomial.

If $p(\xi) \not\equiv 0$, it follows by Lemma 6 that

$$T(r,g) \leq \overline{N}\left(r, \frac{1}{g}\right) + \overline{N}\left(r, \frac{1}{g-p}\right) + S(r,g)$$
$$\leq \frac{1}{2} N\left(r, \frac{1}{g}\right) + S(r,g)$$
$$\leq \frac{1}{2} T(r,g) + S(r,g).$$

Thus we get $T(r, g) = S(r, g)$, a contradiction.

If $p(\xi) \equiv 0$, then we have

$$g(\xi) = \frac{1}{a_1^k} e^{a_1 \xi + a_2}.$$

It follows that $\overline{E}(0, g) = \phi$, $\overline{E}(a, g^{(k)}) \neq \phi$. Thus $\overline{E}(0, g) \neq \overline{E}(a, g^{(k)})$, which contradicts (3.3).

We have proved that $\mathscr{F}$ is normal in D.

Case 2. $c \neq a$, $c \neq b$. Then by Lemma 1, there exist

(a) a number r, $0 < r < 1$;

(b) a sequence of complex numbers $z_n \to 0$, $|z_n| < r < 1$;

(c) a sequence of functions $f_n \in \mathscr{F}$; and

(d) a sequence of positive numbers $\rho_n \to 0$

such that $g_n(\xi) = f_n(z_n + \rho_n \xi) - c$ converges locally uniformly with respect to the spherical metric to a non-constant entire function $g(\xi)$. By Hurwitz's theorem we know that the zeros of $g(\xi)$ are of multiplicity $\geq k \geq 2$.

We claim that $\overline{E}(a - c, g) \subset \overline{E}(0, g^{(k)})$.

Suppose that $g(\xi_0) = a - c$. Then there exist ξ_n, $\xi_n \to \xi_0$, such that (for n sufficiently large)

$$a - c = g_n(\xi_n) = f_n(z_n + \rho_n \xi_n) - c.$$

Thus we get $f_n(z_n + \rho_n \xi_n) = a$. Since f_n and $f_n^{(k)}$ share a, we have

$$f_n^{(k)}(z_n + \rho_n \xi_n) = a.$$

Hence we get

$$g_n^{(k)}(\xi_n) = \rho_n^k f_n^{(k)}(z_n + \rho_n \xi_n) = a \rho_n^k$$

Thus we get $g^{(k)}(\xi_0) = \lim_{n\to\infty} g_n^{(k)}(\xi_n) = 0$, that is, $\overline{E}(a-c, g) \subset \overline{E}(0, g^{(k)})$.

Likewise, we get $\overline{E}(b-c, g) \subset \overline{E}(0, g^{(k)})$.

Hence we deduce that

$$\overline{N}\left(r, \frac{1}{g-(a-c)}\right) + \overline{N}\left(r, \frac{1}{g-(b-c)}\right) \leq N\left(r, \frac{1}{g^{(k)}}\right).$$

Since the zeros of $g(z)$ are of multiplicity $\geq k \geq 2$, it follows from the first and second fundamental theorems of Nevanlinna that

$$\begin{aligned} 2T(r,g) &\leq \overline{N}\left(r, \frac{1}{g}\right) + \overline{N}\left(r, \frac{1}{g-(a-c)}\right) \\ &\quad + \overline{N}\left(r, \frac{1}{g-(b-c)}\right) + S(r,g) \\ &\leq \frac{1}{2}N\left(r, \frac{1}{g}\right) + N\left(r, \frac{1}{g^{(k)}}\right) + S(r,g) \\ &\leq \frac{1}{2}T\left(r, \frac{1}{g}\right) + T\left(r, \frac{1}{g^{(k)}}\right) + S(r,g) \\ &\leq \frac{1}{2}T(r,g) + T\left(r, g^{(k)}\right) + S(r,g) \\ &\leq \frac{3}{2}T(r,g) + S(r,g). \end{aligned}$$

Thus we get that $T(r,g) = S(r,g)$. Hence we conclude that $g^{(k)}(\xi) \equiv 0$. Since the zeros of $g(\xi)$ are of multiplicity $\geq k$, $g(\xi)$ must be a constant, a contradiction. We have proved that $\mathscr{F}$ is normal in D. The proof of the theorem is complete.

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