Stability of Solutions of Nonlinear Diffusion Problems

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1. Introduction

In this paper we discuss the stability of steady-state solutions of nonlinear diffusion equations having the form

\[ u_t(x, t) = F(x, u, u_x, u_{xx}) \quad a < x < b, \quad t > 0 \]

with boundary conditions

\[ u_x(a, t) = f_1(u(a, t)) \quad \text{and} \quad u_x(b, t) = f_2(u(b, t)). \]

It is also shown that the methods used in the proofs of the stability theorems may be used to bound nonsteady state solutions. The notion of stability as used here is analogous to the classical definition of stability as applied to solutions of ordinary differential equations.

The results we give below depend heavily on a variation of a lemma originally given by Westphal [1]. Prodi [2], Narasimhan [3], and Friedman [4] all made use of variations of Westphal’s lemma in order to obtain extensions of results published by Bellman [5]. Bellman did not use Westphal’s lemma but instead used known results from the theory of multiple Fourier series to convert the partial differential equation into an integral equation. All four authors discussed the stability of solutions of problems which were specializations of the following:

\[ u_t = L(u) + F(x, t, u) \]

\[ u(a, t) = f_1(t) \quad \text{and} \quad u(b, t) = f_2(t) \quad \text{for all} \quad t > 0 \]

\[ u(x, 0) = \phi(x) \quad \text{for all} \quad x \in [a, b], \]

where

\[ L(u) = \sum_{i,j=1}^{n} a_{ij}u_{x_i x_j} + \sum_{i=1}^{n} b_i u_{x_i} \]

and \( a_{ij}, b_i \) are constants. Except for Friedman, they treated only cases in which \( F(x, t, 0) = f_1(t) = f_2(t) = 0 \) and gave sufficient conditions for the identically zero solution.
In the latest of a series of papers Lakshmikantham [6-8] generalized a method he originally developed for ordinary differential equations to include parabolic equations. The method is somewhat analogous to Liapunov's direct method. He considers two partial differential equations

\[ u_t = f(t, x, u, u_x, u_{xx}) \]
\[ v_t = g(t, x, v, v_x, v_{xx}), \]

where \( x \) is \( n \) dimensional. He then gives the properties of a function \( v(x, t, u, v) \), the existence of which guarantees that the solutions of the two equations remain close as time increases.

McNabb [9] studied stability of solutions of the same partial differential equation as we consider in this paper, but requires that the value of the function on the boundary be given. The methods we use to establish stability are similar to McNabb's, however he used a method of small disturbances to obtain sufficient conditions for asymptotic stability and instability.

Section 2 below gives a variation of Westphal's lemma which applies to the problem we are considering. Section 3 gives the stability theorems. Section 4 is an example of how the theorems may be used and also shows how the lemma of Section 2 may be used to bound the solution of a boundary value problem. For ease of understanding the results are all given for the case in which the space variable is one dimensional only. An Appendix is added giving a proof of the lemma of Section 2 for the \( n \)-dimensional case. The proofs of the remaining theorems for higher dimensional case require only minor notational changes, hence these proofs are not given.

2. Preliminary Results

The purpose of this section is to present a variation of Westphal's lemma which applies to the problem discussed in this paper. To aid in stating the lemma we introduce the following notation. For a positive real number \( T \) let

\[ R_T = \{ t \mid 0 < t \leq T \} \]
\[ R_{0T} = \{ t \mid 0 \leq t \leq T \}. \]

If \( T = \infty \) let

\[ R_T = \{ t \mid t > 0 \} \]
\[ R_{0T} = \{ t \mid t \geq 0 \}. \]

\textbf{Lemma 1.} Suppose \( u, v, f_1, f_2, g_1, g_2, \) and \( F \) are functions satisfying the following conditions:

(i) \( u(x, t) \) and \( v(x, t) \) are of class \( C' \) for all \( x \in [a, b] \), and \( t \in R_{0T} \), where \( T \) is a positive real number or infinity.
(ii) $u$ and $v$ are twice continuously differentiable with respect to $x$ for all $x \in (a, b)$ and $t \in R_{OT}$.

(iii) $v(x, 0) < u(x, 0)$ for all $x \in [a, b]$.

(iv) $u_x(a, t) = f_1(u(a, t))$ \quad $u_x(b, t) = f_2(u(b, t))$
$v_x(a, t) = g_1(v(a, t))$ \quad $v_x(b, t) = g_2(v(b, t))$

where $f_1, f_2, g_1, g_2$ are all continuous, with closed domains and bounded first derivatives.

(v) $f_1(u) < g_1(u)$ for all $u \in [D(f_1) \cap D(g_1)]$
$f_2(u) > g_2(u)$ for all $u \in [D(f_2) \cap D(g_2)]$.

(vi) $F(x, t, u, u_x, u_{xx})$ is of class $C'$ and is nondecreasing in $u_{xx}$.

(vii) $F(x, t, v, v_x, v_{xx}) - v_t \geq F(x, t, u, u_x, u_{xx}) - u_t$ for all $x \in (a, b)$ and $t \in R_T$.

Then
$v(x, t) < u(x, t)$ for all $x \in [a, b]$ and $t \in R_{OT}$.

**Proof.** We divide the proof into two parts. The first is a proof of the lemma if condition (vii) is replaced by the condition

(vii-a) $F(x, t, v, v_x, v_{xx}) - v_t \geq F(x, t, u, u_x, u_{xx}) - u_t$ for all $x \in (a, b)$ and $t \in R_{OT}$.

**Part 1.** We assume all the hypotheses of the lemma hold with condition (vii) replaced by the stronger condition (vii-a). Deny the conclusion. Define a function $h(x, t)$ by

$h(x, t) = v(x, t) - u(x, t). \quad (2.1)$

Let $t_1$ be the greatest lower bound (glb) of the set

$S = \{t \mid h(x, t) \geq 0 \text{ for some } x \in [a, b]\}$.

We have from the continuity of $h$ with respect to $t$ and the definition of $t_1$ that

$$\sup_{x \in [a, b]} h(x, t_1) = 0. \quad (2.2)$$

Hence there is a point $x_1 \in [a, b]$ such that $h(x_1, t_1) = 0$. To show that $x_1 \neq a$, suppose $x_1 = a$. Then

$$h_x(a, t_1) = v_x(a, t_1) - u_x(a, t_1)$$
$$= g_1(v(a, t_1)) - f_1(u(a, t_1)) > 0.$$

Since $h$ is of class $C'$ we have by application of the mean value theorem that $h(x, t_1) > 0$ for some $x > a$ in contradiction to (2.2). By analogous reasoning
$x_1 \neq b$, hence $x_1 \in (a, b)$. For fixed $t = t_1$, $h(x, t_1)$ is a function of $x$ only; hence it attains an interior maximum at a point $x = x_1$. Therefore we can conclude:

\[
\begin{align*}
h(x_1, t_1) &= 0 \quad \text{hence} \quad \varphi(x_1, t_1) = u(x_1, t_1) \quad (2.3) \\
h'_x(x_1, t_1) &= 0 \quad \text{hence} \quad \varphi'_x(x_1, t_1) = u'_x(x_1, t_1) \quad (2.4) \\
h_{xx}(x_1, t_1) &\leq 0 \quad \text{hence} \quad \varphi_{xx}(x_1, t_1) \leq u_{xx}(x_1, t_1) \quad (2.5) \\
h(t_1, x_1) &\geq 0 \quad \text{hence} \quad \varphi(t_1, x_1) - u(t_1, x_1) \geq 0. \quad (2.6)
\end{align*}
\]

Inequality (2.6) follows from the fact that $h(x, t)$ is of class $C'$ and that $h(x_1, t) < 0$ for $t < t_1$. By hypothesis (vii-a) we have

\[
\varphi_i(x, t) - u_i(x, t) < F(x, t, \varphi, \varphi_x, \varphi_{xx}) - F(x, t, u, u_x, u_{xx})
\]
for all $x \in [a, b]$ and $t \in R_T$. \quad (2.7)

If we use (2.3), (2.4), (2.5), and the fact that $F$ is a nondecreasing function of its last argument, as is assured by condition (vi), we have

\[
F(x_1, t_1, \varphi, \varphi_x, \varphi_{xx}) - F(x_1, t_1, u, u_x, u_{xx}) \leq 0. \quad (2.8)
\]

The inequalities (2.7) and (2.8) imply that

\[
\varphi_i(x_1, t_1) - u_i(x_1, t_1) < 0. \quad (2.9)
\]

If the set $S$ is nonempty, then it has a glb $t_1 > 0$ and at the point $(x_1, t_1)$ we have both (2.6) and (2.9) holding, a contradiction. So the set $S$ does not have a glb and we conclude that $S$ is empty and the lemma as modified holds.

**PART 2.** We assume the hypotheses (i) through (vii) of Lemma 1 hold. Again we deny the conclusion. Then there exists a $t_1 \in R_T$ and an $x_1 \in (a, b)$ such that

\[
\varphi(x_1, t_1) \geq u(x_1, t_1). \quad (2.10)
\]

We define a function $\varphi(x, t)$ by

\[
\varphi(x, t) = v(x, t) + \frac{\epsilon}{(n - 1)(t + 1)^{n-1}}
\]

where $\epsilon > 0$ and $n \geq 2$. (Both will be specified later.) We have

\[
\begin{align*}
\varphi_i(x, t) &= v_i(x, t) - \frac{\epsilon}{(t + 1)^n} \\
\varphi_x(x, t) &= v_x(x, t) \\
\varphi_{xx}(x, t) &= v_{xx}(x, t) \\
\varphi(x, 0) &= \varphi(x, 0) + \frac{\epsilon}{n - 1}
\end{align*}
\]
Since $F$ is of class $C'$, it follows that
\[
F(x, t, w, w_x, w_{xx}) - w_t = F(x, t, v, v_x, v_{xx})
\]
\[
+ \frac{F(x, t, v, v_x, v_{xx})(\epsilon)}{(n-1)(t+1)^{n-1}}
\]
\[
+ O(\epsilon^2) - v_t + \frac{\epsilon}{(t+1)^n}
\]
\[
> F(x, t, v, v_x, v_{xx}) - v_t, \tag{2.11}
\]
if
\[
\frac{\epsilon}{(t+1)^{n-1}} \frac{(F(x, t, v, v_x, v_{xx})}{n-1} + \frac{1}{t+1} > 0 \tag{2.12}
\]
and $\epsilon > 0$ is sufficiently small. Let
\[
\mu = \min_{x \in [a, b] \cap [0, t_1]} F(x, t, v, v_x, v_{xx}),
\]
We now choose $n$ so that
\[
\frac{\mu}{n-1} + \frac{1}{t+1} > 0, \quad \text{or} \quad n > -\mu(t_1 + 1) + 1.
\]
With this value for $n$, inequality (2.12) holds for all $x \in [a, b]$ and $t \in [0, t_1]$. There is a number $\rho > 0$ such that
\[
g_1(v(a, t)) - f_1(v(a, t)) > \rho \quad \text{for all} \quad v \in [D(f_1) \cap D(g_1)], \quad t \in [0, t_1]
\]
and
\[
f_2(v(b, t)) - g_2(v(b, t)) > \rho \quad \text{for all} \quad v \in [D(f_2) \cap D(g_2)], \quad t \in [0, t_1]
\]
since the left side of both inequalities are positive on a closed set. Hence for all $w(a, t) \in D(f_1)$ we have
\[
w_2(a, t) = v_2(a, t)
\]
\[
= g_1(v(a, t))
\]
\[
> f_1(v(a, t)) + \mu
\]
\[
> f_1(v(a, t)) + \frac{\epsilon M}{(n-1)(t+1)^{n-1}}
\]
\[
> f_1\left(v(a, t) + \frac{\epsilon}{(n-1)(t+1)^{n-1}}\right)
\]
\[
= f_1(w(a, t)) \tag{2.13}
\]
if $\varepsilon$ is sufficiently small. Here $M$ is an upper bound for $|f'|$. Similarly we can see that the inequality

$$w_1(b, t) \leq f_2(w(b, t))$$

(2.14)

can also be satisfied for $\varepsilon$ sufficiently small. We now select for $\varepsilon > 0$ a value sufficiently small so that inequalities (2.11), (2.13), (2.14) hold and also so that

$$\min_{x \in [a, b]} [u(x, 0) - v(x, 0)] > \varepsilon.$$ 

(2.15)

We note that there is a positive value of $\varepsilon$ satisfying inequality (2.15) since the left side is the minimum of a continuous positive function on a closed interval. Thus we have from (2.11) and hypothesis (vii),

$$F(x, t, w, w_x, w_{xx}) - w_t > F(x, t, v, v_x, v_{xx}) - v_t$$

$$\geq F(x, t, u, u_x, u_{xx}) - u_t$$

for all $x \in [a, b]$ and $t \in [0, t_1]$. We also have from (2.15) and the fact that $n \geq 2$ that

$$w(x, 0) = v(x, 0) + \frac{\varepsilon}{n - 1} < u(x, 0)$$

for all $x \in [a, b]$.

If we substitute the function $w(x, t)$ for $v(x, t)$ into the statement of our lemma we see that all of the conditions of the lemma are satisfied, with condition (vii) replaced by (vii-a). From Part 1 of this proof it follows that

$$w(x, t) < u(x, t)$$

for all $x \in [a, b]$ and $t \in [0, t_1]$. (2.16)

From inequalities (2.10) and (2.16) and the definition of $w$ we have the contradiction

$$w(x_1, t_1) < u(x_1, t_1) \leq v(x_1, t_1) = w(x_1, t_1) - \frac{\varepsilon}{(n - 1)(t + 1)^{n-1}}.$$ 

Hence there does not exist a $t_1 \in R_T$ such that inequality (2.10) holds and the lemma follows.

3. Stability Theorems

We introduce the following notation. Let Problem D represent the partial differential equation

$$u_t = F(x, u, u_x, u_{xx})$$

for all $x \in [a, b]$ and $t > 0$

together with the boundary conditions

$$u_x(a, t) = f_1(u(a, t)) \quad \text{and} \quad u_x(b, t) = f_2(u(b, t))$$

for all $t > 0$. 

We assume that \( f_1, f_2 \) are continuous with bounded first derivatives, \( F \) is of class \( C' \), and \( F \) is nondecreasing in \( u_{\infty} \). We use the notation \( u(\phi, x, t) \) to represent a solution of Problem D such that
\[
u(\phi, x, 0) = \phi(x) \text{ for all } x \in [a, b],
\]
where \( \phi \) is of class \( C'' \), \( u \) is of class \( C' \) for all \( x \in [a, b] \) and \( t > 0 \) and \( u \) is twice continuously differentiable with respect to \( x \) for all \( x \in (a, b) \) and \( t > 0 \).

In this chapter we present theorems concerned with Liapunov-like stability of Problem D. The following definitions are analogous to the corresponding definitions as they are generally used in ordinary differential equations (ODE's). Similar definitions have been used by other writers.

**Definition 1.** Let \( u(\phi, x, t) \) be a solution of Problem D. We say that \( u \) is a steady-state solution if \( u \) is independent of time, i.e., \( u(\phi, x, t) = \phi(x) \) for all \( t > 0 \).

**Definition 2.** Let \( u(\phi, x, t) \) be a solution of Problem D. Suppose for every \( \epsilon > 0 \) there is a \( \delta > 0 \) such that if the function \( \psi(x) \) satisfies
\[
\max_{x \in [a, b]} | \psi(x) - \phi(x) | < \delta
\]
it is true that
\[
\max_{x \in [a, b]} \max_{t > 0} | u(\phi, x, t) - u(\psi, x, t) | < \epsilon.
\]
Then we say \( u(\phi, x, t) \) is a stable solution to Problem D.

**Definition 3.** Let
\[
A = \{ (x, u) \mid x \in [a, b] \text{ and } \psi_1(x) \leq u \leq \psi_2(x) \},
\]
where \( \psi_1 \) and \( \psi_2 \) are arbitrary functions of class \( C'' \). Let \( B \) be the set of functions defined on the closed interval \( [a, b] \) such that \( \psi \in B \) implies
\[
\{ (x, \psi(x)) \mid x \in [a, b] \} \subset A.
\]
Suppose \( u(\phi, x, t) \) is a solution of Problem D such that if \( \psi \in B \) we have
\[
\lim_{t \to \infty} \left[ \max_{x \in [a, b]} | u(\phi, x, t) - u(\psi, x, t) | \right] = 0.
\]
Then we say \( u(\phi, x, t) \) is an asymptotically stable solution of Problem D and that \( A \) is a region of asymptotic stability.

**Definition 4.** Let \( u(\phi, x, t) \) be a solution of Problem D. Suppose there
exists an $\epsilon > 0$ such that for every $\delta > 0$ there is at least one function $\phi(x)$ satisfying both conditions

$$\max_{x \in [a, b]} | \phi(x) - \phi(x) | < \delta$$

and

$$\max_{x \in [a, b]} | u(\phi, x, t) - u(\phi, x, t) | < \epsilon$$

for some $t > 0$.

Then we say that $u(\phi, x, t)$ is an unstable solution of Problem D.

Our first theorem gives sufficient conditions for a steady-state solution of Problem D to be a stable solution.

**Theorem 1.** Consider Problem D and assume there exists a one parameter family $v(x, \lambda), \lambda \in [\lambda_1, \lambda_2]$, of solutions of the ODE

$$F(x, v, v_x, v_{xx}) = 0$$

satisfying the following four conditions:

(i) There is a number $\lambda' \in (\lambda_1, \lambda_2)$ such that $v_x(a, \lambda') = f_1(v(a, \lambda'))$ and $v_x(b, \lambda') = f_2(v(b, \lambda'))$.

(ii) $v_x(x, \lambda) > 0$, for all $x \in [a, b]$ and $\lambda \in [\lambda_1, \lambda_2]$.

(iii) $v_x(a, \lambda) > f_1(v(a, \lambda))$ and $v_x(b, \lambda) < f_2(v(b, \lambda))$ for $\lambda \in [\lambda_1, \lambda']$.

(iv) $v_x(a, \lambda) < f_1(v(a, \lambda))$ and $v_x(b, \lambda) > f_2(v(b, \lambda))$ for $\lambda \in (\lambda', \lambda_2]$.

Then if $v(x) = v(x, \lambda')$, $u(\phi, x, t)$ is a steady-state solution of Problem D.

**Proof.** Assume the hypotheses hold. We must show that given an $\epsilon > 0$ there is a $\delta > 0$ such that

$$\max_{x \in [a, b]} | u(\phi, x, t) - u(\lambda', x, t) | < \epsilon$$

whenever

$$\max_{x \in [a, b]} | \phi(x) - \phi(x) | < \delta.$$

Let $\epsilon > 0$ be given. Select a number $\lambda \in [\lambda_1, \lambda')$ such that

$$\max_{x \in [a, b]} | v(x, \lambda') - v(x, \lambda) | < \epsilon$$

and a number $\lambda \in (\lambda', \lambda_2)$ such that

$$\max_{x \in [a, b]} | v(x, \lambda) - v(x, \lambda') | < \epsilon.$$

We define a number $\delta$ by

$$\delta = \min \{ \min_{x \in [a, b]} (v(x, \lambda') - v(x, \lambda)), \min_{x \in [a, b]} (v(x, \lambda) - v(x, \lambda')) \}.$$
We note that $\delta > 0$ since $v_\lambda(x, \lambda) > 0$ for all $x \in [a, b]$. Let $\Psi(x)$ be an arbitrary function of class $C^\infty$ satisfying (3.3), then since $\phi(x) = v(x, \lambda')$ we have from (3.6)

$$v(x, \lambda) \leq \phi(x) - \delta < \Psi(x) < \phi(x) + \delta \leq v(x, \lambda) \text{ for all } x \in [a, b].$$

Since $v(x, \lambda)$ is a family of solutions of Eq. (3.1) and $u$ is a solution to Problem D it follows that

$$F(x, v(x, \lambda), v_u(x, \lambda), v_{ux}(x, \lambda)) - v_i(x, \lambda)$$

$$= F(x, u(\Psi, x, t), u_u(\Psi, x, t), u_{ux}(\Psi, x, t)) - u_i(\Psi, x, t)$$

$$= F(x, v(x, \lambda), v_u(x, \lambda), v_{ux}(x, \lambda)) - v_i(x, \lambda).$$

If we let $v(x, \lambda)$ correspond to the function $v(x, t)$ and $u(\Psi, x, t)$ correspond to the function $u(x, t)$ of Lemma 1, we see that all the hypotheses of the lemma are satisfied and it follows that $v(x, \lambda)$ bounds $u(\Psi, x, t)$ from below. We may then let $u(\Psi, x, t)$ correspond to $v(x, t)$ and $v(x, \lambda)$ correspond to $u(x, t)$ of Lemma 1 and it follows that $v(x, \lambda)$ bounds $u(\Psi, x, t)$ from above. Thus it follows that

$$v(x, \lambda) < u(\Psi, x, t) < v(x, \lambda) \text{ for all } x \in [a, b] \text{ and } t > 0. \quad (3.7)$$

By (3.4), (3.5), and (3.7) we have

$$u(\phi, x, t) - \epsilon = v(x, \lambda') - \epsilon < v(x, \lambda) < u(\Psi, x, t) \quad (3.8)$$

and

$$u(\phi, x, t) + \epsilon = v(x, \lambda') + \epsilon > v(x, \lambda) > u(\Psi, x, t) \quad (3.9)$$

Combining (3.8) and (3.9) we have (3.2).

In a situation in which it is difficult or impossible to find a one parameter family satisfying the conditions of Theorem 1 it may still be possible to find an upper bound as we show in the following corollary. A similar corollary could be stated establishing a lower bound.

**Corollary 1.** Suppose there exists a solution $v(x)$, of the ODE (3.1) satisfying the condition

$$v_u(a) < f_1(v(a)) \quad \text{and} \quad v_u(b) > f_2(v(b)).$$

Then if $u(\Psi, x, t)$ is a solution to Problem D where

$$\Psi(x) < v(x) \text{ for all } x \in [a, b],$$

we have

$$u(x, t) < v(x) \text{ for all } x \in [a, b] \text{ and } t > 0.$$
The proof follows immediately from the proof of Theorem 1; \( v(x) \) is an upper bound for \( u(\Psi, x, t) \) for the same reasons that \( v(x, \bar{\lambda}) \) was an upper bound in Theorem 1.

By requiring the function \( F \) of Problem D to satisfy one additional condition we can strengthen the conclusion of Theorem 1 as is shown in the next theorem.

**Theorem 2.** Let all the hypotheses of Theorem 1 hold. In addition suppose that for all \( x \in [a, b] \) and \( \lambda \in [\lambda_1, \lambda_2] \)

\[
F_e(x, v(x, \lambda), v_\lambda(x, \lambda), v_{x_\lambda}(x, \lambda)) \neq 0. \tag{3.10}
\]

Then \( u(\phi, x, t) \), where \( \phi(x) = v(x, \lambda') \), is an asymptotically stable steady-state solution of Problem D and the set

\[
A = \{(x, u) \mid x \in [a, b] \text{ and } v(x, \lambda_1) < u < v(x, \lambda_2)\} \tag{3.11}
\]
is a region of asymptotic stability.

**Proof.** The proof consists of several parts but only one will be given in detail since they are all quite similar. We assume the hypotheses hold. Since condition (3.10) requires that \( F_e \) has the same sign for all \( x \in [a, b] \) and \( \lambda \in [\lambda_1, \lambda_2] \), we assume \( F_e > 0 \) without loss of generality. Let \( A \) be the set defined by (3.11) and let \( B \) be the set of functions such that \( \Psi \in B \) implies \( \{(x, \Psi(x)) \mid x \in [a, b]\} \subseteq A \). We first show that given any \( \epsilon > 0 \) and any \( \Psi \in B \) there exists a \( T' > 0 \) such that

\[
\max_{x \in [a, b]} \left[ u(\Psi, x, t) - u(\phi, x, t) \right] < \epsilon. \tag{3.12}
\]

We bound \( u(\Psi, x, t) \) from above and then show the bound can be decreased as \( t \) increases until it is within \( \epsilon \) of \( v(x, \lambda') \). Let \( \epsilon > 0 \) be given and let \( \lambda \in (\lambda', \lambda_2) \) (see Fig. 1) be such that

\[
v(x, \bar{\lambda}) - v(x, \lambda') < \epsilon \quad \text{for all} \quad x \in [a, b]. \tag{3.13}
\]

We now define three positive numbers \( \mu_1 \), \( \mu_2 \), and \( \mu_3 \) by

\[
\mu_1 = \min_{x \in [a, b]} [v(x, \bar{\lambda}) - v(x, \lambda')],
\]

\[
\mu_2 = \min_{x \in [a, b]} \left\{ F_e(x, v(x, \lambda), v_\lambda(x, \lambda), v_{x_\lambda}(x, \lambda)) \right\}_{\lambda \in (\lambda', \lambda_2)} \]

and

\[
\mu_3 = \max_{x \in [a, b]} v_\lambda(x, \lambda). \]
Let $\lambda^* \in (\lambda', \lambda)$ be such that

$$v(x, \lambda^*) - v(x, \lambda') \leq \frac{1}{2} \mu_1$$
for $x \in [a, b]$.

We now define a positive number $\mu_4$ by

$$\mu_4 = \min_{\lambda \in [\lambda^*, \lambda_0]} \{ [f_1(v(a, \lambda)) - v_2(x, \lambda)], [v_2(b, \lambda) - f_2(v(b, \lambda))] \}. \quad (3.14)$$

Let $H(\lambda)$ be a function defined for $\lambda \in [\lambda, \lambda_0]$ such that for all $h < H(\lambda)$ we have

$$f_1(v(a, \lambda) - h) - f_1(v(a, \lambda)) < \frac{1}{2} \mu_4 \quad (3.15)$$
and also
\[ f_u(v_1, \lambda) - h - f_u(v_2, \lambda) < \frac{1}{2} \mu_1. \]  
(3.16)
Let
\[ \delta_1 = \min_{\lambda_1, \lambda_2} [H(\lambda)]. \]  
(3.17)
Let \(w(x, \lambda)\) be a function defined by
\[ w(x, \lambda) = v(x, \lambda) - \delta, \]
where \(\delta > 0\) will be specified later. We have, since \(F\) is of class \(C'\),
\[ F(x, w, w_x, w_{xx}) = F(x, v, v - \delta, v_x, v_{xx}) \]
\[ = F(x, v, v_x, v_{xx}) - \delta F_v(x, v, v_x, v_{xx}) + O(\delta^2) \]
\[ \leq F(x, v, v_x, v_{xx}) - \delta \mu_2 + O(\delta^2) \]
\[ < F(x, v, v_x, v_{xx}) \]
\[ = 0, \]  
(3.18)
for \(\delta > 0\) sufficiently small. We let \(\delta_2 > 0\) be such that
\[ \delta \leq \min_{x \in [a, b]} (\varphi(x, \lambda_2) - \Psi_f(x)). \]  
Let us now assign a positive value to \(\delta\), sufficiently small so inequality (3.18) holds and also so that
\[ \delta \leq \min \left(\frac{1}{2} \mu_1, \delta_1, \delta_2\right). \]  
(3.19)
From inequality (3.18) it follows that there exists a positive number \(\mu_5\) satisfying
\[ F(x, w(x, \lambda), w_x(x, \lambda), w_{xx}(x, \lambda)) < - \mu_5 \]  
for all \(x \in [a, b] \) and \(\lambda \in [\lambda_1, \lambda_2].\)
Let \(\tilde{w}(x, t)\) be a function defined by
\[ \tilde{w}(x, t) = w(x, \lambda(t)), \]
where
\[ \lambda(t) = \lambda' + (\lambda_2 - \lambda') e^{-pt}, \]
\(p > 0\) to be specified later. We have
\[ F(x, \tilde{w}, \tilde{w}_x, \tilde{w}_{xx}) - \tilde{w}_t = F(x, w, w_x, w_{xx}) + w_x p (\lambda_2 - \lambda') e^{-pt} \]
\[ \leq - \mu_5 + \mu_2 (\lambda_2 - \lambda'). \]  
(3.20)
Let \(p = \mu_2/\mu_5(\lambda_2 - \lambda')\), then the right hand side of inequality (3.20) is zero. Thus the inequality
\[ F(x, \tilde{w}, \tilde{w}_x, \tilde{w}_{xx}) - \tilde{w}_t \leq F(x, u, u_x, u_{xx}) - u_t = 0 \]
is satisfied for all \( x \in [a, b] \) and \( t \in R_T \), where \( T' \) is the solution of the equation \( \lambda(T') = \lambda \). From (3.17) and (3.19), inequality (3.15) yields

\[
    f_1(v(a, \lambda) - \delta) - f_1(v(a, \lambda)) > ( - \frac{1}{2} ) \mu_4. \tag{3.21}
\]

The definition of \( w \) together with (3.21) and (3.14) gives us

\[
    f_1(\omega(a, \lambda)) = f_1(v(a, \lambda) - \delta) > f_1(v(a, \lambda)) - \frac{\mu_4}{2} > v_\omega(a, \lambda) = w_\omega(a, \lambda). \tag{3.22}
\]

Similarly from (3.17), (3.19), inequality (3.16), and (3.14) we obtain

\[
    f_1(\omega(a, \lambda)) = f_1(v(a, \lambda) - \delta) < f_1(v(b, \lambda)) + \frac{\mu_4}{2} < v_\omega(b, \lambda) = w_\omega(b, \lambda). \tag{3.23}
\]

Both (3.22) and (3.23) hold for all \( \lambda \in [\lambda_1, \lambda_2] \). All conditions of Lemma 1 are now satisfied, and since

\[
    \omega(x, 0) > \Phi(x) \quad \text{for all} \quad x \in [a, b],
\]

it follows that

\[
    \omega(x, t) > u(\Psi, x, t) \quad \text{for all} \quad x \in [a, b] \quad \text{and} \quad t \in R_{T'}. \tag{3.24}
\]

From the definition of \( \omega \) and \( T' \) it follows that

\[
    v(x, \lambda) > u(\Psi, x, T') \quad \text{for all} \quad x \in [a, b].
\]

Thus by Corollary 1 we have

\[
    v(x, \lambda) > u(\Psi, x, t) \quad \text{for all} \quad x \in [a, b] \quad \text{and} \quad t > T',
\]

which together with (3.13) gives us (3.12) and completes the first part of the proof.

The next step of the proof is to show that there exists a number \( T'' \) such that

\[
    \max_{x \in [a, b]} \left| u(\phi, x, t) - u(\Psi, x, t) \right| < \epsilon. \tag{3.24}
\]

The proof of this consists of showing that there is a lower bound for \( u(\Psi, x, t) \) which can be increased with time until it is within \( \epsilon \) of \( u(\phi, x, t) \) at some time \( T'' \). We do not give the details of this since it differs from the proof of the existence of \( T' \) only in minor details.

Let \( T \) be the larger of the two numbers \( T' \) and \( T'' \). Then from (3.12) and (3.24) it follows that

\[
    \max_{x \in [a, b]} \left| u(\phi, x, t) - u(\Psi, x, t) \right| < \epsilon.
\]

This completes the proof.
The conclusion of Theorem 2 still holds if (3.10) is replaced by either
\[ F_v = 0 \quad \text{and} \quad F_{v_x}(x, v(x, \lambda), v_x(x, \lambda), v_{xx}(x, \lambda)) \neq 0 \]
for all \( x \in [a, b] \) and \( \lambda \in [\lambda_1, \lambda_2] \). (3.25)
or
\[ F_v = 0, \quad F_{v_x} = 0 \quad \text{and} \quad F_{v_{xx}}(x, v(x, \lambda), v_x(x, \lambda), v_{xx}(x, \lambda)) \neq 0 \]
for all \( x \in [a, b] \) and \( \lambda \in [\lambda_1, \lambda_2] \). (3.26)

The proof of the theorem with (3.10) replaced by (3.25) is essentially the same as the one that was given, the difference being that the function \( W(x, \lambda) \) is defined by \( w(x, \lambda) = v(x, \lambda) - \delta x \). Inequality (3.18) becomes
\[
F(x, w, w_x, w_{xx}) = F(x, v - \delta x, v_x - \delta, v_{xx}) \\
= F(x, v, v_x, v_{xx}) - \delta F_{v_x}(x, v, v_x, v_{xx}) + O(\delta^2) \\
\leq F(x, v, v_x, v_{xx}) - \mu_2 + O(\delta^2) \\
< F(x, v, v_x, v_{xx}) \\
= 0
\]
for \( \delta \) sufficiently small. We have made the assumption that \( F_{v_x} \) is positive and
\[
\mu_2 = \max_{x \in [a,b]} \max_{\lambda \in [\lambda_1, \lambda_2]} F_{v_x}(x, v(x, \lambda), v_x(x, \lambda), v_{xx}(x, \lambda)).
\]
We now select a positive number \( \delta \) such that inequality (3.27) holds and also such that
\[
\max_{x \in [a,b]} |\delta x| \leq \min [(\frac{1}{3} \mu_1), \delta_1, \delta_2].
\]
The remainder of the proof follows as before.

If (3.10) is replaced by (3.26) we make the same type of modification of the proof, the function \( w(x, \lambda) \) being defined by \( w(x, \lambda) = v(x, \lambda) - \delta x^2 \).

The next theorem gives sufficient conditions for the instability of a solution of Problem D. The conditions for instability amount to reversing certain inequalities in the hypotheses of Theorem 2. It turns out, however, that this gives us more than we need, so we break the theorem into two parts weakening our hypotheses as much as possible. The division is a natural one. Theorem 3a may be thought of as giving sufficient conditions for instability from above; Theorem 3b does likewise for instability from below.

**THEOREM 3a.** Consider Problem D and assume there exists a one parameter family \( v(x, \lambda), \lambda \in [\lambda_1, \lambda_2] \), of solutions of Eq. (3.1) satisfying
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(i) \( v(x, \lambda) = f_1(v(a, \lambda')) \) and \( v(b, \lambda') = f_2(v(b, \lambda')) \),

(ii) \( v(x, \lambda) > 0 \) for all \( x \in [a, b] \) and \( \lambda \in [\lambda', \lambda_2] \),

(iii-a) \( v_x(a, \lambda) > f_1(v(a, \lambda)) \) and \( v_x(b, \lambda) < f_2(v(b, \lambda)) \) for all \( \lambda \in [\lambda', \lambda_2] \),

(iv) \( F(v(x, \lambda), v_x(x, \lambda), v_x(x, \lambda)) \neq 0 \) for all \( x \in [a, b] \) and \( \lambda \in [\lambda', \lambda_2] \).

Then \( u(\phi, x, t) \), where \( \phi(x) = v(x, \lambda') \), is an unstable steady-state solution of Problem D.

**Theorem 3b.** Let all the hypotheses of Theorem 3a hold with the interval \([\lambda', \lambda_2]\) replaced by the interval \([\lambda_1, \lambda']\) and condition (iii-a) replaced by

(iii-b) \( v_x(a, \lambda) < f_1(v(a, \lambda)) \) and \( v_x(b, \lambda) > f_2(v(b, \lambda)) \) for all \( \lambda \in (\lambda_1, \lambda_2) \).

Then \( u(\phi, x, t) \), where \( \phi(x) = v(x, \lambda') \), is an unstable steady-state solution of Problem D.

**Proof.** We sketch the proof of Theorem 3a only since the proof for Theorem 3b follows along the same lines. Assume the hypotheses of Theorem 3a hold and that \( F_v > 0 \). Let

\[
A = \{(x, u) \mid x \in [a, b] \text{ and } v(x, \lambda') < u < v(x, \lambda_2)\}.
\]

We must show there exists an \( \epsilon > 0 \) such that for every \( \delta > 0 \) there is a \( \psi \) satisfying the conditions

\[
\max_{x \in [a, b]} | \psi(x) - \phi(x) | < \delta
\]

and

\[
\max_{x \in [a, b]} | u(\psi, x, t) - u(\phi, x, t) | > \epsilon \quad \text{for some } t > 0.
\]

Let

\[
\epsilon = \left( \frac{1}{2} \right) \min [v(x, \lambda_2) - v(x, \lambda')].
\]

We shall show that this \( \epsilon \) satisfies the above condition. Let \( \delta > 0 \) be given. We assume \( \delta \leq \epsilon \) since otherwise there is nothing to prove. Let \( \psi \) be any function such that \( 0 < \psi(x) - v(x, \lambda') < \delta \) for all \( x \in [a, b] \). Let \( \bar{\lambda} \in (\lambda', \lambda_2) \) be such that

\[
\min_{x \in [a, b]} [v(x, \bar{\lambda}) - v(x, \lambda')] \geq \epsilon,
\]

(3.28)

and let \( \lambda^* > \lambda' \) be such that

\( v(x, \lambda^*) < \psi(x) \).

We define three functions

\[
v(x, \lambda) = v(x, \lambda) + h,
\]

\[\lambda(t) = \lambda_2 - (\lambda_2 - \lambda^*) e^{-pt}\]
and
\[
\tilde{w}(x, t) = w(x, \lambda(t)).
\]
We must select positive values for \( h \) and \( p \) sufficiently small so that
\[
\tilde{w}(x, 0) < \Psi(x) \quad \text{for all} \quad x \in [a, b],
\]
\[
\omega_x(a, \lambda) > f_1(\omega(a, \lambda)) \quad \text{and} \quad \omega_x(b, \lambda) < f_2(\omega(b, \lambda)) \quad \text{for all} \quad \lambda \in [\lambda^*, \lambda^2],
\]
and
\[
F(x, \tilde{w}, \tilde{w}_x, \tilde{w}_{xx}) - \tilde{w} \geq F(x, u, u_x, u_{xx}) - u_t
\]
for all \( x \in [a, b] \) and \( t \in R_T \),
where \( T \) is the solution of the equation \( \lambda(T) = \tilde{\lambda} \). We omit the details of showing that we can actually find such values for \( h \) and \( p \) since the procedure is so similar to that followed in the proof of Theorem 2. By application of Lemma 1 we see that
\[
\tilde{w}(x, t) < u(\Psi, x, t) \quad \text{for all} \quad x \in [a, b] \quad \text{and} \quad t \in R_{\bar{T}}.
\]
But for \( t = T \) we have
\[
\tilde{w}(x, T) = w(x, \tilde{\lambda}) < u(\Psi, x, T).
\]
This together with (3.28), yields the desired result and completes the proof.

We are able to modify the hypotheses of Theorems 3a and 3b in the same manner as we did for Theorem 2. That is, condition (iv) may be replaced either (3.25) or (3.26).

4. Example

Consider the PDE
\[
u_t = (1 + u^2) u_{xx} - uu_x^2 = F(x, u, u_x, u_{xx}) \quad \text{for all} \quad x \in (1, 2)
\]
(4.1)
together with the boundary condition
\[
u_x(1, t) = f_1(u) \quad \text{and} \quad \nu_x(2, t) = f_2(u).
\]
(4.2)
The ODE
\[(1 + u^2) u_{xx} - uu_x^2 = 0\]
has as a one parameter family of solutions
\[v(x, \lambda) = \sinh \lambda x.\]
(4.3)
If we differentiate \( v \) with respect to \( x \) and then eliminate \( \lambda \) between the resulting equation and (4.3) we obtain
\[v_x(x, \lambda) = \frac{1}{x} [1 + v^2(x, \lambda)]^{1/2} \sinh^{-1} v(x, \lambda).\]
Suppose

\[ f_1(u) > (1 + u^2)^{1/2} \sinh^{-1} u \quad \text{for} \quad u > 0, \quad f_1(0) = 0, \]  
\[ f_1(u) < (1 + u^2)^{1/2} \sinh^{-1} u \quad \text{for} \quad u < 0, \]  
\[ f_2(u) < \left(\frac{1}{2}\right)(1 + u^2)^{1/2} \sinh^{-1} u \quad \text{for} \quad u > 0, \quad f_2(0) = 0, \]  
\[ f_2(u) > \left(\frac{1}{2}\right)(1 + u^2)^{1/2} \sinh^{-1} u \quad \text{for} \quad u < 0. \]  

If (4.4) through (4.7) hold we have, by Theorem 1, that the identically zero solution is stable. In order to apply Theorem 2 and show that we have asymptotic stability we must check to see if \( F_\phi \neq 0 \). We have

\[ F_\phi(x, \psi, \psi_x, \psi_{xx}) = 2(\psi) \psi_{xx} - \psi_x^2 \]
\[ = 2\lambda^2 \sinh^2 \lambda x - \lambda^2 \cosh^2 \lambda x \]
\[ - \lambda^2 (\sinh^2 \lambda x - 1). \]

Thus \( F_\phi < 0 \) if \( \sinh^2 \lambda x < 1 \) or \( u < 1 \). Hence we can apply Theorem 2 and we conclude that if the above conditions hold, the identically zero solution is asymptotically stable. If inequalities (4.4) and (4.6) or (4.5) and (4.7) were reversed we would conclude from Theorem 3a or 3b that the trivial solution is unstable.

There is still much information regarding PDE (4.1) with boundary conditions (4.2) that can be obtained by methods similar to those we used in proving Theorems 2 and 3. First note that if conditions (4.4) through (4.7) holds, so that the trivial solution is asymptotically stable, we may be interested in the transient part of the solution. Given a particular function \( \Psi(x) \), we may proceed as in proof of Theorem 2 to obtain an upper (lower) bound for \( u(\Psi, x, t) \). Thus for an arbitrary \( \epsilon > 0 \) we would find a \( T \) such that

\[ \max_{x \in [1, a]} u(\Psi, x, t) < \epsilon. \]

In order to apply the theorems on stability or instability it is necessary that certain combinations of inequalities (4.4) through (4.7) all hold or are all reversed. Suppose, for example, inequality (4.4) is reversed while (4.5) through (4.7) hold. Then none of the theorems given in this paper apply to this problem. It may still be possible to find a bounding function (i.e., a function which bounds the solution \( u(\Psi, x, t) \) of the above problem) which will give us the information we seek.

The methods of finding the bounding function is as follows. First find a one parameter family of curves that satisfies the necessary boundary conditions. Then make the parameter a function of time in such a way that inequality

\[ u_t \geq F(x, u, u_x, u_{xx}) \]

or

\[ u_t \leq F(x, u, u_x, u_{xx}) \]
is satisfied. Which of the inequalities we try to satisfy depends on whether we wish to bound the function from above or below.

As an example of the method, consider the function

$$ S(x, a) = ax $$ \hspace{1cm} (4.8) $$

and the related function

$$ \tilde{S}(x, t) = a(t)x $$ \hspace{1cm} (4.9) $$

We differentiate (4.8) to get

$$ S_x(x, a) = a. $$ \hspace{1cm} (4.10) $$

We eliminate $a$ between (4.8) and (4.10) and obtain

$$ S_x(1, a) = S(1, a) \quad \text{and} \quad S_x(2, a) = \left(\frac{1}{2}\right) (S(2, a)). $$

The substitution of (4.9) into (4.1) yields

$$ a'(t) = -a^2(t). $$ \hspace{1cm} (4.11) $$

The general solution of (4.11) is

$$ a(t) = \pm \frac{1}{[2(t + C)]^{1/2}}, $$

and therefore

$$ \tilde{S}(x, t) = \pm \frac{x}{[2(t + C)]^{1/2}} $$ \hspace{1cm} (4.12) $$

is a solution of (4.9). We suppose the functions $f_1$ and $f_2$ are such that

$$ f_1(u) > u \quad \text{and} \quad f_2(u) > \frac{u}{2} \quad \text{if} \quad u > 0 $$ \hspace{1cm} (4.13) $$

and

$$ f_1(u) < u \quad \text{and} \quad f_2(u) < \frac{u}{2} \quad \text{if} \quad u < 0. $$ \hspace{1cm} (4.14) $$

Given any function $\Psi(x)$ we may select values $C_1$ and $C_2$ for $C$ in Eq. (4.12) so that

$$ \tilde{S}_1(x, 0) = -\left[\frac{1}{2C_1}\right]^{1/2} x < \Psi(x) < \left[\frac{1}{2C_2}\right]^{1/2} x = \tilde{S}_2(x, 0). $$

Since $\tilde{S}_1(x, t)$, $\tilde{S}_2(x, t)$, and $u(\Psi, x, t)$ are all solutions of (4.1) it follows by Lemma 1 that

$$ \tilde{S}_1(x, t) < u(\Psi, x, t) < \tilde{S}_2(x, t) \quad \text{for all} \quad x \in [a, b] \quad \text{and} \quad t > 0. $$
Both $S_1$ and $S_2$ go to zero as $t$ goes to infinity so we see that the identically zero solution is asymptotically stable. Note that if inequalities (4.13) or (4.14) hold for all $u$ then the region of asymptotic stability is the point set

$$A = \{(x, u) \mid x \in [a, b], -\infty < u < \infty\}.$$ 

We also see that we are able to obtain upper and lower bounds on the function at any time $t$.

5. Appendix

We wish to generalize Lemma 1 to $n$-dimensions. The following notation is used in the statement of Lemma 1a and its proof. Let $G$ be an open bounded region in $n$-dimensional euclidean space, $B$ the boundary of $G$. For $x \in G$ then $x_1, x_2, \ldots, x_n$ represents its coordinates in some fixed cartesian coordinate system. Let

$$D_T = \{(x, t) \mid x \in G \text{ and } 0 < t \leq T\} \quad \text{if } T < \infty,$$

$$D_T = \{(x, t) \mid x \in G \text{ and } t > 0\} \quad \text{if } T = \infty,$$

$$E_T = \{(x, t) \mid x \in B \text{ and } 0 < t \leq T\} \quad \text{if } T < \infty,$$

$$E_T = \{(x, t) \mid x \in B \text{ and } t > 0\} \quad \text{if } T = \infty.$$ 

For functions $u(x, t)$ and $v(x, t)$ defined for all $(x, t) \in D_T$ we let $p_i = u_{x_i}$, $q_i = v_{x_i}$, $r_{ij} = u_{x_i x_j}$, $s_{ij} = v_{x_i x_j}$, $\rho = (\rho_1, \rho_2, \ldots, \rho_n)$, $q = (q_1, q_2, \ldots, q_n)$, $r = (r_{11}, r_{12}, \ldots, r_{nn})$ and $s = (s_{11}, s_{12}, \ldots, s_{nn})$. Let $u_N(x, t)$ and $v_N(x, t)$ be the derivatives of $u$ and $v$ in the direction of the outward normal to the hypersurface $E_T$.

**Lemma 1a.** Suppose $u, v, f, g,$ and $F$ are functions satisfying the following conditions:

(i) $u(x, t)$ and $v(x, t)$ are of class $C'$ for all $(x, t) \in D_T$ where $T$ is a positive number or infinity.

(ii) $u$ and $v$ are twice continuously differentiable with respect to $x$ for all $(x, t) \in D_T$.

(iii) $u_N(x, t) = f(u(x, t))$, $v_N(x, t) = g(v(x, t))$ for all $(x, t) \in E_T$, where $f$ and $g$ are continuous functions.

(iv) $f(u) < g(u)$ for all $u \in [D(f) \cap D(g)]$.

(v) $F(x, t, u, p, r)$ is of class $C'$ and satisfies the conditions

$$\sum_{i,j=1}^n F_{\xi_i \xi_j} \xi_i \xi_j \geq 0$$ 

for all real numbers $\xi_i$.

(vi) $F(x, t, v, q, s) - v_t \geq F(x, t, u, p, r) - u_t$ for all $(x, t) \in D_T$.
Then if \( v(x, 0) < u(x, 0) \) for all \( x \in G \) we have
\[
v(x, t) < u(x, t) \quad \text{for all} \quad (x, t) \in \overline{D}_T.
\]

**Proof.** As we did for Lemma 1 we divide the proof into two parts. The first part is a proof of the lemma with condition (vi) replaced by the condition

\[(vi-a) \quad F(x, t, v, q, s) - v, \geq F(x, t, u, p, r) - u, .\]

Only the proof of the first part is given since the proof of the second part requires no essential change from the proof of the second part of Lemma 1.

**Part 1.** We assume all the hypotheses hold with condition (vi) replaced by the stronger condition (vi-a). Deny the conclusion. Define a function \( h(x, t) \) by
\[
h(x, t) = v(x, t) - u(x, t).
\]

Let \( t_1 \) be the glb of the set
\[
S = \{ t \mid h(x, t) \geq 0 \text{ for some } x \in G \}.
\]

We have from the continuity of \( h \) with respect to \( t \) and the definition of \( t_1 \) that
\[
\sup_{x \in G} (h(x, t_1)) = 0. \quad (A.1)
\]

Hence there is a point \( x_1 \in G \) such that
\[
h(x_1, t_1) = 0.
\]

To show that \( x_1 \notin B \), suppose \( x_1 \in B \). Then
\[
h_N(x_1, t_1) = v_N(x_1, t_1) - u_N(x_1, t_1)
\]
\[
g(x_1, t_1) - f(x_1, t_1) > 0.
\]

The \( x' \in G \) is a point on the normal to the hypersurface \( B \) at \( x_1 \), \( x' \) sufficiently close to \( x_1 \), we have, from the fact that \( h \) is of class \( C' \) and by application of the mean value theorem, that \( h(x', t) > 0 \). But this is a contradiction to (A.1), thus \( x_1 \in G \). For fixed \( t \), \( h(x, t_1) \) is a function of \( x \) only; hence it attains its interior maximum at the point \( x_1 \). Therefore we conclude
\[
h(x_1, t_1) = 0 \quad \text{hence} \quad v(x_1, t_1) = u(x_1, t_1), \quad (A.2)
\]
\[
h_{x_i}(x_1, t_1) = 0 \quad \text{hence} \quad p_i(x_1, t_1) = q_i(x_1, t_1) \quad \text{for} \quad i = 1, 2, \ldots, n,
\]
\[
\sum_{i, j=1}^n h_{x_i x_j}(x_1, t_1) \xi_i \xi_j \leq 0
\]
hence
\[ h_i(x_1, t_1) \geq 0 \quad \text{hence} \quad v_i(x_1, t_2) \quad u_i(x_1, t_1) \geq 0. \]  
\[ (A.3) \]

From the theory of matrices it follows that
\[ \sum_{i,j=1}^{n} (F_{ij}(r_{ij} - s_{ij}) > 0 \quad \text{at the point} \quad (x_1, t_1), \]
\[ (A.4) \]

hence from (A.2), (A.3) and by application of the mean value theorem we have
\[ F(x_1, t_1, u, p, r) \geq F(x_1, t_1, v, q, s). \]  
\[ (A.5) \]

If we transpose terms and evaluate the functions at the point \((x_1, t_1)\), condition (vi-a) becomes
\[ F(x_1, t_1, v, q, s) - F(x_1, t_1, u, p, r) > v_i - u_t \geq 0 \]
or
\[ F(x_1, t_1, u, p, r) < F(x_1, t_1, v, q, s). \]  
\[ (A.6) \]

Since (A.5) and (A.6) cannot both hold we conclude that the set \(S\) is empty.

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