Orthogonal graphs of odd characteristic and their automorphisms

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Abstract

The (isotropic) orthogonal graph $O(2v + \delta, q)$ over $\mathbb{F}_q$ of odd characteristic, where $v \geq 1$ and $\delta = 0, 1$ or 2 is introduced. When $v = 1$, $O(2 \cdot 1 + \delta, q)$ is a complete graph. When $v \geq 2$, $O(2v + \delta, q)$ is strongly regular and its parameters are computed, as well as its chromatic number. The automorphism groups of orthogonal graphs are also determined.

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1. Introduction

Let $\mathbb{F}_q$ be a finite field with $q$ elements where $q$ is a power of an odd prime $p$ and $n \geq 2$ be an integer. Let $\mathbb{F}_q^n = \{(a_1, \ldots, a_n): a_i \in \mathbb{F}_q, \ i = 1, \ldots, n\}$ be the $n$-dimensional row vector space over $\mathbb{F}_q$. Denote by $e_i$ ($1 \leq i \leq n$) the row vector in $\mathbb{F}_q^n$ whose $i$th component is 1 and all others are 0’s. For any $\alpha_1, \ldots, \alpha_k \in \mathbb{F}_q^n$, denote the subspace of $\mathbb{F}_q^n$ generated by $\alpha_1, \ldots, \alpha_k$ by $[\alpha_1, \ldots, \alpha_k]$. Thus, for a nonzero vector $\alpha = (a_1, \ldots, a_n) \in \mathbb{F}_q^n$, $[\alpha]$ is a one-dimensional subspace of $\mathbb{F}_q^n$ and $[\alpha] = [k\alpha]$ for any $k \in \mathbb{F}_q^n = \mathbb{F}_q \setminus \{0\}$. $[\alpha] = \{(a_1, \ldots, a_n)\}$ is sometimes written as $[a_1, \ldots, a_n]$ for simplicity.

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Let
\[
S_{2\nu+\delta} = \begin{pmatrix}
0 & I^{(v)} \\
I^{(v)} & 0 \\
\Delta
\end{pmatrix},
\]
where \(\delta = 0, 1\) or 2 and
\[
\Delta = \begin{cases}
\emptyset, & \text{if } \delta = 0, \\
1, & \text{if } \delta = 1, \\
\begin{pmatrix} 1 \ -z \end{pmatrix}, & \text{if } \delta = 2,
\end{cases}
\]
and \(\emptyset\) is always omitted if \(\delta = 0\). Sometimes, we write \(S\) for \(S_{2\nu+\delta}\) for simplicity. For any subspace \(P\) of \(\mathbb{F}_q^n\), let \(P^\perp = \{y \in \mathbb{F}_q^n : yS'x = 0\ \text{for all } x \in P\}\). Clearly, \(P^\perp\) is a subspace of \(\mathbb{F}_q^n\) and is called the dual subspace of \(P\) with respect to \(S\). Let \(P\) be any \(m\)-dimensional subspace of \(\mathbb{F}_q^n\), any \(m \times n\) matrix whose rows form a basis of \(P\) is called a matrix representation of \(P\), which is also denoted by \(P\). Then \(P\) is called a totally isotropic or non-isotropic subspace if and only if \(PS'P = 0\) or \(PS'P\) is nonsingular, respectively. A vector \(v \in \mathbb{F}_q^n\) is called isotropic or non-isotropic if \([v]\) is totally isotropic or non-isotropic, respectively. Clearly \(v \in \mathbb{F}_q^n\) is an isotropic or non-isotropic vector if and only if \(vS'v = 0\) or \(vS'v \neq 0\), respectively.

The (isotropic) orthogonal graph with respect to \(S_{2\nu+\delta}\) over \(\mathbb{F}_q\), denoted by \(O(2\nu + \delta, q)\), is the graph with the set of 1-dimensional totally isotropic subspaces of \(\mathbb{F}_q^{2\nu+\delta}\) as the vertex set \(V(O(2\nu + \delta, q))\) and with the adjacency defined by \([\alpha] \sim [\beta]\) if and only if \(\alpha S'\beta \neq 0\) for any nonzero isotropic vectors \(\alpha, \beta\), where \([\alpha] \sim [\beta]\) reads as: \([\alpha]\) and \([\beta]\) are adjacent. Such graph has been mentioned in [4] (Section 8: 2, 4+, and 4-).

The symplectic graphs were studied previously by Godsil and Royle [1, 2], Tang and Wan [6]. The unitary graphs and the orthogonal graphs of characteristic 2 were studied by Wan and Zhou [8, 9]. In the present paper, we study the (isotropic) orthogonal graphs over finite fields of odd characteristic. In Section 2, we show that \(O(2\nu + \delta, q)\) is complete or strongly regular as \(\nu = 1\) or \(\nu \geq 2\), respectively, and compute its parameters as well as its chromatic number. Sections 3–5 are devoted to discussing the groups of automorphisms \(\text{Aut}(O(2\nu + \delta, q))\) of the graph (cf. Theorems 3.3, 4.1 and 5.1). For notations and terminologies, we refer to [2, 7].

2. Strong regularity and chromatic numbers

**Theorem 2.1.** When \(\nu = 1\), \(O(2 \cdot 1 + \delta, q)\), where \(\delta = 0, 1\) or 2, is a complete graph with parameters \((q^\delta + 1, q^\delta)\). When \(\nu \geq 2\), \(O(2\nu + \delta, q)\), where \(\delta = 0, 1\) or 2, is a strongly regular graph with parameters \((n, k, a, c)\), where \(n = (q^{\nu-1})(q^{\nu+\delta-1} + 1)(q - 1)^{-1}, k = q^{2\nu+\delta-2}, a = q^{2\nu+\delta-2} - q^{2\nu+\delta-3} - q^{\nu-1} + q^{\nu+\delta-2}, c = q^{2\nu+\delta-2} - q^{2\nu+\delta-3}\) and eigenvalues \(q^{2\nu+\delta-2}, q^{\nu+\delta-2}\) and \(-q^{\nu-1}\).

**Proof.** We adopt the notations of [7]. By [7, Corollary 6.23], \(n = |V(O(2\nu + \delta, q))| = N(1, 0, 0; 2\nu + \delta, \Delta) = (q^{\nu-1})(q^{\nu+\delta-1} + 1)(q - 1)^{-1}\). For any \([\alpha] \in V(O(2\nu + \delta, q))\), the degree \(k\) of \([\alpha]\) is the number of \([\beta] \in V(O(2\nu + \delta, q))\) such that \(\beta \notin [\alpha]^{\perp}\). Since \([\alpha]\) is a subspace of \(\mathbb{F}_q^{2\nu+\delta}\) of type \((1, 0, 0)\), by [7, Corollary 6.6], \([\alpha]^{\perp}\) is a subspace of \(\mathbb{F}_q^{2\nu+\delta}\) of type \((2\nu + \delta - 1, 2(\nu - 1) + \delta, \nu - 1, \Delta)\). By [7, Theorem 6.33], the number of 1-dimensional isotropic subspaces of \([\alpha]^{\perp}\) is equal to
$N = N(1, 0, 0; 2\nu + \delta - 1, 2(\nu - 1) + \delta, \nu - 1, \Delta; 2\nu + \delta, \Delta)$

$$= q(q^{v-1} - 1)(q^{v+\delta-2} + 1)(q - 1)^{-1} + 1.$$  

Thus

$$k = n - N = (q^{v-1} - 1)(q^{v+\delta-1} + 1)(q - 1)^{-1} - (q(q^{v-1} - 1)(q^{v+\delta-2} + 1)(q - 1)^{-1} + 1)
= q^{2v + \delta - 2}.$$  

In particular, when $v = 1$, $n = q^\delta + 1$ and $k = q^\delta$. This proves that $O(2 \cdot 1 + \delta, q)$ is a complete graph with parameters $(q^\delta + 1, q^\delta)$.

From now on we assume $v \geq 2$. Let $[\alpha], [\beta]$ be two adjacent vertices in $O(2\nu + \delta, q)$ and $[\gamma]$ be any vertex adjacent with both of them. Then $[\alpha, \beta]$ is a subspace of type $(2, 2 \cdot 1, 1)$. By [7, Corollary 6.6], $[\alpha, \beta] \perp$ is a subspace of type $(2\nu + \delta - 2, 2(\nu - 1) + \delta, \nu - 1, \Delta)$. By [7, Theorem 6.33], the number of 1-dimensional isotropic subspaces of $[\alpha, \beta] \perp$ is equal to

$$N' = N(1, 0, 0; 2\nu + \delta - 2, 2(\nu - 1) + \delta, \nu - 1, \Delta; 2\nu + \delta, \Delta)
= (q^{v-1} - 1)(q^{v+\delta-2} + 1)(q - 1)^{-1}.$$  

Note that the value of $a$ is the number of $[\gamma] \in V(O(2\nu + \delta, q))$ such that $[\gamma] \notin [\alpha] \perp \cup [\beta] \perp$. Since $|[\alpha] \perp \cup [\beta] \perp| = |[\alpha] \perp| + |[\beta] \perp| - |[\alpha, \beta] \perp| = 2N - N'$, $a = n - (2N - N') = q^{2v + \delta - 2} - q^{2v + \delta - 3} - q^{v-1} + q^{v+\delta-2}.$

Let $[\alpha], [\beta]$ be two non-adjacent vertices in $O(2\nu + \delta, q)$ and $[\gamma]$ be any vertex adjacent with both of them. Then $[\alpha, \beta]$ is of type $(2, 0, 0)$. By [7, Corollary 6.6], $[\alpha, \beta] \perp$ is of type $(2\nu + \delta - 2, 2(\nu - 2) + \delta, \nu - 2, \Delta)$. By [7, Theorem 6.33], the number of 1-dimensional isotropic subspaces of $[\alpha, \beta] \perp$ is equal to

$$N'' = N(1, 0, 0; 2\nu + \delta - 2, 2(\nu - 2) + \delta, \nu - 2, \Delta; 2\nu + \delta, \Delta)
= q^2(q^{v-2} - 1)(q^{v+\delta-3} + 1)(q - 1)^{-1} + q + 1.$$  

Note that the value of $c$ is the number of $[\gamma] \in V(O(2\nu + \delta, q))$ such that $[\gamma] \notin [\alpha] \perp \cup [\beta] \perp$. So $c = n - (2N - N'') = q^{2v + \delta - 2} - q^{2v + \delta - 3}.$

By the same arguments as in [2, Section 10.2], we have that the eigenvalues of $O(2\nu + \delta, q)$ are $q^{2v + \delta - 2}, q^{v+\delta-2}$ and $-q^{v-1}$. □

Now, we come to determine the chromatic number $\chi(O(2\nu + \delta, q))$ of $O(2\nu + \delta, q)$. We start by stating a bound on the size of independent set of vertices in $O(2\nu + \delta, q)$.

**Proposition 2.2.** (Hoffman [5]) If $X$ is a regular graph on $n$ vertices with valency $k$ and minimum eigenvalue $\theta_{\text{min}}$, then the maximum size $\alpha(X)$ of an independent set of vertices satisfies

$$\alpha(X) \leq \frac{n}{1 - \frac{k}{\theta_{\text{min}}}}.$$
The minimum eigenvalue of $O(2v + \delta, q)$ is $-q^{v-1}$. Applying this proposition to $O(2v + \delta, q)$, we get that

$$\alpha(O(2v + \delta, q)) \leq \frac{q^v - 1}{q - 1}$$

and hence

$$\chi(O(2v + \delta, q)) \geq q^{v+\delta-1} + 1.$$

**Lemma 2.3.** There exist $m = q^{v+\delta-1} + 1$ maximal totally isotropic subspaces $V_1, \ldots, V_m$ of $\mathbb{F}_q^{2v+\delta}$ such that $V_1 \cup V_2 \cup \cdots \cup V_m$ is the set of isotropic vectors of $\mathbb{F}_q^{2v+\delta}$ and $V_i \cap V_j = \{0\}$ for all $i \neq j$.

**Proof.** We distinguish the cases: $\delta = 0, 1$ or $2$.

1. $\delta = 0$. Set $W = [e_{v+1}, \ldots, e_{2v}]$ and

$$W(a_2, \ldots, a_v) = [e_1 - a_2e_{v+2} - \cdots - a_ve_{2v}, e_2 + a_2e_{v+1}, \ldots, e_v + a_ve_{v+1}],$$

where $a_i \in \mathbb{F}_q$, $i = 2, \ldots, v$. It can be checked that $W(a_2, \ldots, a_v)$ and $W$ are maximal totally isotropic. Moreover, each two of them intersect only at $\{0\}$.

2. $\delta = 1$. Similar to the case $\delta = 0$, the $q^v + 1$ maximal totally isotropic subspaces are $[e_{v+1}, \ldots, e_{2v}]$ and

$$[e_1 - 2^{-1}a_{v+1}^2e_{v+1} - a_2e_{v+2} - \cdots - a_{v+1}e_{2v+1}, e_2 + a_2e_{v+1}, \ldots, e_v + a_ve_{v+1}],$$

where $a_i \in \mathbb{F}_q$, $i = 2, \ldots, v + 1$.

3. $\delta = 2$. Similarly, the $q^{v+1} + 1$ maximal totally isotropic subspaces are $[e_{v+1}, \ldots, e_{2v}]$ and

$$[e_1 - 2^{-1}(a_{v+1}^2 - za_{v+2}^2)e_{v+1} - a_2e_{v+2} - \cdots - a_{v+2}e_{2v+2}, e_2 + a_2e_{v+1}, \ldots, e_v + a_ve_{v+1}],$$

where $a_i \in \mathbb{F}_q$, $i = 2, \ldots, v + 2$. q.e.d.

**Proposition 2.4.** $O(2v + \delta, q)$ is $(q^{v+\delta-1} + 1)$-partite. That is, there exist subsets $X_1, \ldots, X_m$ of $V(O(2v + \delta, q))$, where $m = q^{v+\delta-1} + 1$, such that

$$V(O(2v + \delta, q)) = X_1 \cup \cdots \cup X_m$$

and $X_i \cap X_j = \emptyset$ for all $i \neq j$, and there exists no edge of $O(2v + \delta, q)$ joining two vertices of the same subset.

**Proof.** By Lemma 2.3, let $V_1 \cup V_2 \cup \cdots \cup V_m$ be the set of isotropic vectors of $\mathbb{F}_q^{2v+\delta}$, where $V_i \cap V_j = \{0\}$ for all $i \neq j$. Set $X_i = \{[\alpha] : \alpha \in V_i, \alpha \neq 0\}$, $i = 1, 2, \ldots, m$. Then

$$V(O(2v + \delta, q)) = X_1 \cup \cdots \cup X_m,$$

where $X_i \cap X_j = \emptyset$ for all $i \neq j$. As $V_i$ is totally isotropic, there is no edge joining any two vertices in $X_i$. Therefore $O(2v + \delta, q)$ is $(q^{v+\delta-1} + 1)$-partite. q.e.d.
Theorem 2.5. $\chi(O(2v + \delta, q)) = q^{v+\delta-1} + 1$.

Proof. It follows from Proposition 2.4, $\chi(O(2v + \delta, q)) \leq q^{v+\delta-1} + 1$. But we already proved $\chi(O(2v + \delta, q)) \geq q^{v+\delta-1} + 1$. Therefore, $\chi(O(2v + \delta, q)) = q^{v+\delta-1} + 1$. □

3. Automorphism groups (I): The case $\delta = 0$

Recall that a $(2v + \delta) \times (2v + \delta)$ nonsingular matrix $T$ over $\mathbb{F}_q$, where $v \geq 1$ and $\delta = 0, 1$ or 2, is called an orthogonal matrix of order $2v + \delta$ with respect to $S = S_{2v+\delta}$ over $\mathbb{F}_q$ if $T S' T = S$. The set of orthogonal matrices of order $2v + \delta$ over $\mathbb{F}_q$ forms a group with respect to the matrix multiplication, which is called the orthogonal group of degree $2v + \delta$ with respect to $S$ over $\mathbb{F}_q$ and denoted by $O_{2v+\delta}(\mathbb{F}_q)$. The factor group $O_{2v+\delta}(\mathbb{F}_q)/\{I, -I\}$ is called the projective orthogonal group of degree $2v + \delta$ over $\mathbb{F}_q$, denoted by $PO_{2v+\delta}(\mathbb{F}_q)$.

When $v = 1$, $O(2 \cdot 1 + \delta, q)$, where $\delta = 0, 1$ or 2, is a complete graph with parameters $(q^\delta + 1, q^\delta)$ and, hence, its automorphism group is the symmetric group on $q^\delta + 1$ elements. In the following, we assume $v \geq 2$.

Proposition 3.1. Let $T \in O_{2v+\delta}(\mathbb{F}_q)$ and

$$\sigma_T : V(O(2v + \delta, q)) \to V(O(2v + \delta, q))$$

$$[\alpha] \mapsto [\alpha T].$$

Then:

(i) $\sigma_T \in \text{Aut}(O(2v + \delta, q))$.

(ii) For any $T_1, T_2 \in O_{2v+\delta}(\mathbb{F}_q)$, $\sigma_{T_1} = \sigma_{T_2}$ if and only if $T_1 = \pm T_2$.

Proof. (i) Since $T$ is nonsingular, $\sigma_T$ is a bijection. For any $[\alpha], [\beta] \in V(O(2v + \delta, q))$, $\alpha S' \beta = \alpha T S' (\beta T)$, it follows that $[\alpha] \sim [\beta]$ if and only if $\sigma_T([\alpha]) \sim \sigma_T([\beta])$. Hence $\sigma_T \in \text{Aut}(O(2v + \delta, q))$.

(ii) Clearly $T_1 = \pm T_2$ implies $\sigma_{T_1} = \sigma_{T_2}$. Conversely, suppose $\sigma_{T_1} = \sigma_{T_2}$. Let $V = V(O(2v + \delta, q))$, then for any $[\alpha] \in V$, $\alpha T_1 = k \alpha T_2$ for some $k \in \mathbb{F}_q^*$, where $k$ depends on $[\alpha]$.

When $\delta = 0$, taking $[\alpha] = [e_1], \ldots, [e_{2v}] \in V$, we have $T_1 = \text{diag}(k_1, \ldots, k_{2v}) T_2$ for some $k_1, \ldots, k_{2v} \in \mathbb{F}_q^*$. Then taking $[\alpha] = [e_1 + e_2], [e_2 + e_3], \ldots, [e_{2v-1} + e_{2v}] \in V$, we see that $k_1 = k_2 = \cdots = k_{2v}$. Hence $T_1 = k_1 T_2$, $k_1 I = T_1 T_2^{-1} \in O_{2v}(\mathbb{F}_q)$. Thus $k_1 I S'(k_1 I) = S$, which implies $k_1^2 = 1$, so $k_1 = \pm 1$.

When $\delta = 1$, taking $[\alpha] = [e_1], [e_2], \ldots, [e_{2v}], [e_1 - 2^{-1} e_{v+1} + e_{v+1}] \in V$, we get

$$T_1 = \left( \begin{array}{cc} \text{diag}(k_1, \ldots, k_{2v}) \\ \xi \end{array} \right) T_2$$

where $k_1, \ldots, k_{2v+1} \in \mathbb{F}_q^*$ and $\xi = (k_{2v+1} - k_1)e_1 - 2^{-1}(k_{2v+1} - k_{v+1})e_{v+1}$. Then taking $[\alpha] = [e_1 + e_2], [e_2 + e_3], \ldots, [e_{2v-1} + e_{2v}] \in V$, we see that $k_1 = k_2 = \cdots = k_{2v}$. Finally, taking $[\alpha] = [e_1 + e_2 - 2^{-1} e_{v+1} + e_{v+1}] \in V$, we get $k_{2v+1} = k_2$. Hence $k_1 = \cdots = k_{2v+1}$. So $T_1 = k_1 T_2$. As the case $\delta = 0$, we have also $k_1 = \pm 1$.

The case $\delta = 2$ can be treated in the same way as the case $\delta = 1$. □
Corollary 3.2. $O(2v + \delta, q)$ is vertex transitive and edge transitive.

Proof. Vertex transitivity is clear from [7, Lemma 6.8]. For edge transitivity, let $[\alpha_1], [\alpha_2], [\beta_1], [\beta_2] \in V(O(2v + \delta, q))$ such that $[\alpha_1] \sim [\alpha_2]$ and $[\beta_1] \sim [\beta_2]$. We may assume that $\alpha_1 S' \alpha_2 = \beta_1 S' \beta_2$. Then by [7, Lemma 6.8], there exists $T \in O_{2v+\delta}(\mathbb{F}_q)$ such that $\alpha_1 T = \beta_1, \alpha_2 T = \beta_2$. Thus $\sigma_T \in \text{Aut}(O(2v + \delta, q))$ and $\sigma_T([\alpha_1]) = [\beta_1], \sigma_T([\alpha_2]) = [\beta_2]$. Hence $O(2v + \delta, q)$ is edge transitive. \qed

By Proposition 3.1, every orthogonal matrix in $O_{2v+\delta}(\mathbb{F}_q)$ induces an automorphism of $O(2v + \delta, q)$ and two different orthogonal matrices $T_1$ and $T_2$ induce the same automorphism of $O(2v + \delta, q)$ if and only if $T_1 = \pm T_2$. Thus $PO_{2v+\delta}(\mathbb{F}_q)$ can be regarded as a subgroup of $\text{Aut}(O(2v + \delta, q))$.

Now we write $f_i$ for $e_v+i$, $1 \leq i \leq v$. Then $e_i S' f_i = 1$, $e_i S' e_j = 0$, $f_i S' f_j = 0$, $i, j = 1, 2, \ldots, v$, and $e_i S' f_j = 0$, $i \neq j$, $i, j = 1, 2, \ldots, v$.

Let us determine the group of graph automorphisms of $O(2v, q)$ first. We need a concept from group theory (cf. [3] or [10]). Let $K, H$ be abstract groups and $\psi$ be a homomorphism of $K$ into $\text{Aut}(H)$. The semidirect product of $H$ and $K$ corresponding to $\psi$, denoted by $H \rtimes_{\psi} K$, is the group consisting of ordered pairs $(h, k)$, $h \in H$, $k \in K$, with the operation:

$$(h_2, k_2)(h_1, k_1) = (h_2 \psi(k_2)(h_1), k_2 k_1) \quad \text{for } h_i \in H, \ k_i \in K, \ i = 1, 2.$$

Let $\varphi$ be the natural action of $\text{Aut}(\mathbb{F}_q)$ on the group $\mathbb{F}_q^* \times \cdots \times \mathbb{F}_q^*$ ($v$ in number) defined by

$$\varphi((\pi(k_1), \ldots, \pi(k_v))) = (\pi(k_1), \ldots, \pi(k_v)), \quad \text{for all } \pi \in \text{Aut}(\mathbb{F}_q) \text{ and } k_1, \ldots, k_v \in \mathbb{F}_q^*,$$

then the semidirect product of $\mathbb{F}_q^* \times \cdots \times \mathbb{F}_q^*$ by $\text{Aut}(\mathbb{F}_q)$ corresponding to $\varphi$, denoted by $(\mathbb{F}_q^* \times \cdots \times \mathbb{F}_q^*) \rtimes_{\varphi} \text{Aut}(\mathbb{F}_q)$, is the group consisting of all elements of the form $(k_1, \ldots, k_v, \pi)$, where $k_1, \ldots, k_v \in \mathbb{F}_q^*$ and $\pi \in \text{Aut}(\mathbb{F}_q)$, with the multiplication defined by

$$(k_1, \ldots, k_v, \pi)(k_1', \ldots, k_v', \pi') = (k_1 \pi(k_1'), \ldots, k_v \pi(k_v'), \pi \pi').$$

The main result about $\text{Aut}(O(2v, q))$ is the following.

Theorem 3.3. Let $v \geq 2$ and $E$ be the subgroup of $\text{Aut}(O(2v, q))$ defined as follows:

$$E = \{ \sigma \in \text{Aut}(O(2v, q)) : \sigma([e_i]) = [e_i], \ \sigma([f_i]) = [f_i], \ i = 1, \ldots, v \}.$$

Then $\text{Aut}(O(2v, q)) = PO_{2v}(\mathbb{F}_q) \cdot E$. Moreover,

(i) If $v = 2$, $E \cong \text{Sym}(\mathbb{F}_q^*) \times \text{Sym}(\mathbb{F}_q^*)$;

(ii) If $v \geq 3$, then

$$E \cong (\mathbb{F}_q^* \times \cdots \times \mathbb{F}_q^*) \rtimes_{\varphi} \text{Aut}(\mathbb{F}_q)$$

and the isomorphism from $(\mathbb{F}_q^* \times \cdots \times \mathbb{F}_q^*) \rtimes_{\varphi} \text{Aut}(\mathbb{F}_q)$ to $E$ is defined as:

$$h : (k_1, \ldots, k_v, \pi) \mapsto \sigma(k_1 \ldots k_v, \pi)$$
where \( \sigma_{(k_1, \ldots, k_v, \pi)} \) is the map that carries the vertex \([a_1, a_2, \ldots, a_{2v}]\) of \( O(2v, q) \) to the vertex

\[
[p(a_1), k_2 p(a_2), \ldots, k_v p(a_v), k_1 p(a_{v+1}), k_1 k_2^{-1} p(a_{v+2}), \ldots, k_1 k_v^{-1} p(a_{2v})].
\]

**Proof.** Let \( \tau \in \text{Aut}(O(2v, q)) \). Suppose \( \tau([e_i]) = [e'_i], \tau([f_i]) = [f'_i] \), \( i = 1, 2, \ldots, v \). Then \( e'_i S f'_i \neq 0, e'_i S e'_i = 0, f'_i S f'_i = 0, i, j = 1, 2, \ldots, v, \) and \( e'_i S f'_i = 0, i \neq j, i, j = 1, 2, \ldots, v \). We may choose \( e'_i, f'_i \) such that \( e'_i S f'_i = 1, i = 1, \ldots, v \). Let \( A (A'), \) respectively be the \( 2v \times 2v \) matrix whose rows are \( e_1, \ldots, e_v, f_1, \ldots, f_v \) \((e'_1, \ldots, e'_v, f'_1, \ldots, f'_v)\), respectively in order. Then \( A S' A = A' S' A' \). By [7, Lemma 6.8], there exists \( T \in O_{2v}(F_q) \) such that \( A = A'T \). Set \( \tau_1 = \sigma_T \tau \), then \( \tau_1([e_i]) = [e_i], \tau_1([f_i]) = [f_i], i = 1, \ldots, v \), and hence \( \tau_1 \in E \). Thus \( \tau = \tau_1^{-1} \tau_1 \in PO_{2v}(F_q) \). It follows that \( \text{Aut}(O(2v, q)) = PO_{2v}(F_q) \cdot E \).

Now let \( \sigma \in E \) and \( \sigma([a_1, \ldots, a_{2v}]) = [b_1, \ldots, b_{2v}] \). Clearly \( a_i \neq 0 \) if and only if \( [a_1, \ldots, a_{2v}] \not\sim [f_i] \), and \( a_{v+i} \neq 0 \) if and only if \( [a_1, \ldots, a_{2v}] \not\sim [e_i] \). But \( \sigma([e_i]) = [e_i], \sigma([f_i]) = [f_i] \), it follows that \( a_i \neq 0 \) if and only if \( b_i \neq 0, 1 \leq i \leq 2v \). For the vertex \([a_1, a_2, \ldots, a_{2v}] \), if \( a_1 = \cdots = a_{v-1} = 0 \) and \( a_{v} \neq 0 \), then \([a_1, a_2, \ldots, a_{2v}] \) can be uniquely written as \([0, 0, 0, 0, a_1, 0, \ldots, a_{2v}] \), and \( \sigma([a_1, \ldots, a_{2v}]) \) can be uniquely written as \([0, 0, 0, 1, b'_1 + 1, \ldots, b'_{2v}] \). In the following, we denote \([a_1, a_2, a_3, a_4] \) by \( \sum_{i=1}^{v} a_i [e_i] + \sum_{i=1}^{v} a_{v+i} [f_i] \). We will use frequently the fact that for any two vertices \([\alpha], [\beta] \), if \([\alpha] \not\sim [\beta] \), then \( \sigma([\alpha]) \not\sim \sigma([\beta]) \).

(i) \( v = 2 \). Since \( \sigma \in E \) is a bijection from \( V(O(2, q)) \) to itself, we have permutations \( \pi_2 \) and \( \pi_4 \) of \( F_q \) with \( \pi_2(0) = \pi_4(0) = 0 \) such that \( \sigma([1, a, 0, 0]) = [1, \pi_2(a), 0, 0] \) and \( \sigma([1, 0, 0, a]) = [1, 0, \pi_4(a)] \) for \( a \in F_q \).

Clearly \( V(O(2, q)) \) consists of four types of vertices: \([(1, a_1, a_2, a_3), a_2, a_3, a_4 \in F_q, a_3 = a_2 a_4], \) \([0, 1, a_3, 0]: a_3 \in F_q], \) \([0, 0, 1, a_4]: a_4 \in F_q], \) and \([0, 0, 0, 1] \).

Suppose \( \sigma([0, 0, 1, a_4]) = [0, 0, 1, a'_4] \). Assume \( a_4 \neq 0 \). From \( [0, 0, 1, a_4] \not\sim [1, -a_4^{-1}, 0, 0] \), we have \([0, 0, 1, a'_4] \not\sim [1, \pi_2(-a_4^{-1}), 0, 0] \), then \( a'_4 = -\pi_2(-a_4^{-1}) \).

Suppose \( \sigma([0, 1, a_3, 0]) = [0, 1, a'_3, 0] \). From \( [0, 1, a_3, 0] \not\sim [1, 0, 0, a_3] \), we have \([0, 1, a'_3, 0] \not\sim [1, 0, 0, \pi_4(-a_3)] \), then \( a'_3 = -\pi_4(-a_3) \).

Let \( [1, a_2, a_3, a_4] \) \( V(O(2, 2, q)) \), then \( a_3 + a_4 a_4 = 0 \). Suppose \( \sigma([1, a_2, a_3, a_4]) = [1, a'_2, a'_3, a'_4] \), then \( a'_2 + a'_3 a'_4 = 0 \). From \( [1, a_2, a_3, a_4] \not\sim [1, a_2, 0, 0] \), we have \([1, a'_2, a'_3, a'_4] \not\sim [1, \pi_2(a_2), 0, 0] \), then \( a'_2 = -\pi_2(a_2) a'_4 \). Similarly, from \([1, a_2, a_3, a_4] \not\sim [0, 0, a_4] \), we have \( a'_2 = -\pi_4(a_4) a'_2 \). Then \( a'_2 a'_4 = \pi_4(a_4) a'_2 = \pi_2(a_2) a'_4 \). Hence \( a'_2 = \pi_2(a_2), a'_2 = \pi_4(a_4) \).

In a word, for any \( \sigma \in E \), there exist permutations \( \pi_2 \) and \( \pi_4 \) of \( F_q \) such that:

\[
\begin{align*}
\sigma([0, 0, 0, 1]) &= [0, 0, 0, 1], \\
\sigma([0, 0, 1, 0]) &= [0, 0, 1, 0], \\
\sigma([0, 0, 1, a_4]) &= [0, 0, 1, -\pi_2(-a_4^{-1}), 0] \quad \text{if} \ a_4 \neq 0; \\
\sigma([0, 1, a_3, 0]) &= [0, 1, -\pi_4(-a_3), 0], \\
\sigma([1, a_2, a_3, a_4]) &= [1, \pi_2(a_2), -\pi_2(a_2) \pi_4(a_4), \pi_4(a_4)], \quad \text{if} \ a_3 = -a_2 a_4.
\end{align*}
\]

Conversely, for any permutations \( \pi_2 \) and \( \pi_4 \) of \( F_q \) with \( \pi_2(0) = \pi_4(0) = 0 \), if we define a map \( \sigma \) from \( V(O(2, 2, q)) \) to itself by \((*)\), then it is easy to see that \( \sigma \in E \). Moreover, it can also be verified that \( E \cong \text{Sym}(F_q^*) \times \text{Sym}(F_q^*) \).

(ii) \( v \geq 3 \). Clearly \( \sigma_{(k_1, \ldots, k_v, \pi)} \) is well defined and \( \sigma_{(k_1, \ldots, k_v, \pi)} \in E \), thus \( h \) is well defined. It is easy to verify that \( h \) is an injective group homomorphism from \( (F_q^* \times \cdots \times F_q^*) \times \text{Aut}(F_q) \)
to $E$. Thus to prove (ii) of the theorem, it remains to show that every element $\sigma$ of $E$ is of the form $\sigma(k_1, \ldots, k_v, \pi)$.

Let $\sigma \in E$. Since $\sigma$ is a bijection from $V(O(2v, q))$ to itself, we have permutations $\pi_i, i = 2, \ldots, v, v + 2, \ldots, 2v$, of $\mathbb{F}_q$ with $\pi_i(0) = 0$ such that

$$\sigma([e_1] + a_j[e_j]) = [e_1] + \pi_j(a_j)[e_j], \quad 2 \leq j \leq v,$$

$$\sigma([e_1] + a_{\nu + j}[f_j]) = [e_1] + \pi_{\nu + j}(a_{\nu + j})[f_j], \quad 2 \leq j \leq v.$$

**Lemma 3.4.** Let $[1, a_2, \ldots, a_{2v}] \in V(O(2v, q))$ and $\sigma([1, a_2, \ldots, a_{2v}]) = [1, a_2', \ldots, a_{2v}']$. Then $a_j' = \pi_j(a_j)$ and $a_{\nu + j}' = \pi_{\nu + j}(a_{\nu + j}), 2 \leq j \leq v$.

**Proof.** Let $a \in \mathbb{F}_q^*$. Suppose $\sigma([f_1] - a^{-1}[f_j]) = [f_1] + x[f_j]$. Since $[f_1] - a^{-1}[f_j] \not\sim [e_1] + a[e_j], [f_1] + x[f_j] \not\sim [e_1] + \pi_j(a)[e_j]$, which implies $x = -\pi_j(a)^{-1}$. Hence $\sigma([f_1] - a^{-1}[f_j]) = [f_1] - \pi_j(a)^{-1}[f_j]$. Similarly, $\sigma(e_j - a[f_j]) = [e_j] - \pi_{\nu + j}(a)[f_j]$.

Let $2 \leq j \leq v$. If $a_j = 0$, then $a_j' = \pi_j(a_j)$ holds trivially; if $a_j \neq 0$, from $[1, a_2, \ldots, a_{2v}] \not\sim [f_1] - a_j^{-1}[f_j], [1, a_2', \ldots, a_{2v}'] \not\sim [f_1] - \pi_j(a_j)^{-1}[f_j]$, which implies $a_j' = \pi_j(a_j)$. Similarly, from $[1, a_2', \ldots, a_{2v}'] \not\sim [e_j] - a_{\nu + j}[f_1]$, we deduce $a_{\nu + j}' = \pi_{\nu + j}(a_{\nu + j})$. \hfill $\Box$

**Lemma 3.5.**

(i) $\pi_i(1)^{-1}\pi_i = \pi_j(1)^{-1}\pi_j = \pi_{\nu + i}(1)^{-1}\pi_{\nu + i} = \pi_{\nu + j}(1)^{-1}\pi_{\nu + j},$ where $2 \leq i, j \leq v$.

(ii) For any $a \in \mathbb{F}_q, \pi_i(a)\pi_{\nu + i}(1) = \pi_{\nu + i}(a)\pi_i(1) = \pi_j(a)\pi_{\nu + j}(1) = \pi_{\nu + j}(a)\pi_j(1),$ where $2 \leq i, j \leq v$. In particular, $\pi_2(1)\pi_2(1) = \pi_3(1)\pi_3(1) = \cdots = \pi_v(1)\pi_v(1)$.

**Proof.** (i) Let $a \in \mathbb{F}_q$ and $2 \leq i, j \leq v, i \neq j$. From $[e_1] + a[e_i] + a[f_j] \not\sim [e_1] + [e_j] - [f_i]$, we have $[e_1] + \pi_i(a)[e_i] + \pi_{\nu + i}(a)[f_j] \not\sim [e_1] + \pi_i(1)[e_j] + \pi_{\nu + i}(1)(-1)[f_j]$, then $\pi_i(a)\pi_{\nu + i}(-1) + \pi_{\nu + i}(a)\pi_i(1) = 0$. Similarly, from $[e_1] + a[e_i] + a[e_j] \not\sim [e_1] - [f_i] + [f_j]$, we deduce $\pi_i(a)\pi_{\nu + i}(-1) + \pi_j(a)\pi_{\nu + j}(1) = 0$. Thus $\pi_{\nu + j}(a)\pi_j(1) = \pi_j(a)\pi_{\nu + j}(1)$, that is,

$$\pi_j(1)^{-1}\pi_j = \pi_{\nu + j}(1)^{-1}\pi_{\nu + j}, \quad \text{for } 2 \leq j \leq n, \ j \neq i. \quad (3.1)$$

If we interchange the roles of $i$ and $j$ in the above argument, we have

$$\pi_i(1)^{-1}\pi_i = \pi_{\nu + i}(1)^{-1}\pi_{\nu + i}, \quad \text{for } 2 \leq i \leq n, \ i \neq j. \quad (3.2)$$

Suppose $\sigma([e_i] - [e_j]) = [e_i] + x[e_j]$. Since $[e_i] - [e_j] \not\sim [e_1] + [f_i] + [f_j]$, we have $[e_i] + x[e_j] \not\sim [e_1] + \pi_{\nu + i}(1)[f_i] + \pi_{\nu + i}(1)[f_j]$, which implies $x\pi_{\nu + i}(1) + \pi_{\nu + i}(1) = 0$, that is, $x = -\pi_{\nu + i}(1)\pi_{\nu + j}(1)^{-1}$. Hence $\sigma([e_i] - [e_j]) = [e_i] - \pi_{\nu + i}(1)\pi_{\nu + j}(1)^{-1}[e_j]$. Since $[e_1] + a[f_i] + a[f_j] \not\sim [e_i] - [e_j]$, we have $[e_1] + \pi_{\nu + i}(a)[f_i] + \pi_{\nu + j}(a)[f_j] \not\sim [e_i] - \pi_{\nu + i}(1)\pi_{\nu + j}(1)^{-1}[e_j]$, then $\pi_{\nu + i}(a) - \pi_{\nu + j}(a)\pi_{\nu + i}(1)\pi_{\nu + j}(1)^{-1} = 0$. Hence

$$\pi_{\nu + i}(1)^{-1}\pi_{\nu + i} = \pi_{\nu + j}(1)^{-1}\pi_{\nu + j}. \quad (3.3)$$

From Eqs. (3.1)–(3.3), we deduce that:

$$\pi_i(1)^{-1}\pi_i = \pi_{\nu + i}(1)^{-1}\pi_{\nu + i} = \pi_{\nu + j}(1)^{-1}\pi_{\nu + j} = \pi_j(1)^{-1}\pi_j. \quad (3.4)$$
Lemma 3.6. We will show \( \pi_i(a) \pi_{i+1}(1) = \pi_{i+1}(a) \pi_i(1) \) and \( \pi_i(a) \times \pi_{i+1}(1) = \pi_{i+1}(a) \pi_i(1) \), respectively. Since \([e_1] + a[e_j] + a[f_j] \not\sim [e_1] - [e_i] + [f_j], [e_1] + \pi_j(a)[e_j] + \pi_{i+1}(a)[f_j] \not\sim [e_1] + \pi_i(1)[e_j] + \pi_{i+1}(1)[f_j]\). Thus,
\[
\pi_{i+1}(a) \pi_i(1) + \pi_j(a) \pi_{i+j}(1) = 0. \tag{3.5}
\]

It remains to prove that \( \pi_i(-1) = -\pi_i(1) \).

Suppose \( \sigma([f_1] + [f_i] + [f_j]) = [f_1] + x[f_i] + y[f_j] \). Since \([f_1] + [f_i] + [f_j] \not\sim [e_1] - [e_i] \), we have \([f_1] + x[f_i] + y[f_j] \not\sim [e_1] + \pi_i(1)[e_i] \), which implies \( x = -\pi_i(-1) \). Similarly, \( y = -\pi_j(-1) \). Hence \( \sigma([f_1] + [f_i] + [f_j]) = [f_1] - \pi_i(-1)^{-1}[f_i] - \pi_j(-1)^{-1}[f_j] \). Since \([f_1] + [f_i] + [f_j] \not\sim [e_1] - [e_j] \), we have
\[
[f_1] - \pi_i(-1)^{-1}[f_i] - \pi_j(-1)^{-1}[f_j] \not\sim [e_1] - \pi_{i+j}(1)[e_j].
\]

Then \( \pi_i(1)^{-1} - \pi_{i+j}(1)^{-1} \pi_{i+j}(1)^{-1} \pi_j(1)^{-1} = 0 \), that is, \( \pi_{i+j}(1)^{-1} \pi_j(-1) = \pi_{i+j}(1) \pi_i(1) = 0 \). On the other hand, letting \( a = 1 \) in (3.5), we have \( \pi_{i+j}(1) \pi_i(1) + \pi_j(1) \pi_{i+j}(1) = 0 \). Thus \( \pi_{i+j}(1) \pi_i(1) = -\pi_j(1) \pi_{i+j}(1) \). Canceling \( \pi_{i+j}(1) \), we obtain \( \pi_j(-1) = -\pi_j(1) \); then from (3.4), we have \( \pi_i(1) = -\pi_i(1) \). □

Lemma 3.6. Let \( \pi = \pi_2(1)^{-1} \pi_2 \), then \( \pi \) is an automorphism of \( \mathbb{F}_q \).

Proof. We will show \( \pi_i(-a) = -\pi_i(a) \) first. Since \([e_1] - a[e_2] + a[f_3] \not\sim [e_1] + [e_2] + [f_2] \), we have \( \pi_2(-a) \pi_{i+2}(1) + \pi_{i+3}(a) \pi_3(1) = 0 \). By Lemma 3.5(ii), \( \pi_{i+3}(a) \pi_3(1) = \pi_2(a) \pi_{i+2}(1) \). Thus \( -\pi_2(-a) \pi_{i+2}(1) = \pi_2(a) \pi_{i+2}(1) \), which implies \( \pi_2(-a) = -\pi_2(a) \). Therefore, \( \pi_i(-a) = -\pi_i(a) \).

From \([e_1] + (a)[e_2] + a[e_3] \not\sim [e_1] - [f_2] + b[f_3] \), we deduce \( \pi_2(ab) \pi_{i+2}(1) + \pi_{i+3}(a) \times \pi_{i+3}(b) = 0 \). By Lemma 3.5(ii) again,
\[
\pi_2(ab) = \pi_{i+2}(1)^{-1} \pi_3(a) \pi_{i+3}(b) = \pi_{i+2}(1)^{-1} (\pi_3(a) \pi_{i+3}(1)) (\pi_{i+3}(b) \pi_3(1)) (\pi_{i+3}(1)^{-1} \pi_3(1)^{-1}) = \pi_{i+2}(1)^{-1} \pi_2(b) \pi_{i+2}(1) \pi_{i+2}(1)^{-1} = \pi_2(a) \pi_2(b) \pi_2(1)^{-1}.
\]

Then \( \pi(ab) = \pi_2(1)^{-1} \pi_2(ab) = \pi_2(1)^{-1} \pi_2(a) \pi_2(b) \pi_2(1)^{-1} = \pi(a) \pi(b) \).

To prove \( \pi(a + b) = \pi(a) + \pi(b) \), it is enough to assume \( ab \not= 0 \). Suppose \( \sigma([f_1] - a^{-1}[f_2] + a^{-1}b[f_3]) = [f_1] + x[f_2] + y[f_3] \). Since \([f_1] - a^{-1}[f_2] + a^{-1}b[f_3] \not\sim [e_1] + a[e_2] \) and \( \sigma([e_1] + a[e_2]) = [e_1] + \pi_2(a)[e_2] \), we have \( x = -\pi_2(a)^{-1} \). Similarly, \( y = \pi_3(ab)^{-1} \). Thus \( \sigma([f_1] - a^{-1}[f_2] + a^{-1}b[f_3]) = [f_1] - \pi_2(a)^{-1}[f_2] + \pi_3(ab)^{-1}[f_3] \). Since \([e_1] + (a+b)[e_2] + [e_3] \not\sim [f_1] - a^{-1}[f_2] + a^{-1}b[f_3] \), we have \( 1 - \pi_2(a + b) \pi_2(a)^{-1} + \pi_3(1) \pi_3(ab)^{-1} = 0 \). But \( \pi_3(1) \pi_3(ab)^{-1} = (\pi_3(1) \pi_3(ab)^{-1})^{-1} = \pi(ab)^{-1} = \pi(a^{-1}) \pi(b) \), hence
\[
\pi_2(a + b) = \pi_2(a) + \pi_2(a) \pi(a^{-1}) \pi(b) = \pi_2(a) + \pi_2(a) (\pi_2(a) \pi(a^{-1}) \pi(b) = \pi_2(a) + \pi_2(a).
Therefore \( \pi(a + b) = \pi_2(1^{-1}) \pi_2(a + b) = \pi_2(1^{-1}) (\pi_2(a) + \pi_2(b)) = \pi(a) + \pi(b) \).

It is clear that \( \pi \) is injective. Since \( \mathbb{F}_q \) is finite, \( \pi \) is an automorphism of \( \mathbb{F}_q \). \( \square \)

Note that by Lemmas 3.5 and 3.6, we have also \( \pi = \pi_2^{-1}(1) \pi_2 = \cdots = \pi_v^{-1}(1) \pi_v = \pi_{v+2}^{-1}(1) \pi_{v+2} = \cdots = \pi_{v+2}^{-1}(1) \pi_{v+2} \).

Let us return to the proof of Theorem 3.3(ii). We will prove that for any vertex \( [\alpha] = \{a_1, a_2, \ldots, a_{2v}\} \) of \( O(2v, q) \), where \( v \geq 3 \),

\[
\sigma([\alpha]) = [\pi(a_1), k_2 \pi(a_2), \ldots, k_v \pi(a_v), k_1 \pi(a_{v+1}), k_1 k_2^{-1} \pi(a_{v+2}), \ldots, k_1 k_v^{-1} \pi(a_{2v})],
\]

where \( k_1 = \pi_2(1) \pi_{v+2}(1), k_2 = \pi_2(1), \ldots, k_v = \pi_v(1) \). We distinguish the following four cases:

(a) \( a_1 \neq 0 \). We may assume \( a_1 = 1 \) and let \( \sigma([1, a_2, \ldots, a_{2v}]) = [1, a_2', \ldots, a_{2v}'] \). For \( 2 \leq i \leq v \), by Lemma 3.4, \( a_i' = \pi_1(a_i) = k_1 \pi_1(a_i) \) and \( a_i'^{v+1} = \pi_1(a_i) = k_1^{v+1} \pi_1(a_i) \). If \( a_{v+i} \neq 0 \), there exists \( j \), \( 2 \leq j \leq 2v \), \( j \neq v + 1 \), such that \( a_j \neq 0 \). If \( 2 \leq j \leq v \), since \( [1, a_2, \ldots, a_{2v}'] \not\sim [e_1] - a_1'^{v+1} [f_j] \), we have \( a_i'^{v+1} = \pi_1(a_i) = k_1 \pi_1(a_i) = k_1^{v+1} \pi_1(a_i) = k_1 \pi_1(a_i) = k_1 \pi_1(a_i) \). The case \( v < j \leq 2v \) can be treated similarly. Therefore (3.6) holds in this case.

(b) \( a_1 = \cdots = a_{v+i-1} = 0, a_{v+i} \neq 0, 2 \leq i \leq v \). We may assume \( a_{v+i} = 1 \) and let \( \sigma([0, 0, 0, 1, a_{v+i+1}, \ldots, a_{2v}]) = [0, 0, 1, a'_{v+i+1}, \ldots, a'_{2v}] \). If \( a_{v+j} \neq 0, 2 \leq j < v \), then \( [0, 0, 0, 1, a_{v+i+1}, \ldots, a'_{2v}] \not\sim [e_1] - a_1'^{v+1} [e_j] \), which implies \( a_i'^{v+1} = \pi_1(a_i) = k_1 \pi_1(a_i) = k_1^{v+1} \pi_1(a_i) = k_1 \pi_1(a_i) = k_1 \pi_1(a_i) \). If \( a_{v+j} = 0, \) the foregoing formula also holds. Therefore

\[
\sigma([0, 0, 0, 1, a_{v+i+1}, \ldots, a_{2v}]) = [0, 0, 1, k_1 k_i^{-1} \pi(a_{v+i+1}), \ldots, k_1 k_i^{-1} \pi(a_{2v})] = [0, 0, 0, k_1 k_i^{-1} \pi(1), \ldots, k_1 k_i^{-1} \pi(a_{2v})].
\]

(c) \( a_1 = \cdots = a_v = 0, a_{v+1} \neq 0 \). We may assume \( a_{v+1} = 1 \) and let \( \sigma([0, 0, 0, 1, a_{v+2}, \ldots, a_{2v}]) = [0, 0, 0, 1, a'_{v+1}, \ldots, a'_{2v}] \). If \( a_{v+j} \neq 0, 2 \leq j \leq v, [0, 0, 0, 1, a_{v+2}, a_{v+3}, a_{v+4}] \not\sim [e_j] - a_1'^{v+1} [e_j] \), which implies \( a_i'^{v+1} = \pi_1(a_i) = k_1 \pi_1(a_i) = k_1^{v+1} \pi_1(a_i) = k_1 \pi_1(a_i) = k_1 \pi_1(a_i) \). If \( a_{v+j} = 0, \) the foregoing formula holds trivially. Therefore

\[
\sigma([0, 0, 0, 1, a_{v+2}, \ldots, a_{2v}]) = [0, 0, 0, 1, k_1 k_i^{-1} \pi(a_{v+2}), \ldots, k_1 k_i^{-1} \pi(a_{2v})] = [0, 0, 0, k_1 k_i^{-1} \pi(1), k_1 k_i^{-1} \pi(a_{v+2}), \ldots, k_1 k_i^{-1} \pi(a_{2v})].
\]

(d) \( a_1 = \cdots = a_{i-1} = 0, a_i \neq 0, 2 \leq i \leq v \). Assume \( a_i = 1 \) and let \( \sigma([0, 0, 0, 1, a_{i+1}, \ldots, a_{2v}]) = [0, 0, 0, 1, a'_{i+1}, \ldots, a'_{2v}] \). First we consider \( a_j', \) where \( j \) satisfies \( i < j \leq v \). If \( a_j \neq 0 \), we have \( [0, 0, 0, 1, a_{i+1}, \ldots, a_{2v}] \not\sim [f_i] - a_1'^{v+1} [f_j] \). By (b), \( \sigma([f_i] - a_1'^{v+1} [f_j]) = [f_i] - k_i k_j^{-1} \pi(a_j^{-1}) [f_j] \). Thus \( [0, 0, 0, 1, a_{i+1}, \ldots, a_{2v}] \not\sim [f_i] - k_i k_j^{-1} \pi(a_j^{-1}) [f_j] \), which implies \( a_j' = k_i^{-1} k_j \pi(a_j^{-1}) = k_i^{-1} k_j \pi(a_j) \). If \( a_j = 0 \), then \( a_j' = 0 \) and the formula for \( a_j \) also holds.

Then we consider \( a_{v+j}' \), where \( j \) satisfies \( 2 \leq j \leq v \) and \( j \neq i \). Suppose \( \sigma([e_j] - a_{v+j} [f_j]) = [e_j] + x [f_j] \). Since \( [e_j] - a_{v+j} [f_j] \not\sim [e_j] - a_{v+j} [f_j] \), we have \( x = -\pi_1(1)^{-1} \pi_{v+j}(a_{v+j}) = -k_1 k_i^{-1} k_j^{-1} \pi(a_{v+j}) \) and

\[
\sigma([e_j] - a_{v+j} [f_j]) = [e_j] - k_1 k_i^{-1} k_j^{-1} \pi(a_{v+j}) [f_i].
\] (3.7)

Since \( [0, 0, 0, 1, a_{i+1}, \ldots, a_{2v}] \not\sim [e_j] - a_{v+j} [f_j] \), we have \( a_j' = k_i^{-1} k_j^{-1} \pi(a_{v+j}) \).
Now we consider $a'_{v+i}$. From $[0, \ldots, 0, 1, a_{i+1}, \ldots, a_{2v}] \not\sim [e_1] - a_{v+i} [f_i]$, we deduce $[0, \ldots, 0, 1, a'_{i+1}, \ldots, a'_{2v}] \not\sim [e_1] - \pi_{v+i} (a_{v+i}) [f_i]$. Then $a'_{v+i} = \pi_{v+i} (a_{v+i}) = k_1 k_2^{-1} \pi (a_{v+i}).$

Finally we consider $a'_{v+i}$. It is enough to consider the case $a_{v+i} \neq 0$. Then there exists $j$, $i + 1 \leq j \leq v$, such that $a_j \neq 0$ or there exists $j$, $1 \leq j \leq v$, $j \neq i$, such that $a_{v+j} \neq 0$. If the former case occurs, from $[0, \ldots, 0, 1, a_{i+1}, \ldots, a_{2v}] \not\sim [e_1] - a_j^{-1} a_{v+i} [f_i]$ and (3.7), we deduce $a'_{v+i} = a'_j k_1 k_2^{-1} \pi (a_{v+i}) = k_1 k_2^{-1} \pi (a_{v+i})$. If the latter case occurs, we distinguish further: $j = 1, 2 \leq j < i$ or $i > j$.

For $j = 1$, from $[0, \ldots, 0, 1, a_{i+1}, \ldots, a_{2v}] \not\sim [e_1] - a_{v+1} a'_{v+i} [e_1]$, we deduce $[0, \ldots, 0, 1, a_{i+1}, \ldots, a_{2v}] \not\sim [e_1] - k_i \pi (a_{v+1} a'_{v+i}) [e_i]$, which implies $a'_{v+i} = (k_i \pi (a_{v+1} a'_{v+i}))^{-1} a'_{v+1} = k_1 k_2^{-2} \pi (a_{v+i}).$

For $2 \leq j < i$, suppose $\sigma ([e_j] - a_i [e_i]) = [e_j] + x [e_i]$. From $[e_j] - a_i [e_i] \not\sim [e_1] + a_i [f_j] + [f_i]$, by case (a), we deduce $[e_j] + x [e_i] \not\sim [e_1] + k_1 k^{-1} \pi (a_i) [f_j] + k_1 k^{-1} [f_i]$, then $x = -(k_1 k_2^{-1})^{-1} k_1 k^{-1} \pi (a_i) = -k_1 k^{-1} \pi (a_i)$ and

$$\sigma ([e_j] - a_i [e_i]) = [e_j] - k_1 k^{-1} \pi (a_i) [e_i]. \quad (3.8)$$

From $[0, \ldots, 0, 1, a_{i+1}, \ldots, a_{2v}] \not\sim [e_j] - a_{v+i} a_{v+j} [e_i]$ and (3.8), we deduce $a'_{v+i} = (k_j k_1^{-1} \pi (a_{v+i} a_{v+j}))^{-1} a'_{v+j} = k_1 k^{-2} \pi (a_{v+i}).$

If $j > i$, interchanging the roles of $i$ and $j$ in (3.8), we have

$$\sigma ([e_j] - a_j [e_j]) = [e_i] - k_1 k^{-1} \pi (a_j) [e_j]. \quad (3.9)$$

From $[0, \ldots, 0, 1, a_{i+1}, \ldots, a_{2v}] \not\sim [e_j] - a_{v+j} a_{v+i} [e_j]$ and (3.9), we deduce $a'_{v+i} = (k_j k_1^{-1} \pi (a_{v+j} a_{v+i})) a'_{v+j} = k_1 k^{-2} \pi (a_{v+i}).$

Therefore

$$\sigma ([0, \ldots, 0, 1, a_{i+1}, \ldots, a_{2v}])$$

$$= [0, \ldots, 0, 1, k_1^{-1} k_{i+1} \pi (a_{i+1}), \ldots, k_1^{-1} k_v \pi (a_v), k_1 k^{-1} \pi (a_{v+1}), k_1 k^{-1} k_2^{-1} \pi (a_{v+2}), \ldots, k_1 k^{-1} k_2^{-1} \pi (a_{2v})]$$

$$= [0, \ldots, 0, k_1 \pi (1), k_{i+1} \pi (a_{i+1}), \ldots, k_v \pi (a_v), k_1 \pi (a_{v+1}), k_1 k_2^{-1} \pi (a_{v+2}), \ldots, k_1 k_2^{-1} \pi (a_{2v})].$$

Hence (3.6) is proved for all the cases. $\Box$

**Corollary 3.7.**

$$|\text{Aut}(O(2v, q))| = \begin{cases} 2, & \text{if } v = 1; \\ 2((q + 1)!)^2, & \text{if } v = 2; \\ q^{v(v-1)} \prod_{i=1}^{v} (q^i - 1) \prod_{i=0}^{v-1} (q^i + 1)[\mathbb{F}_q : \mathbb{F}_p], & \text{if } v \geq 3; \end{cases}$$

where $p$ is the characteristic of $\mathbb{F}_q$. 
**Proof.** When \( v = 1 \), \( O(2 \cdot 1, q) \) is a complete graph on \( q^0 + 1 = 2 \) vertices, hence \( |\text{Aut}(O(2 \cdot 1, q))| = 2 \). Now let \( v \geq 2 \). Clearly,

\[
|\text{Aut}(O(2 \cdot v, q))| = \frac{|PO_{2v}(\mathbb{F}_q)| |E|}{|PO_{2v}(\mathbb{F}_q) \cap E|}.
\]

By [7, Theorem 6.21], \( |PO_{2v}(\mathbb{F}_q)| = \frac{1}{2} |O_{2v}(\mathbb{F}_q)| = \frac{1}{2} q^{v(v-1)} \prod_{i=1}^{v} (q^i - 1) \prod_{i=0}^{v-1} (q^i + 1) \). Moreover, \( |E| = ((q - 1))^2 \) or \( (q - 1)^v |\text{Aut}(\mathbb{F}_q)| \) according as \( v = 2 \) or \( v \geq 3 \), respectively, and \( PO_{2v}(\mathbb{F}_q) \cap E \) consists of those \( \sigma \in E \) which are reduced from some matrices of the form \( \text{diag}(k_1, \ldots, k_v, k_1^{-1}, \ldots, k_v^{-1}) \), \( k_i \in \mathbb{F}_q^\ast \). Thus, \( |PO_{2v}(\mathbb{F}_q) \cap E| = \frac{1}{2} (q - 1)^v \). Hence

\[
|\text{Aut}(O(2 \cdot 2, q))| = \frac{|PO_{2 \cdot 2}(\mathbb{F}_q)| |E|}{|PO_{2 \cdot 2}(\mathbb{F}_q) \cap E|} = 2((q + 1))^2,
\]

and when \( v \geq 3 \),

\[
|\text{Aut}(O(2v, q))| = q^{v(v-1)} \prod_{i=1}^{v} (q^i - 1) \prod_{i=0}^{v-1} (q^i + 1) |\mathbb{F}_q : \mathbb{F}_p|,
\]

as is well known that \( |\text{Aut}(\mathbb{F}_q)| = [\mathbb{F}_q : \mathbb{F}_p] \). \( \square \)

### 4. Automorphism groups (II): The case \( \delta = 1 \)

**Theorem 4.1.** Let \( v \geq 2 \) and \( E \) be the subgroup of \( \text{Aut}(O(2v + 1, q)) \) defined as follows:

\[
E = \{ \sigma \in \text{Aut}(O(2v + 1, q)) : \sigma([e_i]) = [e_i], \ \sigma([f_i]) = [f_i], \ i = 1, \ldots, v \}.
\]

Then \( \text{Aut}(O(2v + 1, q)) = PO_{2v+1}(\mathbb{F}_q) \cdot E \). Moreover,

\[
E \cong \left( \mathbb{F}_q^\ast \times \mathbb{F}_q^\ast \times \cdots \times \mathbb{F}_q^\ast \times \mathbb{F}_3^\ast \right) \rtimes \psi \text{Aut}(\mathbb{F}_q),
\]

where \( \mathbb{F}_3^\ast = \{-1, 1\} \) is regarded as the unique subgroup of order 2 of \( \mathbb{F}_q^\ast \) and the isomorphism from \( (\mathbb{F}_q^\ast \times \mathbb{F}_q^\ast \times \cdots \times \mathbb{F}_q^\ast \times \mathbb{F}_3^\ast) \rtimes \psi \text{Aut}(\mathbb{F}_q) \) to \( E \) is defined as:

\[
h : (k_1, \ldots, k_v, \delta, \pi) \mapsto \sigma(k_1, \ldots, k_v, \delta, \pi)
\]

where \( \sigma(k_1, \ldots, k_v, \delta, \pi) \) is the map that carries the vertex \([a_1, a_2, \ldots, a_{2v}, a_{2v+1}]\) of \( O(2v + 1, q) \) to the vertex

\[
[\pi(a_1), k_2 \pi(a_2), \ldots, k_v \pi(a_v), k_1 \pi(a_v+1), k_1 k_2^{-1} \pi(a_{v+2}), \ldots, k_1 k_v^{-1} \pi(a_{2v}), \delta \sqrt{k_1} \pi(a_{2v+1})],
\]

where \( \delta = \pm 1 \) and \( \sqrt{k_1} \) is one of the square roots of \( k_1 \).
Proof. Similar to Theorem 3.3, we can prove \( \text{Aut}(O(2\nu + 1, q)) = PO_{2\nu + 1}(\mathbb{F}_q) : E \).

Clearly \( \sigma(1, \ldots, k, \delta, \pi) \) is well defined and \( \sigma(1, \ldots, k, \delta, \pi) \in E \), thus \( h \) is well defined. It is easy to verify that \( h \) is an injective group homomorphism from \( (\mathbb{F}_q^{*2} \times \mathbb{F}_q^* \times \cdots \times \mathbb{F}_q^* \times \mathbb{F}_q^*)_q \times_q \text{Aut}(\mathbb{F}_q) \) to \( E \). Thus to prove the second part of the theorem, it is enough to show that every element \( \sigma \) of \( E \) is of the form \( \sigma(1, \ldots, k, \delta, \pi) \).

**Lemma 4.2.** If \( [a_1, \ldots, a_{2\nu}, a_{2\nu+1}] \in V(O(2\nu + 1, q)) \) and \( [a_1, \ldots, a_{2\nu}] \in V(O(2\nu, q)) \), then \( a_{2\nu+1} = 0 \).

**Proof.** This is clear from the definition of the orthogonal graphs. \( \square \)

Let \( \sigma \in E \). For \( [a_1, \ldots, a_{2\nu}, a_{2\nu+1}] \in V(O(2\nu + 1, q)) \), let \( \sigma([a_1, \ldots, a_{2\nu}, a_{2\nu+1}]) = [b_1, \ldots, b_{2\nu}, b_{2\nu+1}] \). As the case \( \delta = 0 \), \( a_i \neq 0 \) if and only if \( b_i \neq 0 \). Therefore by Lemma 4.2 and the foregoing statement, we have permutations \( \pi_i, i = 2, \ldots, v, v + 2, \ldots, 2v \), of \( \mathbb{F}_q \) with \( \pi_i(0) = 0 \) such that

\[
\sigma([e_1] + a_j[e_j]) = [e_1] + \pi_j(a_j)[e_j], \quad 2 \leq j \leq v,
\]

\[
\sigma([e_1] + a_{v+j}[f_j]) = [e_1] + \pi_{v+j}(a_{v+j})[f_j], \quad 2 \leq j \leq v.
\]

In the following, we distinguish the cases: \( v \geq 3 \) and \( v = 2 \).

(i) \( v \geq 3 \). If we take Lemma 4.2 into account, the following lemmas corresponding to Lemmas 3.4–3.6 can be proved in a similar way.

**Lemma 4.3.** Suppose \([1, a_2, \ldots, a_{2\nu}, a_{2\nu+1}] \in V(O(2\nu + 1, q)) \) and let \( \sigma([1, a_2, \ldots, a_{2\nu}, a_{2\nu+1}]) = [1, a'_2, \ldots, a'_{2\nu}, a'_{2\nu+1}] \). Then \( a'_j = \pi_j(a_j) \) and \( a'_{v+j} = \pi_{v+j}(a_{v+j}) \), \( 2 \leq j \leq v \).

**Lemma 4.4.**

(i) \( \pi_i(1)^{-1} \pi_i = \pi(1)^{-1} \pi_j = \pi_{v+i}(1)^{-1} \pi_{v+i} = \pi_{v+j}(1)^{-1} \pi_{v+j} \), where \( 2 \leq i, j \leq v \).

(ii) For any \( a \in \mathbb{F}_q, \pi_i(a)\pi_i(1) = \pi_{v+i}(a)\pi_i(1) = \pi_j(a)\pi_j(1) = \pi_{v+j}(a)\pi_j(1) \), where \( 2 \leq i, j \leq v \). In particular, \( \pi_2(1)\pi_{v+2}(1) = \pi_3(1)\pi_{v+3}(1) = \cdots = \pi_1(1)\pi_{2v}(1) \).

**Lemma 4.5.** Let \( \pi = \pi_2(1)^{-1} \pi_2 \), then \( \pi \) is an automorphism of \( \mathbb{F}_q \).

As the case \( \delta = 0 \), we have also \( \pi = \pi_2^{-1}(1) = \pi_2^{-1}(1) = \pi_2^{-1}(1) \). By the same method of the proof of Theorem 3.3, we can show that the vertex \([a_1, \ldots, a_{2\nu}, a_{2\nu+1}] \) is mapped into

\[
[a_1, \pi(a_1), k_2\pi(a_2), \ldots, k_v\pi(a_v), k_1\pi(a_{v+1}), k_1k_2^{-1}\pi(a_{v+2}), \ldots, k_1k_v^{-1}\pi(a_{2\nu}), a'_{2\nu+1}]
\]

under \( \sigma \), where \( a'_{2\nu+1} \) is to be determined. Since \([a_1, \ldots, a_{2\nu}, a_{2\nu+1}] \in V(O(2\nu + 1, q)), 2(a_1a_{v+1} + \cdots + a_2a_{v+1}) + a_{2\nu+1}^2 = 0 \) and \( 2k_1(\pi(a_1)\pi(a_{v+1}) + \cdots + \pi(a_2)\pi(a_{v+2})) + a_{2\nu+1}^2 = 0 \), which imply \( a_{2\nu+1}^2 = k_1\pi(a_{2\nu+1}) = k_1\pi(a_{2\nu+1})^2 \). Hence \( k_1 \in \mathbb{F}_q^* \). Let \( \sqrt{k_1} \) be one of its square roots in \( \mathbb{F}_q^* \). Then \( a'_{2\nu+1} = \delta\sqrt{k_1}\pi(a_{2\nu+1}) \), where \( \delta = \pm 1 \).

It remains to prove that all elements of \( E \) are either of the form \( \sigma(k_1, \ldots, k_v, 1, \pi) \) or of the form \( \sigma(k_1, \ldots, k_v, -1, \pi) \). We need a lemma first.
Lemma 4.6. Let $W$ be a maximal totally isotropic space of $\mathbb{F}_q^{2v+1}$, $v \geq 2$, and $X = \{[\alpha]: \alpha \in W, \alpha \neq 0\}$. Then every vertex not in $X$ is adjacent with exactly $q^{v-1}$ vertices in $X$.

Proof. By the transitivity of the orthogonal group (cf. [7]), we can assume $W = \{e_1, \ldots, e_v\}$. Let $[\alpha] \in V(O(2v+1, q)) \setminus X$ and $\alpha = (a_1, a_2, a_3, \ldots, a_{v+1}, a_{v+2})$. Then $(a_{v+1}, \ldots, a_{v+2}) \neq 0$. Let $\beta = x_1e_1 + \cdots + x_ve_v \in \{e_1, \ldots, e_v\}$, $x_i$ not all zero. Then $[\alpha] \not\sim [\beta]$ if and only if $a_{v+1}x_1 + \cdots + a_{v+2}x_v = 0$. The number of nonzero solutions of this equation in $x_1, \ldots, x_v$ is $q^{v-1} - 1$. Thus the number of vertices in $W$ which are adjacent with $[\alpha]$ is $\frac{q^v - 1}{q-1} - \frac{q^{v-1} - 1}{q-1} = q^{v-1}$. \qed

Lemma 4.7. Let $v \geq 2$. If there exists a vertex $[\alpha] = (a_1, \ldots, a_{2v+1})$ of $O(2v+1, q)$ with $a_{2v+1} \neq 0$, such that $\sigma([\alpha]) = \sigma(k_1, \ldots, k_v, 1, \pi)([\alpha])$, then $\sigma = \sigma(k_1, \ldots, k_v, 1, \pi)$.

Proof. It suffices to consider the vertices with their last components nonzero. Without loss of generality, we may assume $a_{2v+1} = 1$ and $\sigma([\alpha]) = \sigma(k_1, \ldots, k_v, 1, \pi)([\alpha])$. Let $V_1 = \{[x_1, \ldots, x_2v, 1] \in V(O(2v+1, q)): x_i \in \mathbb{F}_q\}$ and $[\beta] = [b_1, \ldots, b_{2v}, 1] \in V_1$. Then $\sigma([\beta]) = \sigma(k_1, \ldots, k_v, 1, \pi)([\beta])$, where $\delta = 1$ or $-1$.

If $[\beta] \not\sim [\alpha]$, then

$$b_1a_{v+1} + b_2a_{v+2} + \cdots + b_va_{v+2} + b_{v+1}a_1 + \cdots + b_{2v}a_v + 1 = 0. \quad (4.1)$$

Since $\sigma$ is an automorphism, $\sigma([\alpha]) \not\sim \sigma([\beta])$, i.e.,

$$\pi(b_1)k_1\pi(a_{v+1}) + k_2\pi(b_2)k_1k_2^{-1}\pi(a_2) + \cdots + k_v\pi(b_v)k_1k_2^{-1}\pi(a_{2v})$$

$$+ k_1\pi(b_{v+1})\pi(a_1) + \cdots + k_1k_2^{-1}\pi(b_{2v})k_1k_1^{-1}\pi(a_v) + \delta k_1 = 0. \quad (4.2)$$

From (4.1) and (4.2), we deduce $\delta = 1$.

From the above conclusion, we need only show that the complementary graph $\overline{H}_1$ of the induced subgraph $H_1$ of $O(2v+1, q)$ with vertex set $V_1$, is connected. Let $\gamma_1 = e_1 - 2^{-1}f_1 + e_{2v+1}$ and $\gamma_2 = e_2 - 2^{-1}f_2 + e_{2v+1}$. Then it is clear that $W_1 = \{[\gamma_1, e_2, e_3, \ldots, e_v] \text{ and } W_2 = \{[\gamma_2, f_1, f_3, \ldots, f_v] \text{ are two maximal totally isotropic subspaces, and } W_1 \cap W_2 = \{0\}. \text{ As in Lemma 4.6, let } X_i = \{[\alpha]: \alpha \in W_i, \alpha \neq 0\}, \text{ where } i = 1 \text{ or } 2. \text{ Then any two distinct vertices in each } X_i \text{ are not adjacent. Clearly, } [\gamma_1 + e_2] = [e_1 + e_2 - 2^{-1}f_1 + e_{2v+1}] \in X_1 \cap V_1,$$ $[\gamma_2 - 2^{-1}f_1] = [e_2 - 2^{-1}f_1 - 2^{-1}f_2 + e_{2v+1}] \in X_2 \cap V_1, \text{ and they are not adjacent. For every } [\gamma] = [c_1, \ldots, c_{2v}, 1] \in V_1 \setminus ((X_1 \cup X_2) \cap V_1), \text{ we claim that there exists } [\theta] \in (X_1 \cup X_2) \cap V_1 \text{ such that } [\gamma] \not\sim [\theta]. \text{ If not, then all the elements of } X_i \cap V_i, \text{ } i = 1, 2, \text{ are adjacent with } [\gamma]. \text{ Note that } |X_i \cap V_i| = q^{v-1} \text{ and by Lemma 4.6, the number of vertices in } X_i \text{ which are adjacent with } [\gamma] \text{ is exactly } q^{v-1} \text{, so } [\gamma] \text{ is not adjacent with } [e_i], \text{ } 2 \leq i \leq v, \text{ and also not adjacent with } [f_i], \text{ } i = 1, 3, \ldots, v. \text{ Then } c_i = 0, \text{ } 1 \leq i \leq 2v + 1, \text{ } i \neq 2 \text{ and } i \neq v + 1. \text{ But now } [\gamma] = c_2[e_2] + c_{v+1}[f_1] + [e_{2v+1}] \text{ is not a vertex of } O(2v+1, q). \text{ This is a contradiction, and our claim is proved. Therefore } \overline{H}_1 \text{ is connected. Hence } \sigma = \sigma(k_1, \ldots, k_v, 1, \pi). \quad \Box$

The proof of theorem for the case $v \geq 3$ is complete.

(ii) $v = 2$. Now we have only permutations $\pi_2$ and $\pi_4$.

Lemma 4.8. $\pi_2(1)^{-1} \pi_2 = \pi_4(1)^{-1} \pi_4$. 
\textbf{Proof.} Taking Lemma 4.2 into account, as the case \(v = 2, \delta = 0\), we can prove

\[
\begin{align*}
\sigma([0, 0, 1, a_4, 0]) &= [0, 0, 1, -\pi_2(-a_4^{-1})^{-1}, 0], \quad \text{if } a_4 \neq 0, \\
\sigma([0, 1, a_3, 0, 0]) &= [0, 1, -\pi_4(-a_3), 0, 0]
\end{align*}
\]

and

\[
\sigma([1, a_2, a_3, a_4, 0]) = [1, \pi_2(a_2), -\pi_2(a_2)\pi_4(a_4), \pi_4(a_4), 0], \quad \text{where } a_2a_4 + a_3 = 0.
\]

Let \([1, a_2, a_3, a_4, a_5] \in V(O(5, q))\), then \(2(a_3 + a_2a_4) + a_2^2 = 0\). Suppose \(\sigma([1, a_2, a_3, a_4, a_5]) = [1, a'_2, a'_3, a'_4, a'_5]\). For \(a_2 \neq 0\), from \([1, a_2, a_3, a_4, a_5] \not\sim [0, 0, 1, -a_2^{-1}, 0]\), we have \(a'_2 = \pi_2(a_2)\). For \(a_2 = 0\), we have \(a'_2 = 0\) and the formula \(a'_2 = \pi_2(a_2)\) also holds. From \([1, a_2, a_3, a_4, a_5] \not\sim [0, 1, -a_4, 0, 0]\), we have \([a'_2, a'_3, a'_4, a'_5] \not\sim [0, 1, -\pi_4(a_4), 0, 0]\), then \(a'_4 = \pi_4(a_4)\).

Next suppose \(a_2, a_4 \in \mathbb{F}_q^*\) and we will calculate \(a'_3\) in two ways. First from \([1, a_2, a_3, a_4, a_5] \not\sim [1, -a_4^{-1}a_3, 0, 0, 0]\), we have \(a'_3 = -\pi_2(-a_4^{-1}a_3)\pi_4(a_4)\). On the other hand, from \([1, a_2, a_3, a_4, a_5] \not\sim [1, 0, 0, -a_2^{-1}a_3, 0]\), we have \(a'_3 = -\pi_2(a_2)\pi_4(-a_2^{-1}a_3)\). Hence

\[
\pi_2(a_2)\pi_4(-a_2^{-1}a_3) = \pi_4(a_4)\pi_2(-a_2^{-1}a_3)
\]

for all \(a_2, a_4 \in \mathbb{F}_q^*\) and \(a_3 \in \mathbb{F}_q\) such that \(-2(a_3 + a_2a_4) \in \mathbb{F}_q^{*2} \cup \{0\}\).

Let \(a \in \mathbb{F}_q^*\) and \(a \neq 1\), we will show \(\pi_2(1)\pi_4(a) = \pi_4(1)\pi_2(a)\). We distinguish the following two cases: \(-a \in \mathbb{F}_q^{*2} \) or not.

(i) \(-a \notin \mathbb{F}_q^{*2} \). There is one and only one of \(-2(-a)(1 - a) \) and \(-2(1 - a) \) in \(\mathbb{F}_q^{*2} \). If \(-2(-a)(1 - a) \in \mathbb{F}_q^{*2} \), substituting \(a_2 = a_4 = a, a_3 = -a\) into (4.3), we have \(\pi_2(a)\pi_4(1) = \pi_4(a)\pi_2(1)\). If \(-2(1 - a) \in \mathbb{F}_q^{*2} \), substituting \(a_2 = a_4 = 1, a_3 = -a\) into (4.3), we also have \(\pi_2(1)\pi_4(a) = \pi_4(1)\pi_2(a)\).

(ii) \(-a \in \mathbb{F}_q^{*2} \). We distinguish further: \(2 \in \mathbb{F}_q^{*2} \) or not.

If \(2 \in \mathbb{F}_q^{*2} \), then \(-2a \in \mathbb{F}_q^{*2} \). Choose \(a_5 \in \mathbb{F}_q\) such that \([1, 0, a, 0, a_5] \in V(O(5, q))\). Suppose \(\sigma([1, 0, a, 0, a_5]) = [1, 0, a_5', 0, a_5']\). We will calculate \(a'_3\) in two ways. Since \([1, 0, a, 0, a_5] \not\sim [1, -a, a, 0], [1, 0, a', 0, a'_5] \not\sim [1, \pi_2(1), -\pi_2(1)\pi_4(a), \pi_4(a), 0]\), which implies \(a'_3 = \pi_2(1)\pi_4(a)\). Similarly, from \([1, 0, a, 0, a_5] \not\sim [1, a, -a, 1, 0]\), we have \(a'_3 = \pi_4(1)\pi_2(a)\). Hence \(\pi_2(1)\pi_4(a) = \pi_4(1)\pi_2(a)\).

If \(2 \notin \mathbb{F}_q^{*2} \), we distinguish further: \(a - 1 \in \mathbb{F}_q^{*2} \) or not. If \(a - 1 \notin \mathbb{F}_q^{*2} \), substituting \(a_2 = a_4 = 1, a_3 = -a\) into (4.3), we have \(\pi_2(1)\pi_4(a) = \pi_4(1)\pi_2(a)\). If \(a - 1 \in \mathbb{F}_q^{*2} \), substituting \(a_2 = 1, a_3 = -2^{-1}a, a_4 = 2^{-1}\) into (4.3), we have

\[
\pi_2(1)\pi_4(2^{-1}a) = \pi_4(2^{-1})\pi_2(a).
\]

Suppose \([1, 0, a_3, a_4, a_5] \in V(O(5, q)) \) and let \(\sigma([1, 0, a_3, a_4, a_5]) = [1, 0, a'_3, a'_4, a'_5]\). Then \(a'_4 = \pi_4(a_4)\). Before continuing, note the following claim:

\textbf{Claim.} Let \([1, 0, a_3, a_4, a_5] \) and \([1, 0, a_3, b_4, a_5] \in V(O(5, q))\). If \(\sigma([1, 0, a_3, a_4, a_5]) = [1, 0, a'_3, a'_4, a'_5]\) and \(\sigma([1, 0, a_3, b_4, a_5]) = [1, 0, a_3'', b'_4, a_5'']\), then \(a'_3 = a_3''\).
Proof. Since \([1, 0, a_3, a_4, a_5] \in V(O(5, q))\) and \([1, 0, a_3, b_4, a_5] \in V(O(5, q))\), we have \([1, 0, a'_3, a'_4, a'_5] \) and \([1, 0, a_3'', b'_4, a_5''] \in V(O(5, q))\), which implies

\[
2a'_3 + a'_5^2 = 0, \tag{4.5}
\]

\[
2a''_3 + a''_5^2 = 0. \tag{4.6}
\]

Clearly, \([1, 0, a_3, a_4, a_5] \not\sim [1, 0, a_3, b_4, a_5]\), then \([1, 0, a'_3, a'_4, a'_5] \not\sim [1, 0, a_3'', b'_4, a_5'']\), which implies

\[
a'_3 + a'_5'' + a_5'' = 0. \tag{4.7}
\]

From (4.5)–(4.7), we deduce \(a'_3 = a''_3\). \(\square\)

Since \(-a \in \mathbb{F}_q^{*2}\), we may choose \(a_5 \in \mathbb{F}_q\) such that \(a + a_5^2 = 0\). Then \([1, 0, 2^{-1}a, 2^{-1}a, a_5]\) and \([1, 0, 2^{-1}a, 2^{-1}, a_5]\) are all vertices of \(O(5, q)\). By the claim we may suppose \(\sigma([1, 0, 2^{-1}a, 2^{-1}, a_5]) = [1, 0, a'_3, a'_4, a'_5]\) and \(\sigma([1, 0, 2^{-1}a, 2^{-1}, a_5]) = [1, 0, a''_3, a''_4, a''_5]\). Since \([1, 0, 2^{-1}a, 2^{-1}, a_5] \not\sim [1, 1, -a, a, 0]\), we have

\[
[1, 0, a'_3, a'_4, a'_5] \not\sim [1, \pi_2(1), -\pi_2(1), \pi_4(a), \pi_4(a), 0],
\]

which implies

\[
a'_3 - \pi_2(1)\pi_4(a) + \pi_4(2^{-1}a)\pi_2(1) = 0. \tag{4.8}
\]

Similarly, since \([1, 0, 2^{-1}a, 2^{-1}, a_5] \not\sim [1, a, -a, 1, 0]\), we have

\[
a'_3 - \pi_2(a)\pi_4(1) + \pi_2(a)\pi_4(2^{-1}) = 0. \tag{4.9}
\]

From (4.4), (4.8) and (4.9), we have \(\pi_2(1)\pi_4(a) = \pi_4(1)\pi_2(a)\).

Therefore, we have proved \(\pi_2(1)\pi_4(a) = \pi_4(1)\pi_2(a)\) for all \(a \in \mathbb{F}_q^*\) and \(a \neq 1\). Clearly, when \(a = 0\) or \(1\), the lemma holds trivially. Thus the proof of the lemma is complete. \(\square\)

Let \(\pi = \pi_2(1)^{-1}\pi_2\), then \(\pi(1) = 1\) and (4.3) can be rewritten as:

\[
\pi(a_2)\pi(-a_2^{-1}a_3) = \pi(a_4)\pi(-a_4^{-1}a_3), \tag{4.10}
\]

where \(a_2, a_4 \in \mathbb{F}_q^*\) and \(a_3 \in \mathbb{F}_q\) such that \(-2(a_3 + a_2a_4) \in \mathbb{F}_q^{*2} \cup \{0\}\). We will show \(\pi \in \text{Aut}(\mathbb{F}_q)\).

First we have

**Lemma 4.9.** \(\pi(-a) = -\pi(a)\) for all \(a \in \mathbb{F}_q\).

Proof. Substituting \(a_2 = a_3 = -1, a_4 = 1\) into (4.10), we have \(\pi(-1)^2 = 1\). Hence \(\pi(-1) = -1\).

If \(a = 0\), \(\pi(-a) = -\pi(a)\) holds trivially. Next let \(a \neq 0\). We distinguish the cases: \(a \in \mathbb{F}_q^{*2}\) or not. When \(a \in \mathbb{F}_q^{*2}\), substituting \(a_2 = -1, a_3 = -a, a_4 = a\) into (4.10), we have \(\pi(-1)\pi(-a) = \pi(a)\pi(1)\), that is, \(\pi(-a) = -\pi(a)\). When \(a \notin \mathbb{F}_q^{*2}\), we distinguish further: \(2(a + 1) \in \mathbb{F}_q^{*2}\) or not.

If \(2(a + 1) \notin \mathbb{F}_q^{*2}\), then \(2a(a + 1) \in \mathbb{F}_q^{*2}\). Substituting \(a_2 = a_3 = -a, a_4 = a\) into (4.10), we have
\[ \pi(\omega) = \pi(\omega) \] that is, \( \pi(\omega) = -\pi(\omega) \). If \( 2(a + 1) \in \mathbb{F}_q^* \), substituting \( a_2 = -1 \), \( a_3 = -a, a_4 = 1 \) into (4.10), we have \( \pi(-1)\pi(-a) = \pi(\omega) \), then \( \pi(-a) = -\pi(\omega) \). Therefore, \( \pi(-a) = -\pi(\omega) \) for \( a \in \mathbb{F}_q \).

**Lemma 4.10.** If \( 2 \notin \mathbb{F}_q^* \), then \( \pi(2a) = \pi(2)\pi(a) \) for all \( a \in \mathbb{F}_q \).

**Proof.** Suppose \( 2 \notin \mathbb{F}_q^* \). If \( a = 0 \), \( \pi(2a) = \pi(2)\pi(a) \) holds trivially. Now we assume \( a \neq 0 \). We distinguish the following two cases: \( a \in \mathbb{F}_q^* \) or not. If \( a \in \mathbb{F}_q^* \), substituting \( a_2 = 2a, a_3 = 2a, a_4 = -2 \) into (4.10), we have \( \pi(2a)\pi(-1) = \pi(-2)\pi(a) \), which implies \( \pi(2a) = \pi(2)\pi(a) \). If \( a \notin \mathbb{F}_q^* \), then \( 2a \notin \mathbb{F}_q^* \). Substituting \( a_2 = 1, a_3 = -2a, a_4 = a \) into (4.10), we have \( \pi(2a) = \pi(a)\pi(2) \).

**Lemma 4.11.** \( \pi(ab) = \pi(a)\pi(b) \) for all \( a, b \in \mathbb{F}_q \).

**Proof.** It is enough to assume \( ab \neq 0, a \neq 1, b \neq 1 \). We distinguish the following two cases: \( 2ab(1 - a) \in \mathbb{F}_q^* \) or not. When \( 2ab(1 - a) \in \mathbb{F}_q^* \), substituting \( a_2 = a, a_3 = -ab, a_4 = ab \) into (4.10), we have \( \pi(a)\pi(b) = \pi(ab) \). When \( 2ab(1 - a) \notin \mathbb{F}_q^* \), we distinguish further: \( 2 \notin \mathbb{F}_q^* \) or not. If \( 2 \notin \mathbb{F}_q^* \), then \( 2a(2b)(1 - a) \notin \mathbb{F}_q^* \). By the previous case, we have \( \pi(2ab) = \pi(a)\pi(2b) \). Noting that here \( 2 \notin \mathbb{F}_q^* \), by Lemma 4.10, we have \( \pi(ab) = \pi(a)\pi(b) \). On the other hand, if \( 2 \in \mathbb{F}_q^* \), we distinguish further: \( a \in \mathbb{F}_q^* \) or not. When \( a \notin \mathbb{F}_q^* \), \( 2b(1 - a) \notin \mathbb{F}_q^* \). Substituting \( a_2 = -1, a_3 = ab, a_4 = b \) into (4.10), we have \( \pi(-1)\pi(ab) = \pi(b)\pi(-a) \), which implies \( \pi(ab) = \pi(a)\pi(b) \). When \( a \notin \mathbb{F}_q^* \), by the symmetry of \( a \) and \( b \), we can assume further \( b \notin \mathbb{F}_q^* \). By the repeated use of the following lemma (if necessary), this case will be reduced to the case \( a \notin \mathbb{F}_q^* \) or the trivial situation. So it suffices to prove the following lemma.

**Lemma 4.12.** \( \pi(a^2) = \pi(a)^2 \) for all \( a \in \mathbb{F}_q \).

**Proof.** It suffices to assume \( a \neq 0, 1 \). By the proof of Lemma 4.11, we need only consider the case: \( 2 \in \mathbb{F}_q^* \), \( a \in \mathbb{F}_q^* \) and \( 1 - a \notin \mathbb{F}_q^* \). Since \( 2 \notin \mathbb{F}_q^* \) and \( a \notin \mathbb{F}_q^* \), choose \( a_5 \in \mathbb{F}_q \) such that \( [0, 1, 0, -a, a_5] \in V(\mathcal{O}(5, q)) \). Suppose \( \sigma([0, 1, 0, -a, a_5]) = [0, 1, 0, a_4', a_5'] \), we will calculate \( a_4' \) in two ways. Since \( [0, 1, 0, -a, a_5] \prec [1, a, -a', a_2', 0], [0, 1, 0, a_4', a_5'] \prec [1, \pi_2(a), -\pi_2(a)\pi_4(a^2), \pi_4(a^2), 0] \), from which we can deduce \( a_4' = -\pi_2(a)^{-1}\pi_4(a^2) \). Similarly from \( [0, 1, 0, -a, a_5] \prec [1, 1, -a, a, 0] \), we have \( a_4' = -\pi_2(1)^{-1}\pi_4(a) \). Hence \( -\pi_2(a)^{-1} \times \pi_4(a^2) = -\pi_2(1)^{-1}\pi_4(a) \), which implies \( \pi(a^2) = \pi(a)^2 \).

**Lemma 4.13.** \( \pi \in \text{Aut}(\mathbb{F}_q) \).

**Proof.** It suffices to prove \( \pi(1 + a) = 1 + \pi(a) \) for \( a \in \mathbb{F}_q \). It holds trivially when \( a = -1 \). Thus one can assume \( a \neq -1 \). From \( [1, 1 + a, -2, 0, 2] \prec [1, a, -2a, 2, 0] \), by the proof of Lemma 4.8 we have

\[
[1, \pi_2(1 + a), -\pi_2(1 + a)\pi_4(2(1 + a)^{-1}), 0, a_5'] \prec [1, \pi_2(a), -\pi_2(a)\pi_4(2), \pi_4(2), 0],
\]

then

\[
-\pi_2(a)\pi_4(2) + \pi_2(1 + a)\pi_4(2) - \pi_2(1 + a)\pi_4(2(1 + a)^{-1}) = 0,
\]
which implies
\[-\pi(a)\pi(2) + \pi(1 + a)\pi(2) - \pi(1 + a)\pi\left(2(1 + a)^{-1}\right) = 0.\]

Therefore \(\pi(1 + a) = 1 + \pi(a).\) \(\square\)

Next we will show that for any vertex \([a_1, a_2, a_3, a_4, a_5],\)
\[
\sigma(\{a_1, a_2, a_3, a_4, a_5\}) = \left[\pi(a_1), k_2\pi(a_2), k_1\pi(a_3), k_1k_2^{-1}\pi(a_4), a'_5\right],
\] (4.11)

where \(k_1 = \pi_2(1)\pi_4(1), k_2 = \pi_2(1)\) and \(a'_5 \in \mathbb{F}_q\) is to be determined. We distinguish the following cases:

(a) \([1, a_2, a_3, a_4, a_5] \in V(O(5, q)).\)

Suppose \(\sigma([1, a_2, a_3, a_4, a_5]) = [1, a'_2, a'_3, a'_4, a'_5].\) By the proof of Lemma 4.8, we have \(a'_2 = \pi_2(a_2) = k_2\pi(a_2), a'_4 = \pi_4(a_4) = k_1k_2^{-1}\pi(a_4)\) and \(a'_3 = -\pi_2(-a_4^{-1})\pi_4(a_4) = k_1\pi(a_3).\) Therefore (4.11) holds.

(b) \([0, 1, a_3, a_4, a_5] \in V(O(5, q)).\)

Let \(\sigma([0, 1, a_3, a_4, a_5]) = [0, 1, a'_3, a'_4, a'_5].\) From \([0, 1, a_3, a_4, a_5] \not\sim [1, 0, 0, -a_3, 0],\) we deduce \(a'_3 = \pi_4(a_3) = k_1k_2^{-1}\pi(a_3).\) For \(a'_4,\) it is enough to assume \(a_4 \neq 0.\) We distinguish \(a_3 = 0\) or not. If \(a_3 \neq 0,\) from \([0, 1, a_3, a_4, a_5] \not\sim [1, -a_3a_4^{-1}, 0, 0, 0],\) we have \([0, 1, a'_3, a'_4, a'_5] \not\sim [1, -\pi_2(a_3a_4^{-1}), 0, 0, 0],\) which implies \(a'_4 = \pi_2(a_3a_4^{-1})^{-1}a'_3 = k_1k_2^{-2}\pi(a_4).\) If \(a_3 = 0,\) from \([0, 1, 0, a_4, a_5] \not\sim [1, a_4, -a_4, 0],\) we have \([0, 1, 0, a'_4, a'_5] \not\sim [1, \pi_2(1), -\pi_2(1)\pi_4(-a_4), \pi_4(-a_4), 0],\) then \(a'_4 = -\pi_2(1)\pi_4(-a_4) = k_1k_2^{-2}\pi(a_4).\) Therefore
\[
\sigma(\{0, 1, a_3, a_4, a_5\}) = [0, 1, k_1k_2^{-1}\pi(a_3), k_1k_2^{-2}\pi(a_4), a'_5] = [0, k_2\pi(1), k_1\pi(a_3), k_1k_2^{-1}\pi(a_4), a'_5].
\]

(c) \([0, 0, 1, a_4, 0] \in V(O(5, q)).\)

Assume \(a_4 \neq 0.\) As the case of \(\nu = 2\) and \(\delta = 0,\) we have \(\sigma([0, 0, 1, a_4, 0]) = [0, 0, 1, -\pi_2(-a_4^{-1}), 0] = [0, 0, 1, k_2^{-1}\pi(a_4), 0] = [0, 0, k_1\pi(1), k_1k_2^{-1}\pi(a_4), 0].\) Hence (4.11) holds.

(d) \([0, 0, 0, 1, 0] \in V(O(5, q)).\) This case is trivial.

Notice that Lemmas 4.6, 4.7 hold also for \(\nu = 2.\) Thus one may conclude that every element of \(E\) is of the form \(\sigma(k_1, k_2, \delta, \pi)\) and the proof of theorem for the case \(\nu = 2\) is complete. Therefore the theorem is proved. \(\square\)

**Corollary 4.14.** When \(\nu = 1,\) \(|\text{Aut}(O(2 \cdot 1 + 1, q))| = (q + 1)!.\) When \(\nu \geq 2,\) \(|\text{Aut}(O(2\nu + 1, q))| = q^{2\nu} \prod_{i=1}^{\nu} (q^i - 1) \prod_{i=1}^{\nu} (q^i + 1) [\mathbb{F}_q: \mathbb{F}_p].\)

**Proof.** The proof is similar to that of Corollary 3.7 and, hence, is omitted. \(\square\)

5. Automorphism groups (III): The case \(\delta = 2\)

**Theorem 5.1.** Let \(\nu \geq 2\) and \(E\) be the subgroup of \(\text{Aut}(O(2\nu + 2, q))\) defined as follows:
\[
E = \{\sigma \in \text{Aut}(O(2\nu + 2, q)) : \sigma([e_i]) = [e_i], \sigma([f_i]) = [f_i], i = 1, \ldots, \nu\}.\]
Then $\text{Aut}(O(2\nu + 2, q)) = PO_{2\nu + 2}(\mathbb{F}_q) \cdot E$. Moreover, $E$ consists of all $\sigma(k_1, \ldots, k_\nu, \pi, x_1, x_2, y_1, y_2)$ which maps any vertex $[a_1, \ldots, a_{2\nu + 2}]$ of $O(2\nu + 2, q)$ to

$$\left[ \pi(a_1), k_2 \pi(a_2), \ldots, k_\nu \pi(a_\nu), k_1 \pi(a_{\nu + 1}), k_1 k_2^{-1} \pi(a_{\nu + 2}), \ldots, k_1 k_\nu^{-1} \pi(a_{2\nu}), a'_{2\nu + 1}, a'_{2\nu + 2} \right],$$

(5.1)

where $k_1, \ldots, k_\nu \in \mathbb{F}_q^\times, \pi \in \text{Aut}(\mathbb{F}_q)$, $x_1, x_2, y_1, y_2 \in \mathbb{F}_q$ and satisfying $x_1^2 - z x_2^2 = k_1$, $y_1^2 - z y_2^2 = k_1 \pi(-z)$ and

$$a'_{2\nu + 1} = k_1 (y_2 \pi(a_{2\nu + 2}) + x_2 \pi (a_{2\nu + 2})) / (x_1 y_2 - x_2 y_1),$$

$$a'_{2\nu + 2} = k_1 z^{-1} (y_1 \pi(a_{2\nu + 1}) + x_1 \pi (a_{2\nu + 1})) / (x_1 y_2 - x_2 y_1).$$

**Proof.** Similar to Theorem 3.3, we can prove $\text{Aut}(O(2\nu + 2, q)) = PO_{2\nu + 2}(\mathbb{F}_q) \cdot E$. It can be readily verified that the map $\sigma(k_1, \ldots, k_\nu, \pi, x_1, x_2, y_1, y_2)$ defined by (5.1) is in $E$. Conversely, we will show that every element of $E$ is of the form $\sigma(k_1, \ldots, k_\nu, \pi, x_1, x_2, y_1, y_2)$.

**Lemma 5.2.** If $[a_1, \ldots, a_{2\nu}, a_{2\nu + 1}, a_{2\nu + 2}] \in V(O(2\nu + 2, q))$ and $[a_1, \ldots, a_{2\nu}] \in V(O(2\nu, q))$, then $a_{2\nu + 1} = a_{2\nu + 2} = 0$.

**Proof.** This is clear from the definition of orthogonal graphs. \(\Box\)

Let $\sigma \in E$. Suppose the image of $[a_1, \ldots, a_{2\nu}, a_{2\nu + 1}, a_{2\nu + 2}]$ under $\sigma$ is $[b_1, \ldots, b_{2\nu}, b_{2\nu + 1}, b_{2\nu + 2}]$. Then as the cases $\delta = 0$ and $\delta = 1$, we have $a_i \neq 0$ if and only if $b_i \neq 0$, $1 \leq i \leq 2\nu$. Therefore by Lemma 5.2 and the foregoing statement, we have permutations $\pi_i, i = 2, \ldots, \nu, \nu + 2, \ldots, 2\nu$, of $\mathbb{F}_q$ with $\pi_i(0) = 0$ such that

$$\sigma ([e_1] + a_j [e_j]) = [e_1] + \pi_j(a_j) [e_j], \quad 2 \leq j \leq \nu,$$

$$\sigma ([e_1] + a_{\nu + j} [f_j]) = [e_1] + \pi_{\nu + j}(a_{\nu + j}) [f_j], \quad 2 \leq j \leq \nu.$$

In the following, we distinguish the cases: $\nu \geq 3$ and $\nu = 2$.

(i) $\nu \geq 3$. If we take Lemma 5.2 into account, the following lemmas corresponding to Lemmas 3.4–3.6 can be proved in a similar way.

**Lemma 5.3.** Suppose $[1, a_2, \ldots, a_{2\nu}, a_{2\nu + 1}, a_{2\nu + 2}] \in V(O(2\nu + 2, q))$ and let

$$\sigma ([1, a_2, \ldots, a_{2\nu}, a_{2\nu + 1}, a_{2\nu + 2}]) = [1, a'_{2}, \ldots, a'_{2\nu}, a'_{2\nu + 1}, a'_{2\nu + 2}].$$

Then $a'_{2} = \pi_2(a_2)$ and $a'_{2\nu + j} = \pi_{\nu + j}(a_{\nu + j}), 2 \leq j \leq \nu$.

**Lemma 5.4.**

(i) $\pi_1(1)^{-1} \pi_1 = \pi_1(1)^{-1} \pi_j = \pi_{\nu + i}(1)^{-1} \pi_{\nu + i} = \pi_{\nu + j}(1)^{-1} \pi_{\nu + j}$, where $2 \leq i, j \leq \nu$.

(ii) For any $a \in \mathbb{F}_q$, $\pi_i(a) \pi_{\nu + i}(1) = \pi_{\nu + i}(a) \pi_i(1) = \pi_j(a) \pi_{\nu + j}(1) = \pi_{\nu + j}(a) \pi_j(1)$, where $2 \leq i, j \leq \nu$. In particular, $\pi_2(1) \pi_{\nu + 2}(1) = \pi_3(1) \pi_{\nu + 3}(1) = \cdots = \pi_\nu(1) \pi_{2\nu}(1)$.
Lemma 5.5. Let \( \pi = \pi_2(1)^{-1}\pi_2 \), then \( \pi \) is an automorphism of \( \mathbb{F}_q \).

Similar to the proof of Theorem 3.3, any vertex \( [a_1, \ldots, a_{2v}, a_{2v+1}, a_{2v+2}] \in V(O(2v + 2, q)) \), \( v \geq 2 \), is mapped to

\[
\{ (a_1\pi(a_1), k_2\pi(a_2), \ldots, k_v\pi(a_v), k_1\pi(a_{v+1}), k_1k_2^{-1}\pi(a_{v+2}), \ldots, k_1k_v^{-1}\pi(a_{2v}), a'_{2v+1}, a'_{2v+2}) \},
\]

under \( \sigma \), where \( k_1 = \pi_2(1)\pi_2^{-1}(1) \), \( k_2 = \pi_2(1), \ldots, k_v = \pi_v(1) \) and \( a'_{2v+1} = a_{2v+1} - za_{2v+2} \). Since \( [a_1, \ldots, a_{2v}, a_{2v+1}, a_{2v+2}] \in V(O(2v + 2, q)) \), we have

\[
2(a_1a_{v+1} + \cdots + a_va_{2v}) + a_{2v+1}^2 - za_{2v+2}^2 = 0,
\]

\[
2k_1(2(a_1\pi(a_1) + \cdots + \pi(a_v)\pi(a_{2v})) + a'_{2v+1}^2 - za'_{2v+2}^2 = 0.
\]

Then \( a'_{2v+1} = za'_{2v+2} = k_1(2a_{2v+1} - za_{2v+2}) \). In particular, when \( (a_{2v+1}, a_{2v+2}) = (1, 0) \) or \( (0, 1) \), we have the following lemmas:

Lemma 5.6. Suppose \([a_1, \ldots, a_{2v}, 1, 0] \in V(O(2v + 2, q)) \) and

\[
\sigma([a_1, \ldots, a_{2v}, 1, 0]) = [\pi(a_1), \ldots, k_1k_v^{-1}\pi(a_{2v}), x_1, x_2],
\]

where \((x_1, x_2)\) is one of the solutions of the equation \( X_1^2 - zX_2^2 = k_1 \). Then for any vertex \([b_1, \ldots, b_{2v}, 1, 0] \) of \( O(2v + 2, q) \), we have

\[
\sigma([b_1, \ldots, b_{2v}, 1, 0]) = [\pi(b_1), \ldots, k_1k_v^{-1}\pi(b_{2v}), x_1, x_2].
\]

Proof. Let \([b_1, \ldots, b_{2v}, 1, 0] \in V(O(2v + 2, q)) \) and suppose

\[
\sigma([b_1, \ldots, b_{2v}, 1, 0]) = [\pi(b_1), \ldots, k_1k_v^{-1}\pi(b_{2v}), y_1, y_2],
\]

where \((y_1, y_2)\) is one of the solutions of the equation \( X_1^2 - zX_2^2 = k_1 \). We will show \((y_1, y_2) = (x_1, x_2)\).

If \([b_1, \ldots, b_{2v}, 1, 0] \not= [a_1, \ldots, a_{2v}, 1, 0] \), then \([\pi(b_1), \ldots, k_1k_v^{-1}\pi(b_{2v}), y_1, y_2] \not= [\pi(a_1), \ldots, k_1k_v^{-1}\pi(a_{2v}), x_1, x_2] \). From these non-adjacent relations, we deduce that \( b_1a_{v+1} + \cdots + b_{2v}a_v + 1 = 0 \) and \( k_1\pi(b_1a_{v+1} + \cdots + b_{2v}a_v) + x_1y_1 - zx_2y_2 = 0 \), respectively. Then \( x_1y_1 - zx_2y_2 = k_1 \). But \( x_1^2 - x_2^2 = k_1 \) and \( y_1^2 - y_2^2 = k_1 \), therefore \((x_1 - y_1)^2 - z(x_2 - y_2)^2 = 0 \). Since \( z \) is a fixed non-square element, \( x_1 = y_1 \) and \( x_2 = y_2 \).

It remains to show that the complementary graph \( \overline{H}_2 \) of the induced subgraph \( H_2 \) of \( O(2v + 2, q) \) with vertex set \( V_2 = [[c_1, \ldots, c_{2v}, 1, 0] \in V(O(2v + 2, q))] \) is connected. It is clear that \( \overline{H}_2 \) is isomorphic to the graph \( \overline{H}_1 \) in Lemma 4.7. Since \( \overline{H}_1 \) is connected, \( \overline{H}_2 \) is connected, too. \( \Box \)

Lemma 5.7. Suppose \([a_1, \ldots, a_{2v}, 0, 1] \in V(O(2v + 2, q)) \) and

\[
\sigma([a_1, \ldots, a_{2v}, 0, 1]) = [\pi(a_1), \ldots, k_1k_v^{-1}\pi(a_{2v}), y_1, y_2],
\]
where \((y_1, y_2)\) is one of the solutions of the equation \(X_1^2 - zX_2^2 = k_1\pi(-z)\). Then for any vertex \([b_1, \ldots, b_{2v}, 0, 1]\) of \(O(2v + 2, q)\), we have

\[
\sigma([b_1, \ldots, b_{2v}, 0, 1]) = [\pi(b_1), \ldots, k_1k_v^{-1}\pi(b_{2v}), y_1, y_2].
\]

**Proof.** The proof is similar to that of Lemma 5.6 and is omitted. \(\square\)

Now assume that for every vertex \([a_1, \ldots, a_{2v}, 1, 0]\) of \(O(2v + 2, q)\),

\[
\sigma([a_1, \ldots, a_{2v}, 1, 0]) = [\pi(a_1), \ldots, k_1k_v^{-1}\pi(a_{2v}), x_1, x_2],
\]

where \((x_1, x_2)\) is one of the solutions of the equation \(X_1^2 - zX_2^2 = k_1\), and for every vertex \([b_1, \ldots, b_{2v}, 0, 1]\) of \(O(2v + 2, q)\),

\[
\sigma([a_1, \ldots, a_{2v}, 0, 1]) = [\pi(a_1), \ldots, k_1k_v^{-1}\pi(a_{2v}), y_1, y_2],
\]

where \((y_1, y_2)\) is one of the solutions of the equation \(X_1^2 - zX_2^2 = k_1\pi(-z)\). Clearly, \(x_1^2 - zx_2^2 = k_1\) and \(y_1^2 - zy_2^2 = k_1\pi(-z)\). Moreover, we have

**Lemma 5.8.**

(i) \(x_1y_1 = x_2y_2z\).

(ii) \(x_1y_2 - x_2y_1 \neq 0\).

**Proof.** (i) Choose \([a_1, \ldots, a_{2v}, 1, 0]\) and \([b_1, \ldots, b_{2v}, 0, 1]\) \(\in V(O(2v + 2, q))\) such that they are not adjacent, that is, \(a_1b_{v+1} + \cdots + a_{2v}b_v = 0\). Since \(\sigma \in E\),

\[
[\pi(a_1), \ldots, k_1k_v^{-1}\pi(a_{2v}), x_1, x_2] \not\sim [\pi(b_1), \ldots, k_1k_v^{-1}\pi(b_{2v}), y_1, y_2],
\]

that is, \(k_1\pi(a_1b_{v+1} + \cdots + a_{2v}b_v) + x_1y_1 - zx_2y_2 = 0\). Hence \(x_1y_1 - x_2y_2z = 0\).

(ii) Since

\[
k_1^2\pi(-z) = (x_1^2 - zx_2^2)(y_1^2 - zy_2^2) = (x_1y_1 - zx_2y_2)^2 - z(x_1y_2 - x_2y_1)^2 = -z(x_1y_2 - x_2y_1)^2,
\]

we have \(x_1y_2 - x_2y_1 \neq 0\). \(\square\)

Finally, we will show \(\sigma = \sigma(\pi, k_1, k_v, \ldots, k_{2v}, x_1, x_2, y_1, y_2)\). Let \([\alpha] = [a_1, \ldots, a_{2v}, a_{2v+1}, a_{2v+2}] \in V(O(2v + 2, q))\). Suppose \(\sigma([\alpha]) = [\pi(a_1), \ldots, k_1k_v^{-1}\pi(a_{2v}), a'_{2v+1}, a'_{2v+2}]\). Choose \([\gamma_1] = [c_1, \ldots, c_{2v}, 1, 0] \in V(O(2v + 2, q))\) such that \([\gamma_1] \not\sim [\alpha]\), then \(a_1c_{v+1} + \cdots + a_{2v}c_v + a_{2v+1} = 0\). Since \(\sigma \in E\), we have \(\sigma([\gamma_1]) \not\sim \sigma([\alpha])\), that is, \(k_1\pi(a_1c_{v+1} + \cdots + a_{2v}c_v) + x_1a'_{2v+1} - zx_2a'_{2v+2} = 0\). Hence we have

\[
x_1a'_{2v+1} - zx_2a'_{2v+2} = k_1\pi(a_{2v+1}). \tag{5.2}
\]

Choose \([\gamma_2] = [d_1, \ldots, d_{2v}, 0, 1] \in V(O(2v + 2, q))\), such that \([\gamma_2] \not\sim [\alpha]\), then \(a_1d_{v+1} + \cdots + a_{2v}d_v - za_{2v+2} = 0\). Since \(\sigma \in E\), we have \(\sigma([\gamma_2]) \not\sim \sigma([\alpha])\), then \(k_1\pi(a_1d_{v+1} + \cdots + a_{2v}d_v) + y_1a'_{2v+1} - zy_2a'_{2v+2} = 0\). Hence we have

\[
y_1a'_{2v+1} - zy_2a'_{2v+2} = -k_1\pi(za_{2v+2}). \tag{5.3}
\]
Solving (5.2) and (5.3), we obtain

\[ a'_{2v+1} = k_1(y_2\pi(a_{2v+1}) + x_2\pi(za_{2v+2}))/ (x_1y_2 - x_2y_1), \]

\[ a'_{2v+2} = k_1z^{-1}(y_1\pi(a_{2v+1}) + x_1\pi(za_{2v+2}))/ (x_1y_2 - x_2y_1). \]

Hence the proof of the theorem for the case \( v \geq 3 \) is complete.

(ii) \( v = 2 \). Now we have only permutations \( \pi_2 \) and \( \pi_4 \).

**Lemma 5.9.**

(i) \( \pi_2(1)^{-1}\pi_2 = \pi_4(1)^{-1}\pi_4 \).

(ii) Let \( \pi = \pi_2(1)^{-1}\pi_2 \). Then \( \pi(ab) = \pi(a)\pi(b) \), \( \pi(-a) = -\pi(a) \) for \( a, b \in \mathbb{F}_q \).

**Proof.** Let \([1, a_2, a_3, a_4, a_5, a_6] \in V(O(2 \cdot 2 + 2, q)) \). Suppose its image under \( \sigma \) is \([1, a'_2, a'_3, a'_4, a'_5, a'_6] \). If \( a_2 \neq 0 \), from \([1, a_2, a_3, a_4, a_5, a_6] \not\sim [0, 0, 1, -a_2^{-1}, 0, 0] \), we deduce \([1, a'_2, a'_3, a'_4, a'_5, a'_6] \not\sim [0, 0, 1, -\pi_2(a_2)^{-1}, 0, 0] \), which implies \( a'_2 = \pi_2(a_2) \). If \( a_2 = 0 \), then \( a'_2 = 0 \) and the formula \( a'_2 = \pi_2(a_2) \) holds trivially. Moreover, from \([1, a_2, a_3, a_4, a_5, a_6] \not\sim [0, 1, -a_4, 0, 0, 0] \), we deduce \([1, a'_2, a'_3, a'_4, a'_5, a'_6] \not\sim [0, 1, -\pi_4(a_4), 0, 0, 0] \), hence \( a'_4 = \pi_4(a_4) \).

Next let \( a_2, a_4 \in \mathbb{F}_q^* \) and we will calculate \( a'_3 \) in two ways. From \([1, a_2, a_3, a_4, a_5, a_6] \not\sim [1, -a_4^{-1}a_3, 0, 0, 0, 0] \), we have \( a'_3 = -\pi_2(-a_4^{-1}a_3)\pi_4(a_4) \). Similarly, from \([1, a_2, a_3, a_4, a_5, a_6] \not\sim [0, 0, -a_2^{-1}a_3, 0, 0, 0] \), we have \( a'_3 = -\pi_2(a_2)\pi_4(-a_2^{-1}a_3) \). Hence

\[ \pi_2(-a_4^{-1}a_3)\pi_4(a_4) = \pi_2(a_2)\pi_4(-a_2^{-1}a_3) \]  \hspace{1cm} (5.4)

where \( a_2, a_4 \in \mathbb{F}_q^* \) and \( a_3 \in \mathbb{F}_q \).

(i) Substituting \( a_2 = a_4 = 1 \) into (5.4), we have \( \pi_2(-a_3)\pi_4(1) = \pi_2(1)\pi_4(-a_3) \) for \( a_3 \in \mathbb{F}_q \).

Hence \( \pi_2(1)^{-1}\pi_2 = \pi_4(1)^{-1}\pi_4 \).

(ii) Let \( \pi = \pi_2(1)^{-1}\pi_2 \). Substituting \( a_3 = -a_4 \) into (5.4), we have

\[ \pi_2(1)\pi_4(a_4) = \pi_2(a_2)\pi_4(a_2^{-1}a_4). \]  \hspace{1cm} (5.5)

Then substituting \( a_4 = 1 \) into (5.5), we have \( \pi_2(1)\pi_4(1) = \pi_2(a_2)\pi_4(a_2^{-1}) \). Hence

\[ \pi(a_2^{-1}) = \pi_4(1)^{-1}\pi_4(a_2^{-1}) = \pi_2(1)\pi_2(a_2)^{-1} = \pi(a_2)^{-1}. \]  \hspace{1cm} (5.6)

From (5.5) and (5.6), we deduce that \( \pi(a_2a_4) = \pi(a_2)\pi(a_4) \). Therefore \( \pi(ab) = \pi(a)\pi(b) \), for \( a, b \in \mathbb{F}_q \). In particular, \( 1 = \pi(1) = \pi(-1)^2 \), then \( \pi(-1) = -1 \). Hence for \( a \in \mathbb{F}_q \), \( \pi(-a) = \pi(-1)\pi(a) = -\pi(a) \). \( \square \)

Note that \([a_1, a_2, a_3, a_4, a_5] \in V(O(5, q)) \) if and only if \([a_1, a_2, a_3, a_4, a_5, 0] \in V(O(2 \cdot 2 + 2, q)) \). Parallel to the case \( v = 2, \delta = 1 \), we have \( \pi \in \text{Aut}(\mathbb{F}_q) \) and

\[ \sigma([a_1, a_2, a_3, a_4, a_5, a_6]) = [\pi(a_1), k_2\pi(a_2), k_1\pi(a_3), k_1k_2^{-1}\pi(a_4), a'_5, a'_6], \]  \hspace{1cm} (5.7)

where \( k_1 = \pi_2(1)\pi_4(1), k_2 = \pi_2(1) \) and \( a'_5^2 - za'_6^2 = k_1\pi(a_5^2 - za_6^2) \).
Notice that Lemmas 5.6, 5.7 hold also for \( \nu = 2 \). Thus one may conclude that every element of \( E \) is of the form \( \sigma(k_1, k_2, x_1, x_2, y_1, y_2, \pi) \) and the proof of theorem for the case \( \nu = 2 \) is complete. Therefore the theorem is proved.

**Corollary 5.10.** When \( \nu = 1 \), \( |\text{Aut}(O(2 \cdot 1 + 2, q))| = (q^2 + 1)! \). When \( \nu \geq 2 \), \( |\text{Aut}(O(2\nu + 2, q))| = q^{\nu(\nu + 1)} \prod_{i=1}^{\nu} (q^i - 1) \prod_{i=1}^{\nu+1} (q^i + 1)[F_q : F_p] \).

**Proof.** The proof is similar to that of Corollary 3.7 and, hence, is omitted.

**References**