Theoretical Computer Science 103 (1992) 39–49
Elsevier

Semicommutations and algebraic languages*

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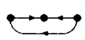
Abstract

Clerbout, M. and Y. Roos, Semicommutations and algebraic languages, Theoretical Computer Science 103 (1992) 39–49.

We give a decidable necessary and sufficient condition for a semicommutation function f_θ to transform any rational language into an algebraic language. This condition is the following: The semicommutation graph of f_θ has no subgraph isomorphic to or 

1. Introduction

The free partially commutative monoids study was initiated by Cartier and Foata [1], whose aim was to solve some combinatory problems. Trace languages, which are subsets of a free partially commutative monoid, were proposed by Mazurkiewicz [11] as tools for the description of concurrent program behaviour. Important results have been found and several syntheses have been written about this subject (see [5, 6, 9, 12, 13, 16, 17, 19]).

A partially commutative alphabet is a couple (A, θ), in which A is an alphabet and θ, the independence relation, is a symmetric and irreflexive binary relation over A. Associated with the commutation relation θ, an application $f_\theta : 2^{A^*} \to 2^{A^*}$ can be defined by: *For every language L over the alphabet A, $f_\theta(L)$ is the set of words which are equivalent to some word of L for the congruence generated by θ.* Thus, f_θ is a unary operation over languages, which is named partial commutation function associated with θ.

More recently, we introduced the notion of semicommutation, which generalizes the notion of partial commutation: a semicommutation is an irreflexive independence

* This work is supported by the P.R.C. Mathématiques et Informatique, and by the EBRA project Algebraic and Syntactic Methods in Computer Science.

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relation over A (see [2, 3]). Since then, a lot of papers have been dealing with their properties (see [8, 10, 15, 18]).

When new operators, as semicommutation functions, are defined, a natural question is: *Do these operators preserve regularity*? Clearly, the answer is *no*: for any semicommutation function f_θ such that θ is not empty, there exist regular languages such that their image by f_θ is not regular. A more interesting question is the following: *If R is a regular language and f_θ a semicommutation function, is it decidable to know whether the image of R by f_θ is a regular language*? As a matter of fact, only decidable sufficient conditions have been found to ensure that the image by a semicommutation function of a regular language remains regular (see [4, 14, 15]).

In this paper, we answer the following question: *If f_θ is a semicommutation function, is it decidable to know whether the image of any regular language by f_θ is algebraic*? We name such functions *algebrico rational functions* and we give a decidable characterization of semicommutation functions which are algebrico rational.

2. Preliminaries

2.1. Notations

In the following text X is the used alphabet; u, v and w are words in X^*; Y is a subset of X.

$|w|$ is the length of the word w;

$|w|_x$ is the number of occurrences of the letter x that appear in the word w;

$|w|_Y$ is the number of occurrences of the letters of Y that appear in the word w;

$\mathrm{alph}(w)=\{x\in X \mid |w|_x\neq 0\}$ is the *alphabet* of the word w;

$\mathrm{com}(w)=\{u\in X^* \mid \forall x\in X, |w|_x=|u|_x\}$ is the *commutative closure* of the word w, and if $L\subseteq X^*$,

$$\mathrm{com}(L)=\bigcup_{w\in L}\mathrm{com}(w).$$

$F(w)$ is the set of *factors* of the word w, that is,

$$F(w)=\{u\in X^* \mid \exists v, v'\in X^*, \text{ with } w=vuv'\},$$

and if $L\subseteq X^*$, we extend the definition by

$$F(L)=\bigcup_{w\in L}F(w).$$

$\mathrm{LF}(w)$ is the set of *left factors* of the word w, that is,

$$\mathrm{LF}(w)=\{u\in X^* \mid \exists v\in X^*, \text{ with } w=uv\},$$

and if $L\subseteq X^*$, we extend the definition by

$$\mathrm{LF}(L)=\bigcup_{w\in L}\mathrm{LF}(w).$$

$\Pi_Y(w)$ is the *projection* of the word w over the *subalphabet* Y, i.e. the image of w by the homomorphism Π_Y which is defined by

$$\forall x \in X, \quad \text{if } x \in Y \text{ then } \Pi_Y(x) = x, \text{ else } \Pi_Y(x) = \varepsilon.$$

$u \sqcup\!\sqcup v$ is the *shuffle* of the two words u and v, that is,

$$u \sqcup\!\sqcup v = \{u_1 v_1 u_2 v_2 \ldots u_n v_n \mid u_i \in X^*,\ v_i \in X^* \text{ and } u = u_1 u_2 \ldots u_n,\ v = v_1 v_2 \ldots v_n\}.$$

$u \sqcap\!\sqcap v$ is the *synchronized shuffle* (see [7]) of the two words u and v, that is,

$$u \sqcap\!\sqcap v = \{w \in (\text{alph}(u) \cup \text{alph}(v))^* \mid \Pi_{\text{alph}(u)}(w) = u \text{ and } \Pi_{\text{alph}(v)}(w) = v\}.$$

$D_1^*(x, y)$ is the *Dyck language* on the alphabet $\{x, y\}$, that is,

$$D_1^*(x, y) = \{w \in \{x, y\}^* \mid |w|_x = |w|_y\}.$$

$D_1'^*(x, y)$ is the *semi-Dyck language* on $\{x, y\}$, that is,

$$D_1'^*(x, y) = \{w \in D_1^*(x, y) \mid \forall u \in \text{LF}(w),\ |u|_x \geq |u|_y\}.$$

Finally, Rat will denote the family of *rational languages*, Alg the family of *algebraic languages* and Ocl the one-counter languages family, which is the smallest set of languages which contains $D_1'^*(x, y)$ and which is closed under rational transductions, product, union and star.

2.2. Semicommutations

A *semicommutation relation* defined over an alphabet X is an irreflexive relation: it is a subset of $X \times X \setminus \{(x, x) \mid x \in X\}$.

With each semicommutation relation θ, we associate a rewriting system $S = \langle X, P \rangle$, which is named *semicommutation system* in which P is the set $\{xy \to yx \mid (x, y) \in \theta\}$. We shall write $u \underset{\theta}{\to} v$ if there is a rule $xy \to yx$ in P and two words w and w' such that $u = wxyw'$ and $v = wyxw'$. We shall write $u \overset{*}{\underset{\theta}{\to}} v$ if there are words $w_1, w_2, \ldots, w_n$ $(n \geq 1)$ such that $w_1 = u$, $w_n = v$, and for each $i < n$, $w_i \underset{\theta}{\to} w_{i+1}$. Then we shall write that there is a *derivation* from u to v.

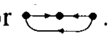
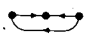
With each semicommutation θ we associate its *commutation graph*, which is the directed graph defined by (X, θ), where X is the vertex set and θ the edge set.

With each semicommutation relation θ, we associate a *semicommutation function* $f_\theta : 2^{A^*} \to 2^{A^*}$, which is defined by

$$\forall L \subset X^*, \quad f_\theta(L) = \bigcup_{w \in L} \left\{ u \in X^* \,\middle|\, w \overset{*}{\underset{\theta}{\to}} u \right\}.$$

3. Algebrico rational functions

Definition 3.1. A semicommutation function f defined on an alphabet X is *algebrico rational* if and only if for any rational language R included in X^*, the language $f(R)$ is algebraic.

Our main result is the following characterization of algebrico rational functions: A semicommutation function f defined on an alphabet X is algebrico rational if and only if the semicommutation graph of f has no subgraph isomorphic to  or . The proof of this result is based on an induction on the cardinality of the alphabet X. Clearly, when X contains only one letter, the semicommutation graph of any semicommutation function f defined on X cannot contain a subgraph isomorphic to or , and it is easy to verify that f is algebrico rational. To give an idea of what happens in the general case, we will also examine the case of a two-letter alphabet.

Let $X=\{a,b\}$. As in the case of a one-letter alphabet, we have to verify that each semicommutation function defined on X is algebrico rational. There are four semicommutation functions on a two-letter alphabet: the identity (no commutation at all), com (the total commutation), $f_{ab\to ba}$ associated with the rule $ab\to ba$, and $f_{ba\to ab}$ associated with the rule $ba\to ab$. At first, we give a necessary and sufficient condition for a word w' to be in the image of a word w of X^* by $f_{ab\to ba}$ (the proof of this result is in [2]).

Lemma 3.2. *Let w and w' be two words of X^*. $w'\in f_{ab\to ba}(w)$ if and only if $w'\in\mathrm{com}(w)$ and $\forall(u,v)\in \mathrm{LF}(w)\times LF(w')$, $|u|=|v|\Rightarrow|u|_b\leqslant|v|_b$.*

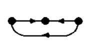
Then we can state the following proposition.

Proposition 3.3. *Any semicommutation function defined on $X=\{a,b\}$ is algebrico rational.*

Proof. If f is the identity, the result is obvious: $f(R)=R\in\mathrm{Rat}\subset\mathrm{Alg}$. Latteux [9] proved that if $f=\mathrm{com}$, $\forall R\in\mathrm{Rat}$, $\mathrm{com}(R)\in\mathrm{Ocl}$. So we have to establish that $f_{ab\to ba}(R)$ is context-free, for each rational language R. The proof is symmetric for $f_{ba\to ab}$. Let h be the morphism defined on $\{a,b,\bar{b}\}$ by $h(a)=a$, $h(b)=b$, $h(\bar{b})=\varepsilon$ and let g be the morphism defined on the same alphabet by $g(a)=a$, $g(b)=\varepsilon$, $g(\bar{b})=b$. We have: $\forall u\in X^*, f(u)=g(h^{-1}(u)\cap(D_1'^*(\bar{b},b)\sqcup\!\sqcup a^*))$. Indeed, set $u'\in f(u)$ and let us denote by $\bar{u}'$ the word u' where each occurrence of the letter b has been marked. ($\bar{u}'=m(u')$ with $m:\{a,b\}\mapsto\{a,\bar{b}\}$, $m(a)=a$, $m(b)=\bar{b}$.) Set $v=u\sqcap\bar{u}'\cap(\bar{b}^*b^*a)^*$. By Lemma 3.2, it is clear that $\Pi_{\{b,\bar{b}\}}(v)\in D_1'^*(\bar{b},b)$; so, $v\in h^{-1}(u)\cap D_1'^*(\bar{b},b)\sqcup\!\sqcup a^*$ and $u'=g(v)$. On the other hand, if $u'\in h^{-1}(u)\cap(D_1'^*(\bar{b},b)\sqcup\!\sqcup a^*)$, each left factor α of $\mathrm{LF}(u')$ satisfies $|\alpha|_{\bar{b}}\geqslant|\alpha|_b$. So, $g(u')\in f(\Pi_{\{a,b\}}(u'))=f(u)$. As $D_1'^*(\bar{b},b)\in\mathrm{Ocl}$, which is a family closed under rational transduction, each rational language has its image by f in Ocl, so f is algebrico rational. □

Remark. If L is an algebraic language, there is a rational language R such that $\mathrm{com}(L)=\mathrm{com}(R)$ (see [9]); so $\mathrm{com}(L)$ is an algebraic language. However, $f_{ab\to ba}(L)$ is not always context-free: Set $L=\{(ba)^n b^n,\ n\geqslant 0\}$. $L\in\mathrm{Alg}$, but $f_{ab\to ba}(L)\cap b^*a^*b^* =\{b^{n+k}a^n b^{n-k},\ n\geqslant k\geqslant 0\}=L_1$. And $\mathrm{LF}(b^*L_1)=\{b^n a^p b^q,\ n\geqslant p\geqslant q\geqslant 0\}\notin\mathrm{Alg}$.

Let us now suppose that the cardinality of the alphabet X is greater than 2.

Definition 3.4. Let f be a semicommutation function defined on the alphabet X. We say that f satisfies the (C) condition if the semicommutation graph of f has no subgraph isomorphic to or .

We may also express this condition in the following way: A semicommutation function f on X associated with the semicommutation relation C_0 satisfies the (C) condition if and only if

$$(y,z)\in C_0 \;\Rightarrow\; ((y,x_1)\in C_0 \text{ and } (x_2,z)\in C_0 \;\Rightarrow\; x_1=z \text{ or } x_2=y).$$

Proposition 3.5. *If a semicommutation function is algebrico rational then it satisfies the (C) condition.*

Proof. Let f be a semicommutation function defined on $X=\{a,b,c\}$ by the commutation graph shown in Fig. 1. Set $R=(abc)^*$. Then

$$f(R)\cap c^*b^*a^* = \{c^n b^n a^n \mid n\in N\}\notin \text{Alg}.$$

Let g be the function defined on $\{a,b,c,d\}$ by the semicommutation graph shown in Fig. 2. Set $R'=(cd)^*(ab)^*$. Then

$$g(R')\cap d^*b^*c^*a^* = \{d^n b^p c^n a^p \mid n,p\in N\}\notin \text{Alg}.$$

A function which does not satisfy the (C) condition would never be algebrico rational. □

We shall now prove the converse of this proposition. We will consider two cases.

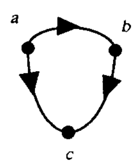


Fig. 1.

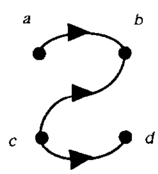


Fig. 2.

Definition 3.6. We say that a semicommutation function f defined on X satisfies the (P) *property* if and only if there exists a letter x in X such that for each letter y in X, $yx \in f(xy)$ or such that for each letter y in X, $xy \in f(yx)$.

So, let f be a semicommutation function defined on X which satisfies both the (C) condition and the (P) property, i.e. there exists a letter x in X such that

$$\forall y \in X, \quad yx \in f(xy).$$

The other case ($xy \in f(yx)$) would be studied in the same way. Let us explain what the function f does:

$$\forall y_1, y_2 \in X \setminus \{x\}, \quad y_1 y_2 \notin f(y_2 y_1).$$

Because the commutation graph of f already contains $y \longleftarrow x \longrightarrow y_2$ it is impossible to add an arrow between y_1 and y_2 since f satisfies the (C) condition. However, we may have commutations of the kind $yx \to xy$, $y \in X \setminus \{x\}$. Thus, the alphabet X may be partitioned into three disjoint subsets: $X = X_1 \cup X_2 \cup \{x\}$, with $X_1 = \{y \in X \setminus \{x\}, xy \in f(yx)\}$, $X_2 = \{y \in X, xy \notin f(yx)\}$. It means that, in a word $w \in X^*$, the occurrences of the letter x are going to move on left or right, in each factor of w which is in X_1^*, but an occurrence of x may move over a letter of X_2 only from left to right. When adding in w the new positions of marked occurrences of the letter x ($\bar{x}$ instead of x) from a word w' in $f(w)$, we get words of $D_1^*(x, \bar{x}) \sqcup\!\sqcup X_1^*$ and words of $D_1'^*(x, \bar{x}) \sqcup\!\sqcup (X_1 \cup X_2)^*$. This is what is formalized in the following lemmas.

Lemma 3.7. *Let f be a semicommutation function such that f satisfies (C) and (P). Let $u, u' \in X^*$. Then $u \xrightarrow{*} u'$ if and only if the following conditions hold*:

(1) $\Pi_{X-\{x\}}(u) = \Pi_{X-\{x\}}(u')$.

(2) $\Pi_x(u) = \Pi_x(u')$.

(3) *for all prefixes $u_1 \in X^* X_2$ of u and $u'_1 \in X^* X_2$ of u', we have*

$$\Pi_{X-\{x\}}(u_1) = \Pi_{X-\{x\}}(u'_1) \;\Rightarrow\; |u_1|_x \geqslant |u'_1|_x.$$

Proof. ($\Rightarrow$): It is clear that conditions (1) and (2) are necessary. We prove condition (3) by induction on the length of the derivation $u \xrightarrow{n} u'$. The value $n = 0$ being obvious, consider $u \xrightarrow{n} u' \to u''$. Two cases may arise:

- $u' = w_1 xy w_2$ and $u'' = w_1 yx w_2$, with $y \in X_1 \cup X_2$.
- $u' = w_1 yx w_2$ and $u'' = w_1 xy w_2$, with $y \in X_1$.

Let u_1, u'_1 and $u''_1 \in X^* X_2$ be prefixes of u, u' and u''. If $\Pi_{X-\{x\}}(u_1) = \Pi_{X-\{x\}}(u'_1)$ then, by induction hypothesis, $|u_1|_x \geqslant |u'_1|_x$. Moreover, in both cases, if $y \in X_1$ then $\Pi_{X-\{x\}}(u'_1) = \Pi_{X-\{x\}}(u''_1)$ implies that $|u'_1|_x = |u''_1|_x$. Now if $y \in X_2$ and $w = \Pi_{X-\{x\}}(u'_1) = \Pi_{X-\{x\}}(u''_1)$ then $|u'_1|_x = |u''_1|_x$ if $w \neq \Pi_{X-\{x\}}(w_1 y)$, and $|u'_1|_x = |u''_1|_x + 1$ otherwise.

($\Leftarrow$): If $u, u' \in X^*$ satisfy the three conditions then $|u| = |u'|$. Let w be the longest common prefix of u and u'. We argue on the integer $n = |u| - |w| = |u'| - |w|$. If $n = 0$

then $u=u'$ and there is nothing to prove; so we assume $n>0$. Because of condition (3) we have only the following different cases:

- $u=wx^r yu_2$ and $u'=wyu'_2$ with $r>0$ and $y\in X_1\cup X_2$.
- $u=wyu_2$ and $u'=wx^r yu'_2$ with $r>0$ and $y\in X_1$.

In the first case we have $u\xrightarrow{*}u''=wyx^r u_2$. Furthermore, u' and u'' satisfy the three conditions of the statement; so, by induction hypothesis, we have $u''\xrightarrow{*}u'$; thus, $u\xrightarrow{*}u'$. In the second case we have $wyx^r u'_2=u''\xrightarrow{*}u'=wx^r yu'_2$. Since u and u'' satisfy the three conditions, we obtain $u\xrightarrow{*}u''$; thus, $u\xrightarrow{*}u'$. □

Notation. If w is a word of X^*, $\bar{w}$ denotes the image of w by the morphism which marks the letter x: $m: X\mapsto X\cup\{\bar{x}\}$, $m(x)=\bar{x}$, and $\forall y\in X\setminus\{x\}$, $m(y)=y$.

Now we are able to prove the following lemma.

Lemma 3.8. *Given $u,u'\in X^*$, let $v\in X^*$ be the word in $u \sqcap \bar{u}' - X^*\bar{x}xX^*$ when it exists. Then $u\xrightarrow{*}u'$ if and only if $v\in((D_1^*(x,\bar{x})\sqcup\!\sqcup X_1^*)(D_1'^*(x,\bar{x})\sqcup\!\sqcup(X_1\cup X_2)^*))^*$.*

Proof. ($\Rightarrow$): By Lemma 3.7, if $u\xrightarrow{*}u'$ holds then the word v exists. Furthermore, we factorize $v=v_1v_2\ldots v_m$ in a unique way as follows. For all prefixes w of v consider the difference $\delta(w)=|w|_x-|w|_{\bar{x}}$. The factors $v_1,v_2,\ldots$ correspond to the positions in the word v where $\delta(w)$ changes sign. Formally, the factor v_1 is the longest prefix of v such that for all prefixes w of v_1, $|\Pi_x(w)|\leqslant|\Pi_{\bar{x}}(w)|$ holds. Set $v=v_1v'$ and define v_2 as the longest prefix of v' such that for all prefixes w of v', $|\Pi_x(w)|\geqslant|\Pi_{\bar{x}}(w)|$ holds. Set $v=v_1v_2v''$ and apply this procedure recursively to v''.

It suffices to verify that for $k=0,1,\ldots$ the word v_{2k+1} belongs to X_1^*. Assume by contradiction that this is not the case, i.e. for some $k=0,1,\ldots$ and some words $t\in(X_1^*\cup\{x\}\cup\{\bar{x}\})^*$, $v'\in X^*$ and $y\in X_2$ we have

$$\Pi_{X-\{x,\bar{x}\}}(v_{2k+1})=tyv'.$$

Then $u_1=\Pi_{X-\{\bar{x}\}}(v_1v_2\ldots v_{2k}ty)$ is a prefix of u and $u'_1=\Pi_{X-\{x\}}(v_1v_2\ldots v_{2k}ty)$ is a prefix of u' that satisfy $\Pi_{X-\{x\}}(u_1)=\Pi_{X-\{x,\bar{x}\}}(v_1v_2\ldots v_{2k}ty)=\Pi_{X-\{\bar{x}\}}(u'_1)$, implying that $|u_1|_x\geqslant|u'_1|_x$. This contradicts the definition of v_{2k+1}.

($\Leftarrow$): If a word v exists then u and u' satisfy conditions (1) and (2) of Lemma 3.7. Furthermore, because of the form of v, if a prefix $u_1\in X^*X_2$ of u and a prefix $u'_1\in X^*X_2$ of u' satisfy

$$\Pi_{X-\{x\}}(u_1)=\Pi_{X-\{x\}}(u'_1)$$

then there exists a prefix w of v such that

$$\Pi_{X-\{x\}}(u_1)=\Pi_{X-\{x\}}(u'_1)=\Pi_{X-\{x,\bar{x}\}}(w).$$

By hypothesis, we have $\Pi_{\{x,\bar{x}\}}(w)\in D_1^*(x,\bar{x})\mathrm{LF}(D_1'^*(x,\bar{x}))$, which is exactly condition (3). □

In this light, we can state that if f satisfies both the (P) property and the (C) condition then f is an algebraic relation.

Proposition 3.9. *Let f be a semicommutation function defined on the alphabet X, which satisfies the (C) condition and for which there exists a letter x such that $\forall y \in X$, $yx \in f(xy)$. Then we can find morphisms h and g and two subsets of X, X_1 and X_2, such that*

$$\forall u \in X^*, \quad f(u) = g(h^{-1}(u) \cap ((D_1^*(x,\bar{x}) \sqcup\!\sqcup X_1^*)(D_1'^*(x,\bar{x}) \sqcup\!\sqcup (X_1 \cup X_2)^*))^*).$$

So f is algebrico rational.

Proof (*sketch*). Set $X_1 = \{y \in X \setminus \{x\} \mid xy \in f(yx)\}$ and $X_2 = X \setminus (X_1 \cup \{x\})$. Let h and g be the morphisms defined on $X \cup \{\bar{x}\}$ by

$$\forall y \in X_1 \cup X_2, \quad h(y) = y, \; g(y) = y,$$

$$h(x) = x, \qquad g(x) = \varepsilon,$$

$$h(\bar{x}) = \varepsilon, \qquad g(\bar{x}) = x.$$

Set $L = ((D_1^*(x,\bar{x}) \sqcup\!\sqcup X_1^*)(D_1'^*(x,\bar{x}) \sqcup\!\sqcup (X_1 \cup X_2)^*))$. It is easy to see that

$$\forall u \in X^*, \quad f(u) = g(h^{-1}(u) \cap L^*).$$

So f is algebrico rational. □

Now we study the case where the (P) property is not satisfied.

Notation. If w is a word of X^*, and $t \in N^+$, we write $w(t)$ as the left factor of length t of w.

Lemma 3.10. *Let $u \in X^+$, $a \notin X$, $w \in a(u \sqcup\!\sqcup a^i)$, $w' \in u \sqcup\!\sqcup a^i a^+$. Then we can find $t_0 \in N^+$ such that*

(1) $\text{com}(w(t_0)) = \text{com}(w'(t_0))$;
(2) $\forall s \in \{1, \ldots, t_0 - 1\}$, $|w'(s)|_a < |w(s)|_a$.

Proof. Let t_0 be the smallest element of $\{t \mid t \in N^+ \text{ and } |w'(t)|_a \geqslant |w(t)|_a\}$. t_0 exists since, if $t = |w|$, we have $|w'(t)|_a \geqslant |w(t)|_a$. Then $|w'(t_0)|_a = |w(t_0)|_a$ and, thus, $|\Pi_X(w'(t_0))| = |\Pi_X(w(t_0))|$, which implies that $\Pi_X(w'(t_0)) = \Pi_X(w(t_0))$ since these two words are left factors of u. Therefore, $\text{com}(w(t_0)) = \text{com}(w'(t_0))$. From the definition of t_0 the second assertion is satisfied. □

For a given semicommutation function f, if no letter may commute with each of the others, shuffles will be local. So, to get the image of a word by f, it is sufficient to make shuffles on factors which are defined on a smaller alphabet. This motivates the following lemma.

Lemma 3.11. *Let f be a semicommutation function defined on X, which satisfies the (C) condition but not the (P) property. Then, for any word u of* X^*, *for any word v of* $f(u)$, *we can find decompositions* $u=u_1u_2$ *and* $v=v_1v_2$ *with* $u_1\neq\varepsilon$, $\mathrm{alph}(u_1)\subsetneq X$ *and* $v_1\in f(u_1)$.

Proof. If $\mathrm{alph}(u)\subsetneq X$, the result is obvious. If not, set $u=au'$, $v=dv'$, $a\in X$, $d\in X$. Then
either $a=d$: we can choose $u_1=v_1=a$;
or $a\neq d$: we set $D=\{z\in X\setminus\{a\}\mid za\in f(az)\}$ and $Y=X\setminus(D\cup\{a\})$.
Then
- $d\in D$; so $D\neq\emptyset$;
- $Y\neq\emptyset$ since f does not verify the (P) property;
- if z_1 and z_2 are in D, the graph of f contains a subgraph $z_1 \leftarrow a \rightarrow z_2$; thus, there is no commutation between z_1 and z_2 because f satisfies the (C) condition.

We have to consider two different cases.

First case: *There do not exist letters* $y\in Y$ *and* $z\in D$ *such that* $zy\in f(yz)$. Then let us set $u=awyu''$ and $v=dw'y'v''$ with $w,w'\in(D\cup\{a\})^*$ and $y,y'\in Y$. No occurrence of letter a in aw can overstep y because $\forall x\in Y$, $xa\notin f(ax)$. So we have $|aw|_a\leqslant|w'|_a$, i.e. $|w'|_a>|w|_a$.

On the other hand, for two letters (d_1,y_1) in $D\times Y$, it is possible to have $y_1d_1\in f(d_1y_1)$, but, since $d_1y_1\notin f(y_1d_1)$, we get $\Pi_D(dw')\in\mathrm{LF}(\Pi_D(dw))$. Then

$$u=u'yu'',\quad u'\in a(\Pi_D(dw')\sqcup a^i)(w''\sqcup a^{i'}),\ \ \Pi_D(w)=\Pi_D(dw')w'',$$

$$v=v'y'v'',\quad v'\in\Pi_D(dw')\sqcup a^j,$$

where $j=|w'|_a$, $i+i'=|w|_a$; so $j>i$.

From Lemma 3.10, it follows that $u'=u_1u'_2$ and $v'=v_1v'_2$, where $u_1\neq\varepsilon$ and $\mathrm{com}(u_1)=\mathrm{com}(v_1)$ (so $v_1\in f(u_1)$). Moreover, $\mathrm{alph}(u_1)\subset D\cup\{a\}\subsetneq X$. Hence, the couple (u_1,v_1) answers the problem.

Second case: *There exist letters* $y\in Y$ *and* $d_1\in D$ *such that* $d_1y\in f(yd_1)$. Then $D=\{d\}$; if two different letters d_1 and d_2 belong to D, we will find in the semicommutation graph of f the subgraph $y \leftarrow d_1 \rightarrow a \leftarrow d_2$, which contradicts the hypothesis.

Then we set $E=\{z\in X\setminus\{d\}\mid dz\in f(zd)\}$ and $Z=X\setminus(E\cup\{d\})$. We have
- $E\neq\emptyset$ because $y\in E$;
- $Z\neq\emptyset$ because f does not verify the (P) property;
- if z_1 and z_2 are two different letters of E, there is no commutation between z_1 and z_2 because the graph of f already contains $z_1 \leftarrow d \rightarrow z_2$ and f satisfies the (C) condition.

Let us set $u=awzu''$ and $v=dw'y'v''$, with $w,w'\in(E\cup\{d\})^*$ and $z,z'\in Z$. No occurrence of the letter d in u'' can overstep z and z', but it is possible to have rules such as $dt\rightarrow td$ for a letter t in Z. So we get $|aw|_d\geqslant|dw'|_d$, i.e. $|w|_d>|w'|_d$.

On the other hand, $\forall x\in E\setminus\{a\}$, $\forall z\in Z$, the graph of f contains $a \leftarrow d \rightarrow x$. No arrow like $x \leftarrow z$ can be added and $za\notin f(az)$ because $D=\{d\}$ and $d\notin Z$. Thus, $\forall x\in E$,

$\forall z \in Z$, $zx \notin f(xz)$. But we can find a rule as $z_1 x_1 \to x_1 z_1$ for $(x_1, z_1) \in E \times Z$. So we have $\Pi_E(aw) \in \mathrm{LF}(\Pi_E(dw')) = \mathrm{LF}(\Pi_E(w'))$. Then we write

$$u = u'zu'', \quad u' \in \Pi_E(aw) \sqcup\!\sqcup\, d^j,$$

$$v = v'z'v'', \quad v' \in d(\Pi_E(aw) \sqcup\!\sqcup\, d^j)(w'' \sqcup\!\sqcup\, d^{i'}), \ \Pi_E(dw') = \Pi_E(aw)w'',$$

where $j = |aw|_d$, $i + i' = |w'|_d$; so $j > i$.

According to Lemma 3.10, we get $u' = u_1 u_2''$ and $v' = v_1 v_2''$, where $u_1 \neq \varepsilon$, $\mathrm{com}(u_1) = \mathrm{com}(v_1)$ (so $v_1 \in f(u_1)$), and $\mathrm{alph}(u_1) \subset E \cup \{d\} \subsetneq X$. The couple (u_1, v_1) agrees with the question, proving the result. □

We are now able to state the main result of the paper.

Proposition 3.12. *Let f be a semicommutation function defined on the alphabet X, and satisfying the (C) condition. Then f is algebrico rational.*

Proof. We proceed by induction on the cardinality of the alphabet X, denoted by $\mathrm{card}(X)$. If $\mathrm{card}(X) = 2$, the result is true; see Proposition 3.3. If $\mathrm{card}(X) > 2$ then if f satisfies the (P) property, the result is true because of Proposition 3.9. If not, we are going to show that for each rational language R, $f(R) \in \mathrm{Alg}$. Let $R \in \mathrm{Rat}$. We can define a deterministic automaton $M = (X, Q, q_0, *, F)$ which accepts R. If $q, q' \in Q$, we set $R_{q,q'} = \{w \in X^*, q * w = q'\}$. Let s be the substitution defined on $Q \times Q$ by

$$\forall (q, q') \in Q \times Q, \quad s((q, q')) = \bigcup_{x \in X} f(R_{q,q'} \cap (X \setminus \{x\})^*).$$

By induction hypothesis, s is an algebraic substitution. Let K be the rational language defined on $Q \times Q$ by

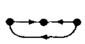
$$K = \{(q_0, q_1)(q_1, q_2) \ldots (q_{p-1}, q_p) \mid p \geqslant 1, \forall i \in \{1, \ldots, p\}, q_i \in Q, q_p \in F\}.$$

We can easily show that $f(R) = s(K)$, and the proof is complete since the image of a rational language by an algebraic substitution is an algebraic language. □

As a matter of fact, Propositions 3.5 and 3.12 permit us to state that the image of a rational language by a semicommutation function is always in Ocl. So we state the following proposition.

Proposition 3.13. *Let f be a semicommutation function defined on X. The following assertions are equivalent:*

(1) *f is algebrico rational.*

(2) *The semicommutation graph of f does not contain any subgraph isomorphic to* *or*  .

(3) *For each rational language R, $f(R) \in$* Ocl.

(4) *For each rational bounded language R, $f(R) \in$* Alg.

Proof. (1)⇒(2) by proposition 2. (2)⇒(3) because constructions which give us the image by f of a rational language use $D_1'^*$ and D_1^* which are in Ocl and operations under which Ocl is closed. (3)⇒(4) is obvious. (4)⇒(1) when looking at the proof of Proposition 3.5. □

In the particular case of partial commutation (associated with irreflexive and symmetrical relations), the results of Propositions 3.5 and 3.12 become Proposition 3.14.

Proposition 3.14. *A partial commutation function is algebrico rational if and only if its commutation graph does not contain a path whose length is* 3.

Acknowledgment

We thank the anonymous referees for suggesting several improvements.

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