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Optimality Principles of Dynamic Programming in Differential Games

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1. INTRODUCTION

A two-person zero-sum differential game is a dynamical system whose dynamics are described in the general case by a system of differential inclusions

$$\dot{x}(t) \in F_1(t, x(t)), \qquad x(t_*) = x_*, \qquad x \in R^n, \tag{1.1}$$

$$\dot{y}(t) \in F_2(t, y(t)), \qquad y(t_*) = y_*, \qquad y \in R^k, \tag{1.2}$$

with phase constraints of the form

$$x(t) \in N_1(t), \qquad y(t) \in N_2(t), \qquad t \geqslant t_*, \tag{1.3}$$

where $N_1(\cdot)$, $N_2(\cdot)$ are set-valued maps with closed graphs. By a solution of (1.1) (resp. (1.2)) we mean an absolutely continuous function $x(\cdot)$ (resp. $y(\cdot)$) satisfying (1.1) (resp. (1.2)) almost everywhere. The game stops when, for the first time $t \geqslant t_*$ (henceforth abbreviated by $T_M = T_M\{t_*, x_*, y_*; x(\ ), y(\ )\}$) the triplet $(t, x(t), y(t))$ hits a prescribed terminal set $M \subset R^{n+k+1}$. At time T_M player I receives from player II the payoff

$$\begin{aligned} P &= P\{t_*, x_*, y_*; x(\cdot), y(\cdot)\} \\ &= g(T_M, x(T_M), y(T_M)) + \int_{t_*}^{T_M} h(t, x(t), y(t))\, dt. \end{aligned} \tag{1.4}$$

Concerning the information available to both players during the course of the game, we assume they can employ any lower strategy; this concept will be introduced in this paper (Definition 2.1) and is more general than the notion of a lower π-strategy introduced in [4, p. 400]. Under Axioms (A1)–(A6) we prove the optimality principle of dynamic programming for

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player I and player II in differential games. This principle was originally formulated and proved by Bellman for optimal control problems under the assumption that the optimal cost function was twice differentiable [1].

2. Assumptions and Strategies

Throughout the paper, t_*, x_*, y_* are specified values with (t_*, x_*, y_*) being the initial point. We shall consider admissible points (t_*, x_*, y_*) only, i.e., those for which $x_* \in N_1(t_*)$, $y_* \in N_2(t_*)$. When $N_1(t) = R^n$, $N_2(t) = R^k$, $t \geqslant t_*$, we actually deal with a differential game without phase constraints. We make the following assumptions on the game:

(A1) the graphs of set-valued maps $N_1(\cdot)$, $N_2(\cdot)$, and the terminal set M are closed in R^{n+1}, R^{k+1}, R^{n+k+1}, respectively;

(A2) the functions $g(\cdot)$ and $h(\cdot)$ occurring in (1.4) are continuous;

(A3) for any $(t, x) \in N_1$, the set $F_1(t, x)$ is convex; we assume the same for F_2;

(A4) (i) for each $x \in N_1(t)$, the set $F_1(t, x)$ is compact; (ii) for each x, $F_1(t, x)$ is measurable in t; (iii) there exists $x^* \in R^n$ and an integrable function $a(t) \geqslant 0$ such that, for $t \geqslant t_*$, the following condition with H denoting the Hausdorff metric holds: $H[F_1(t, x^*), \{0\}] \leqslant a(t)$; (iv) $F_1(t, x)$ satisfies a Lipschitz condition in x, i.e., $H[F_1(t, x_1), F_1(t, x_2)] \leqslant b(t) \cdot \|x_2 - x_1\|$, where $b(t)$ is an integrable function; the same is true for $F_2(t, y)$.

Observe that (A4) will hold true if: (i) $F_1(t, x) = f_1(t, x, U)$, $F_2(t, y) = f_2(t, x, V)$; (ii) the functions $f_1(\cdot), f_2(\cdot)$ are continuous and Lipschitz in the spatial variable; (iii) U and V are compact subsets of some Euclidean spaces. Concerning regularity of lower and upper values of the game to be defined later, we assume less than continuity. Namely, we make the following two assumptions:

(A5) the lower value $\underline{W}(t, x, y)$ is lower semi-continuous in y;

(A6) the upper value $\bar{W}(t, x, y)$ is upper semi-continuous in x.

There are several papers giving sufficient conditions for continuity of lower and upper values of a differential game; see, for example, [2, 3]. These papers, however, deal with lower and upper values defined in a slightly different manner than $\underline{W}(t, x, y)$, $\bar{W}(t, x, y)$. Clearly, in several situations those concepts coincide. In our setting, which is general enough, the functions $\underline{W}(t, x, y)$, $\bar{W}(t, x, y)$ may not be continuous because closedness is the only assumption imposed on the constraint sets $N_1 = \{(t, x): x \in N_1(t)\}$ and $N_2 = \{(t, x): y \in N_2(t)\}$. However, it was shown in [6, Propositions 2.1, 2.2] that for generalized pursuit-evasion differential games and games of fixed time duration, the lower value $\underline{V}(t, x, y)$, defined in a very similar manner as $\underline{W}(t, x, y)$, is lower semi-continuous in y.

It was also shown in [6, pp. 160–161] that the upper semi-continuity in x and lower semi-continuity in y imply that, for each pair of trajectories $x(\cdot)$, $y(\cdot)$, the lower value $\underline{V}(t, x(t), y(t))$ is continuous from the left in t, although it may not be continuous from the right [6, p. 162]. The reason for this was that $N_1(\cdot)$, $N_2(\cdot)$ were not continuous in the Hausdorff metric sense. If there were, this author would expect the continuity of $\underline{V}(t, x(t), y(t))$ and $\underline{W}(t, x(t), y(t))$ along each pair of admissible trajectories. The problem of continuity of $\underline{W}(t, x, y)$ and $\bar{W}(t, x, y)$ is less obvious and will not be treated here.

For each point $x_0 \in N_1(t_0)$ ($y_0 \in N_2(t_0)$), $t_0 \geqslant t_*$, denote by $X(t_0, x_0)$ (resp. $Y(t_0, y_0)$) the space of all admissible trajectories of player I (II) defined on $[t_0, \infty)$ and emanating from $x_0(y_0)$. It is known [5] that, under Axiom (A4), the system of differential inclusions has a solution through each initial point (t_0, x_0) (resp. (t_0, y_0)) on $[t_0, \infty)$.

By a finite partition $\pi = (t_i)$ of a ray $[a, \infty)$ we mean a partition, such that, for each $[a, \bar{t}]$, finitely many partition points t_i belong to $[a, \bar{t}]$.

DEFINITION 2.1. By a lower strategy $\alpha = \alpha(t_0, x_0, y_0)$ for player I we shall understand a sequence of maps

$$\alpha_i(t_i, y(\ )) \rightarrow (t_{i+1}, x(s)), \qquad t_i < s \leqslant t_{i+1}, i \geqslant 0,$$

such that the resulting partition is finite and α_i is non-anticipatory.

It follows from this definition that at time t_0 player I chooses the first partition point t_1 and the trajectory he will be moving along on the segment $[t_0, t_1]$. At time t_1, based on the knowledge of $x(s)$, $y(s)$, $t_0 \leqslant s \leqslant t_1$, he chooses the consecutive partition point t_2 and the trajectory he will be moving along with on $[t_1, t_2]$, and so on.

In a similar way, the notion of a lower strategy β for player II is understood. It is obvious that each lower π-strategy [4, p. 400] is a particular case of a lower strategy introduced here because, in the case of lower π-strategy, the partitions do not depend on the opponent's trajectory. Denote the space of all lower strategies for player I (II) by $A(t_0, x_0, y_0)$ ($B(t_0, x_0, y_0)$).

The notions of lower and upper values of the game are defined in a standard way, namely

$$\underline{W}(t_0, x_0, y_0) = \underline{W} = \sup\{P^+(\alpha) \colon \alpha \in A(t_0, x_0, y_0)\}, \tag{2.1}$$

$$\bar{W}(t_0, x_0, y_0) = \bar{W} = \inf\{P^-(\beta) \colon \beta \in B(t_0, x_0, y_0)\}, \tag{2.2}$$

where $P^+(\alpha)(P^-(\beta))$ is the least (greatest) value of the payoff P that is ensured by a strategy $\alpha(\beta)$, that is to say,

$$P^+(\alpha)=\inf\{P[t_0, x_0, y_0; \alpha(y(\cdot)), y(\cdot)]: y(\cdot)\in Y(t_0, y_0)\}, \quad (2.3)$$

$$P^-(\beta)=\sup\{P[t_0, x_0, y_0; x(\cdot), \beta(x(\cdot))]: x(\cdot)\in X(t_0, x_0)\}. \quad (2.4)$$

The game is said to have a value W if $W=\underline{W}=\bar{W}$. Clearly, each lower strategy α (the same refers to any lower strategy β) is a non-anticipating operator from $Y(t_0, y_0)$ into $X(t_0, x_0)$, which means that if $y_1(t)=y_2(t)$, $t_0\leqslant t\leqslant \bar{t}$ then $\alpha[y_1(\cdot)](t)=\alpha[y_2(\cdot)](t)$, $t_0\leqslant t\leqslant \bar{t}$. The spaces of such defined strategies (non-anticipating operators) will be denoted by $A_1(t_0, x_0, y_0)$ and $B_1(t_0, x_0, y_0)$, respectively. Thus $A(t_0, x_0, y_0)\subset A_1(t_0, x_0, y_0)$ and $B(t_0, x_0, y_0)\subset B_1(t_0, x_0, y_0)$. Arguing as in [4, p. 400], one can show that a unique outcome results from two given strategies $\alpha\in A_1(t_0, x_0, y_0)$, $\beta\in B(t_0, x_0, y_0)$ as well as from $\alpha\in A(t_0, x_0, y_0)$, $\beta\in B_1(t_0, x_0, y_0)$, i.e., a pair $(x^*(\cdot), y^*(\cdot))$ of trajectories with the property

$$\alpha(y^*(\cdot))=x^*(\cdot), \qquad \beta(x^*(\cdot))=y^*(\cdot). \quad (2.5)$$

Therefore, we get the inequalities

$$\underline{W}(t_0, x_0, y_0)\leqslant \underline{V}=\sup_{\alpha\in A_1(t_0, x_0, y_0)} P^+(\alpha),$$

$$\bar{V}=\inf_{\beta\in B_1(t_0, x_0, y_0)} P^-(\beta)\leqslant \bar{W}(t_0, x_0, y_0).$$

Remark 2.1. Each trajectory $\bar{x}(\cdot)\in X(t_0, x_0)$ (respectively $\bar{y}(\cdot)\in Y(t_0, y_0)$) may be treated as a trivial strategy $\bar{\alpha}\in A(t_0, x_0, y_0)$ ($\bar{\beta}\in B(t_0, x_0, y_0)$).

To see this, set $\bar{\alpha}(y(\cdot))=\bar{x}(\cdot)$, $y(\cdot)\in Y(t_0, y_0)$, $t_0\leqslant t<\infty$ (similarly for $\bar{\beta}$).

3. Basic Results

The following set-valued map $W(\cdot)$: $[t_*, \infty)$ plays an important role in our considerations.

Set $(x, y, r)\in W(t)$, $t_*\leqslant t<\infty$, if and only if $x\in N_1(t)$, $y\in N_2(t)$, and $\underline{W}(t, x, y)+r\leqslant \underline{W}(t_*, x_*, y_*)$. Thus, if $x\in N_1(t)$, $y\in N_2(t)$, and $(x, y, r)\notin W(t)$ then, by (A5), $\underline{W}(t, x, y')+r>\underline{W}(t_*, x_*, y_*)+\varepsilon$ for a certain positive number ε and y' belonging to some neighbourhood of y. Clearly, $(x_*, y_*, 0)\in W(t_*)$.

LEMMA 3.1. *Let assumptions* (A3)–(A5) *hold. Then, for every* $(x, y, r) \in W(t)$, *every* $\delta > 0$, *and every strategy* $\alpha \in A(t, x, y)$ *there exists an admissible trajectory* $\bar{y}(\cdot) \in Y(t, y)$ *such that either* $(t+\bar{\delta}, \alpha(\bar{y}(\cdot))(t+\bar{\delta}), \bar{y}(t+\bar{\delta})) \in M$ *for some* $0 \leqslant \bar{\delta} \leqslant \delta$, *or else* $(\alpha(\bar{y}(\cdot))(t+\delta), \bar{y}(t+\delta), r_\alpha(t+\delta)) \in W(t+\delta)$, *i.e.*,

$$\underline{W}(t+\delta, \alpha(\bar{y}(\cdot))(t+\delta), \bar{y}(t+\delta)) + r_\alpha(t+\delta) \leqslant \underline{W}(t_*, x_*, y_*), \quad (3.1)$$

where

$$r_\alpha(t+\delta) = \int_t^{t+\delta} h(s, \alpha(\bar{y}(\cdot))(s), \bar{y}(s))\, ds + r.$$

Proof. Let us assume that, for each $y(\cdot) \in Y(t, y)$ and $0 \leqslant \bar{\delta} \leqslant \delta$, we have $(t+\bar{\delta}, \alpha(\bar{y}(\cdot))(t+\bar{\delta}), \bar{y}(t+\bar{\delta})) \notin M$. We will prove our result by assuming that the assertion is false and arriving at a contradiction. Notice that it is sufficient to consider a trivial strategy $\alpha \in A(t, x, y)$; by Remark 2.1, α may be identified with a trajectory $x(\cdot) \in X(t, x)$. We thus get a point $(x, y, r) \in W(t)$, a trajectory $\bar{x}(\cdot) \in X(t, x)$, and a number $\delta > 0$ with the property that

$$\bar{x}(t+\delta), y(t+\delta), r(t+\delta)) \notin W(t+\delta), \qquad y(\cdot) \in Y(t, y),$$

that is to say, $\underline{W}(t+\delta, \bar{x}(t+\delta), y(t+\delta)) + r(t+\delta) \geqslant \underline{W}(t_*, x_*, y_*) + \varepsilon_{y(\cdot)}$ for some $\varepsilon_{y(\cdot)}$, where

$$r(t+\delta) = r + \int_t^{t+\delta} h(s, \bar{x}(s), y(s))\, ds. \quad (3.2)$$

In addition, $(t+\bar{\delta}, \bar{x}(t+\bar{\delta}), y(t+\bar{\delta}) \notin M$ for each $y(\cdot) \in Y(t, y)$ and $0 \leqslant \bar{\delta} \leqslant \delta$. Since $\underline{W}(t, x, y)$ is l.s.c. in y, there must exist for each $\bar{y}(\cdot) \in Y(t, y)$ a neighbourhood $E_{\bar{y}(\cdot)}$ of $\bar{y}(t+\delta)$ such that $\underline{W}(t+\delta, \bar{x}(t+\delta), y(t+\delta)) + r(t+\delta) \geqslant \underline{W}(t_*, x_*, y_*) + \frac{1}{2} \cdot \varepsilon_{\bar{y}(\cdot)}$ for all $y(\cdot)$ satisfying $y(t+\delta) \in E_{\bar{y}(\cdot)}$. The set $K = \{y: y = (t+\delta), y(\cdot) \in Y(t, y)\}$ is compact because the space of all solutions of (1.2) considered on the segment $[t, t+\delta]$ is compact in the Banach space of continuous mappings equipped with the sup norm [4]. Therefore, one can find finitely many open sets, say $E_{y_1(\cdot)}, ..., E_{y_m(\cdot)}$, which cover K. In this way, we have shown that, for each $y(\cdot) \in Y(t, y)$, we have $\underline{W}(t+\delta, \bar{x}(t+\delta), y(t+\delta)) \geqslant \underline{W}(t_*, x_*, y_*) + \bar{\varepsilon}$, $\bar{\varepsilon} = \min\{\varepsilon_{y_1(\cdot)}, ..., \varepsilon_{y_m(\cdot)}\}$. By the definition of the lower value, there exist lower strategies $\alpha_{y(t+\delta)} \in A(t+\delta, \bar{x}(t+\delta), y(t+\delta))$ with

$$P^+(\alpha_{y(t+\delta)}) + r(t+\delta) \geqslant \underline{W}(t_*, x_*, y_*) + \tfrac{1}{2}\bar{\varepsilon}. \quad (3.3)$$

Now, it is easy to define a lower strategy $\alpha_0 \in A(t, x, y)$ satisfying

$$P^+(\alpha_0) + r \geqslant \underline{W}(t_*, x_*, y_*) + \tfrac{1}{2} \cdot \bar{\varepsilon} \quad (3.4)$$

which yields $\underline{W}(t, x, y)+r \geqslant \underline{W}(t_*, x_*, y_*)+\frac{1}{2}\bar{\varepsilon}$, a contradiction to the assumption $(x, y, r) \in W(t)$. To this end, for each $y(\cdot) \in Y(t, y)$, denote by $y^\delta(\cdot)$ the restriction of $y(\cdot)$ to $[t+\delta, \infty)$ and set

$$\alpha_0(y(\cdot))(t)=\begin{cases}\bar{x}(s), & t \leqslant s \leqslant t+\delta \\ \alpha_{y(t+\delta)}(y^\delta(\cdot))(s), & t+\delta \leqslant s<\infty.\end{cases}$$

It is obvious that (3.4) holds. ∎

PROPOSITION 3.1. *If $(x, y_n, r_n) \in W(t)$, $n=1, 2, \ldots$, with $t_* \leqslant t<\infty$, and the sequence (y_n, r_n) converges to (y, r), then $(x, y, r) \in W(t)$.*

Proof. Since $y_n \in N_2(t)$, $n=1, 2, \ldots$, y must also belong to $N_2(t)$. Therefore, assuming that $(x, y, r) \notin W(t)$, we obtain $\underline{W}(t, x, y)+r> \underline{W}(t_*, x_*, y_*)+\varepsilon$, for some $\varepsilon>0$. Making use of the lower semicontinuity of $\underline{W}(t, x, y)$ with respect to y, we get $\underline{W}(t, x, y')+r>\underline{W}(t_*, x_*, y_*)+\frac{1}{2}\varepsilon$ for y' belonging to some neighbourhood E of y, which contradicts the assumption $(x, y_n, r_n) \in W(t)$ because the latter is equivalent to $\underline{W}(t, x, y_n)+r_n \leqslant \underline{W}(t_*, x_*, y_*)$. ∎

THEOREM 3.1 (the optimality principle of dynamic programming for player I in a differential game). *If $(t_*, x_*, y_*) \notin M$, then there exists a number $\bar{\delta}$ such that, for each $0 \leqslant \delta \leqslant \bar{\delta}$, we have*

$$\sup_{\alpha \in A(t_*, x_*, y_*)} \min_{\bar{y}(\cdot) \in Y(t_*, y_*)}\left[\underline{W}(t_*+\delta, \alpha(\bar{y}(\cdot))(t_*+\delta), \bar{y}(t_*+\delta)) + \int_{t_*}^{t_*+\delta} h(s, \alpha(\bar{y}(\cdot))(s), \bar{y}(s))\, ds\right]=\underline{W}(t_*, x_*, y_*). \tag{3.5}$$

Proof. Since the terminal set M is closed and $X(t_*, x_*)$ as well as $Y(t_*, y_*)$ is a set of equibounded functions on each finite interval [5, Corollary 2.1], one can find a number $\bar{\delta}>0$ such that, for all $t_* \leqslant t \leqslant t_*+\bar{\delta}$, $x(\cdot) \in X(t_*, x_*)$, $y(\cdot) \in Y(t_*, y_*)$, we have $(t, x(t), y(t)) \notin M$. By invoking Lemma 3.1 with $(t, x, y)=(t_*, x_*, y_*)$, $r=0$, we obtain that (cf. (3.1))

$$\sup_{\alpha \in A(t_*, x_*, y_*)} \min_{\bar{y}(\cdot) \in Y(t_*, y_*)} \underline{W}(t_*+\delta, \alpha(\bar{y}(\cdot))(t_*+\delta), \bar{y}(t_*+\delta)) + r_\alpha(t_*+\delta) \leqslant \underline{W}(t_*, x_*, y_*). \tag{3.6}$$

Let $X_\varepsilon(t_*, x_*)$, $Y_\varepsilon(t_*, y_*)$ stand for the restrictions of $X(t_*, x_*)$, $Y(t_*, y_*)$ to $[t_*, t_*+\varepsilon]$. Denote by $x_\varepsilon(\cdot)$, $y_\varepsilon(\cdot)$ generic elements of $X_\varepsilon(t_*, x_*)$, $Y_\varepsilon(t_*, y_*)$, respectively, and by $y^\varepsilon(\cdot)$ the restriction of $y(\cdot) \in Y(t_*, y_*)$ to $[t_*+\varepsilon, \infty)$. Thus $y(\cdot)$ can be written as the pair of functions

$(y_\varepsilon(\cdot), y^\varepsilon(\cdot))$. Notice that each strategy $\alpha \in A(t_*, x_*, y_*)$ and each $\varepsilon > 0$ uniquely determine the strategy α_ε: $Y_\varepsilon(t_*, y_*) \to X_\varepsilon(t_*, x_*)$ from the equation $\alpha_\varepsilon(y(\cdot))(t) = \alpha(y(\cdot))(t)$, $t_* \leqslant t \leqslant t_* + \varepsilon$. They also determine the family of strategies $\alpha^\varepsilon_{\bar{y}(\cdot)} \in A(t_* + \varepsilon, \alpha(\bar{y}(\cdot))(t_* + \varepsilon), \bar{y}(t_* + \varepsilon))$ according to the formulae

$$\alpha^\varepsilon_{\bar{y}(\cdot)}(y^\varepsilon(\cdot))(t) = \alpha[y(\cdot)](t), \qquad t_* + \varepsilon \leqslant t < \infty$$

for $y(\cdot)$ satisfying $y_\varepsilon(\cdot) = \bar{y}_\varepsilon(\cdot)$. Observe that $\alpha^\varepsilon_{\bar{y}(\cdot)}$ usually means something different than $\alpha^\varepsilon_{\bar{\bar{y}}(\cdot)}$ even if $\bar{y}(t_* + \varepsilon) = \bar{\bar{y}}(t_* + \varepsilon)$ and, in addition, $\alpha(\bar{y}(\cdot))(t_* + \varepsilon) = \alpha(\bar{\bar{y}}(\cdot))(t_* + \varepsilon)$. To prove (3.5), observe that

$$\begin{aligned}\underline{W}(t_*, x_*, y_*) = &\sup_{\alpha_\varepsilon} \sup_{\alpha^\varepsilon_{y(\cdot)}} \min_{\bar{y}_\varepsilon(\cdot)} \min_{\bar{y}^\varepsilon(\cdot)} \\ &\times \{P[t_*, x_*, y_*; (\alpha_\varepsilon, \alpha^\varepsilon_{y(\cdot)})(\bar{y}_\varepsilon(\cdot), \bar{y}^\varepsilon(\cdot)), \bar{y}(\cdot)]\},\end{aligned}$$

where $y(\cdot)$ ranges over $Y(t_*, y_*)$ and $\bar{y}_\varepsilon(\cdot)$ (resp. $\bar{y}^\varepsilon(\cdot)$) ranges over $Y_\varepsilon(t_*, y_*)$ (resp. $Y(t_* + \varepsilon, \bar{y}(t_* + \varepsilon))$. This means that the strategies $\alpha^\varepsilon_{y(\cdot)}$ range over the spaces $A(t_* + \varepsilon, \alpha_\varepsilon(y_\varepsilon(\cdot))(t_* + \varepsilon), y_\varepsilon(t_* + \varepsilon))$. The number in the right hand side of the last equality (henceforth denoted by C) is the best result which can be obtained by player I assuming that he announces his choice of a strategy α at the beginning of the game (at time t_*). Therefore $C \leqslant D$, where

$$D = \sup_{\alpha_\varepsilon} \min_{\bar{y}_\varepsilon(\cdot)} \sup_{\alpha^\varepsilon_{y(\cdot)}} \min_{\bar{y}^\varepsilon(\cdot)} \{P[t_*, x_*, y_*; (\alpha_\varepsilon, \alpha^\varepsilon_{y(\cdot)})(\bar{y}_\varepsilon(\cdot), \bar{y}^\varepsilon(\cdot)), \bar{y}(\cdot)]\} \tag{3.7}$$

is the best result which can be obtained by player I under the following information pattern: at t_* player I announces α_ε, i.e., the "first part" of his strategy $\alpha = (\alpha_\varepsilon, \alpha^\varepsilon_{y(\cdot)})$, where $y(\cdot)$ ranges over $Y(t_*, y_*)$ and next (between t_* and $t_* + \varepsilon$) player II announces the path $\bar{y}(\cdot)$ he will be moving along. At $t_* + \varepsilon$ player I, knowing $\bar{y}(\cdot)$, announces $\alpha^\varepsilon_{\bar{y}(\cdot)}$, i.e., the "second part" of his strategy. The equality $C = D$ means that player I will gain nothing if player II announces his choice $\bar{y}_\varepsilon(\cdot)$ at any time between t_* and $t_* + \varepsilon$ including $t_* + \varepsilon$. In (3.7), the sup over all $\alpha^\varepsilon_y(\cdot)$ may be replaced by the sup over all $\alpha^\varepsilon_{\bar{y}(\cdot)}$ because before choosing $\alpha^\varepsilon_{y(\cdot)}$ player I knows $\bar{y}_\varepsilon(\cdot)$, the choice of his opponent on $[t_*, t_* + \varepsilon]$. Note that, for fixed α_ε, and $\bar{y}_\varepsilon(\cdot)$, we have

$$\begin{aligned}&\sup_{\alpha^\varepsilon_{y(\cdot)}} \min_{\bar{y}^\varepsilon(\cdot)} P[t_*, x_*, y_*; (\alpha_\varepsilon, \alpha^\varepsilon_{\bar{y}(\cdot)})(\bar{y}(\cdot)), \bar{y}(\cdot)] \\ &\quad = \int_{t_*}^{t_* + \varepsilon} h(s, \alpha_\varepsilon(\bar{y}(\cdot))(s), \bar{y}(s))\, ds \\ &\quad\quad + \underline{W}(t_* + \varepsilon, \alpha_\varepsilon(\bar{y}(\cdot))(t_* + \varepsilon), \bar{y}_\varepsilon(t_* + \varepsilon))\end{aligned}$$

because the sup over $\alpha^{\varepsilon}_{\bar{y}(\cdot)}$ and the min over $\bar{y}^{\varepsilon}(\cdot)$ do not refer to the first integral in the right hand side of the last equality. Taking infimum over $\bar{y}_{\varepsilon}(\cdot)$ and supremum over α_{ε}, one sees that the number in the left hand side of (3.5) equals D. Thus (3.5) is equivalent to the equality $C = D$. The proof is thus completed since $C \leqslant D$ and (3.6) states $D \leqslant C$. ∎

Since the set of all outcomes resulting from a given strategy $\alpha \in A(t_*, x_*, y_*)$ and all $y(\cdot) \in Y(t_*, y_*)$ is the same as the set of all outcomes resulting from α and all strategies $\beta \in B(t_*, x_*, y_*)$ (cf. Remark 2.1), we arrive at

$$\sup_{\alpha \in A(t_*, x_*, y_*)} \min_{\beta \in B(t_*, x_*, y_*)} \underline{W}(t_* + \delta, \alpha(\beta)(t_* + \delta), \beta(\alpha)(t_* + \delta)) + \int_{t_*}^{t_* + \delta} h(s, \alpha(\beta)(s), \beta(\alpha)s)\, ds = W_*,$$

where $W_* = \underline{W}(t_*, x_*, y_*)$ and $\alpha(\beta)$, $\beta(\alpha)$ stand for $x^*(\cdot)$, $y^*(\cdot)$, resp. with $[x^*(\cdot), y^*(\cdot)]$ being the unique outcome resulting from (α, β). According to the notation employed in the proof of Theorem 3.1, it would be more natural to replace α and β in the last equality by α_{ε} and β_{ε}, respectively. Using Theorem 3.1 once again, one can easily prove the equality

$$\sup_{\alpha_{\varepsilon_1}} \min_{\beta_{\varepsilon_1}} \sup_{\alpha_{\varepsilon_2}} \min_{\beta_{\varepsilon_2}} \underline{W}(t_* + \varepsilon_1 + \varepsilon_2, \alpha(\beta)(t_* + \varepsilon_1 + \varepsilon_2), \beta(\alpha)(t_* + \varepsilon_1 + \varepsilon_2)) + \int_{t_*}^{t_* + \varepsilon_1 + \varepsilon_2} h(s, \alpha(\beta)(s), \beta(\alpha)(s)\, ds = \underline{W}(t_*, x_*, y_*),$$

with an appropriate interpretation of $\alpha = (\alpha_{\varepsilon_1}, \alpha_{\varepsilon_2}, \alpha^{\varepsilon_1 + \varepsilon_2})$, $\beta = (\beta_{\varepsilon_1}, \beta_{\varepsilon_2}, \beta^{\varepsilon_1 + \varepsilon_2})$. Thus, player I will gain nothing if player II announces β_{ε_1} between t_* and $t_* + \varepsilon_1$ including $t_* + \varepsilon_1$ and β_{ε_2} between $t_* + \varepsilon_1$ and $t_* + \varepsilon_1 + \varepsilon_2$, including $t_* + \varepsilon_1 + \varepsilon_2$. We can, clearly, generalize this remark for any natural number $n \geqslant 3$.

Arguing as in Theorem 3.1 with (A5) replaced by (A6) one can prove Theorem 3.2.

THEOREM 3.2 (the optimality principle of dynamic programming for player II in a differential game). *If $(t_*, x_*, y_*) \notin M$, then there exists a number $\bar{\delta}$ such that, for each $0 \leqslant \delta \leqslant \bar{\delta}$,*

$$\inf_{\beta \in B(t_*, x_*, y_*)} \max_{\bar{x}(\cdot) \in X(t_*, x_*)} (t_* + \delta, \bar{x}(t_* + \delta), \beta(\bar{x}(\cdot))(t_* + \delta)) + \int_{t_*}^{t_* + \delta} h(s, \bar{x}(s), \beta(\bar{x}(\cdot))(s))\, ds = \bar{W}(t_*, x_*, y_*).$$

Remarks similar to those following Theorem 3.1 apply here.

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