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On the Expressiveness of Higher Dimensional Automata

(Extended Abstract)

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Abstract

In this paper I compare the expressive power of several models of concurrency based on their ability to represent causal dependence. To this end, I translate these models, in behaviour preserving ways, into the model of higher dimensional automata, which is the most expressive model under investigation. In particular, I propose four different translations of Petri nets, corresponding to the four different computational interpretations of nets found in the literature.
I also extend various equivalence relations for concurrent systems to higher dimensional automata. These include the history preserving bisimulation, which is the coarsest equivalence that fully respects branching time, causality and their interplay, as well as the ST-bisimulation, a branching time respecting equivalence that takes causality into account to the extent that it is expressible by actions overlapping in time. Through their embeddings in higher dimensional automata, it is now well-defined whether members of different models of concurrency are equivalent.

Keywords: Concurrency, expressiveness, causality, higher dimensional automata, Petri nets, event structures, history preserving bisimulation, ST-bisimulation.

1 A hierarchy of concurrency models

Figure 1 lists the main models of concurrency proposed in the literature, ordered by expressive power. Of all models, the labelled variant is understood.

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$$\begin{tikzcd}
 & \textit{higher dimensional automata } [8] & & \\
\textit{automata } [9] \arrow[ur] & & \textit{Petri nets} \arrow[ul] & \\
 & \textit{configuration structures } [11] \arrow[ul] \arrow[ur] \arrow[r, leftrightarrow] & \textit{pure Petri nets} & \\
 & \textit{Winskel's event structures } [28,29] \arrow[u] & & \\
 & \textit{prime event structures } [28,29] \arrow[u] \arrow[r, leftrightarrow] & \left\{\begin{array}{l}\textit{bundle e.s. } [16] \\ \textit{flow e.s. } [5] \\ \textit{stable e.s. } [28,29] \\ \textit{safe Petri nets}\end{array}\right. & \\
 & \textit{synchronisation trees } [18] \arrow[u] & &
\end{tikzcd}$$

Fig. 1. A hierarchy of concurrency models, ordered by expressive power up to history preserving bisimulation

Here I only treat models of processes that perform actions a, b, c, . . . whose internal structure is not further examined, and real-time and stochastic aspects of processes are completely ignored. Furthermore, I only study the representation of *processes*, not the representation of *operators* on processes. I restrict myself to models that take branching time fully into account, and hence skip models that represent processes by the sets of their executions. Thus, the expressive power of the models differs only to the extent that certain forms of causal dependence are expressible. I limit myself to models that take a fully *asynchronous* view on parallelism: whenever a number of actions can happen simultaneously, this must be because they are causally independent, and hence they can also happen in any order. Because of this, I do not include Petri nets with inhibitor arcs, or Chu spaces [23].

There is an arrow from model A to model B in Figure 1 iff there exists a translation from processes representable in model A to processes representable in model B that fully respects branching time, causality, and their interplay. Thus a process and its translation ought to be *history preserving bisimulation equivalent* [24,10].

Part of the contribution of this paper is a definition of what this means. Forms of history preserving bisimulation were defined on behaviour structures in [24], on stable event structures in [10], and on Petri nets under the individual token interpretation in [4]; however, it has never been formally defined what it means for an event structure, for instance, to be history preserving bisimulation equivalent to a Petri net.

In Section 2 I introduce the model of higher dimensional automata. Then, in Sections 3, 4 and 5, I define behaviour preserving translations from the other models of Figure 1 into this model; for the model of Petri nets I do this in four different ways, corresponding to the four different computational interpretations of nets found in the literature. In Section 7 I define history preserving bisimulation equivalence on higher dimensional automata. The embeddings of the other models of concurrency into higher dimensional automata make this definition apply to processes representable in any of these models: two processes are equivalent iff their representations as higher dimensional automata are. Naturally, I have to ensure that the new definition agrees with the existing ones on the models where it has already been defined. Demonstrating this is deferred to the full version of this paper.

With this tool in hand, the hierarchy of Figure 1 is justified in Section 8. In particular, counterexamples will be presented to illustrate the strictness of the expressiveness ordering.

Besides history preserving bisimulation equivalence, I also define *interleaving bisimulation equivalence* [18,10] on higher dimensional automata, and thus on the other models, as well as *ST-bisimulation equivalence* [13], a branching time respecting semantic equivalence that takes causality into account to the extent that it can be expressed by the possibility of durational actions to overlap in time. If I compare the models merely up to interleaving bisimulation equivalence they turn out to be all equally expressive. If I compare them up to ST-bisimulation equivalence, I conjecture that just two equivalence classes of models remain: the models that do not take causality into account—*in Figure 1 just the model of synchronisation trees*—and the models that do—*in Figure 1 all other models*. It follows that the more interesting hierarchy of Figure 1 is due to causal subtleties that evaporate when considering processes up to ST-bisimulation equivalence.

2 Higher Dimensional Automata

One of the most commonly used models of concurrency is that of *automata*, also known as *process graphs*, *state transition diagrams* or *labelled transition systems*. In ordinary automata, the parallel composition P of two actions a and b is displayed in the same way as a system M that executes a and b in either order, ending in the same state each way, such that a and b are mutually exclusive (see Figure 2). Nevertheless, it is often important to tell these systems apart. This happens for instance when a and b take time: the total running time of M is at least the sum of the running times of a and b, whereas P may be as quick as the maximum of the running times of a and b. Another occasion where it is essential to distinguish between P and M is when designing systems using *action refinement*, as described in [10]. In many other models of concurrency P and M are represented distinctly. For the model of Petri nets, this is illustrated in Figure 2.

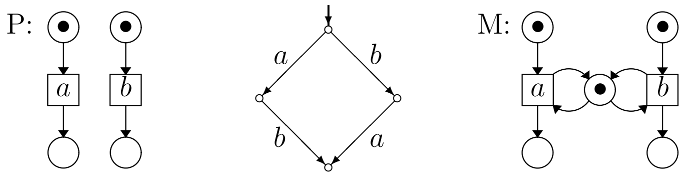


Fig. 2. The automaton in the middle represents both *parallelism*, as does the Petri net P, and *mutual exclusion*, as does the Petri net M.

Throughout the years, people have wondered whether the elegance of automata could be combined with the expressiveness of models like Petri nets or event structures, that are able to capture causal relationships between action occurrences. These relationships include the causal independence of a and b in P, the dependence of b on a in the left branch of the automaton representing M, and the dependence of a on b in the right branch. As a result, several models of concurrency have been proposed that are essentially automata, upgraded with some extra structure to express causal independence [26,2,24,27,30].

In [6] it was pointed out that ordinary automata, without such extra structure, are already sufficiently expressive to capture these causal relationships. All that is needed is a reading of automata that assumes squares to represent concurrency, and nonconfluent branching to represent a choice or conflict. Under this *concurrent interpretation* the automaton of Figure 2 represents parallelism only, whereas mutual exclusion is represented by the automaton of Figure 3. The square of Figure 2 is now seen as the *product* of the transi-

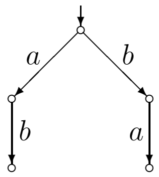


Fig. 3. Representation of mutual exclusion in ordinary automata under the concurrent interpretation

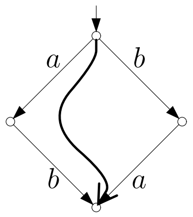


Fig. 4. Travelling through the inside of a square

tions a and b that form its sides. As each of these two transitions has 3 cells, a *start node*, an *end node*, and the *edge* between them, their product has 9 cells, namely the 4 nodes and 4 edges displayed in Figure 2, and the *inside of the square*. The latter represents the concurrent execution of a and b.

Modelling concurrency by means of squares (or cubes, hypercubes etc. in case of three or more concurrent actions) is particularly useful when actions are thought to have a duration or structure. A concurrent execution of a and b in the process $a\|b$ can then be thought of as a continuous path through the surface of the square, starting at the top and terminating at the bottom node, while being nondecreasing when projected on any edge. The execution displayed in Figure 4 for instance passes through a point in which 50% of a and about 17% of b has happened. As long as we want to identify all such paths (when abstracting from timing information and structural knowledge of a and b) we simply represent their equivalence class as the square. In [14] this representation of concurrent systems turned out to be essential in finding and explaining an essential counterexample in the study of semantic equivalences.

The price to pay for this simple solution is that it is no longer possible to represent the system M of Figure 2 in such a way that both executions ab and ba end up in the same state. In order to overcome this deficiency, and to distinguish *choice*, as displayed in Figure 3, from *mutual exclusion*, as in M in Figure 2, Vaughan Pratt [22] preferred for every square, cube, hypercube etc. to indicate explicitly whether it represents concurrency or not. An n-dimensional hypercube that represents the concurrent execution of the n transitions that span its sides is thought of as being "filled in", which is represented by an additional n-dimensional cell in the automaton. Those that represent mutual exclusion stay empty. This gives rise to what he baptised *higher dimensional automata*.

A totally sequential implementation of a system, in which no two actions happen in parallel, can be represented by an ordinary automaton, which is a higher dimensional automaton in which there are no higher dimensional cells, so no squares are filled in. At the other side of the spectrum, the very same ordinary automata under the concurrent interpretation of [6] form the special case of higher dimensional automata in which *every* (nondegenerate) square, cube or hypercube is filled in. In between, Pratt's higher dimensional automata can model systems that feature both parallelism and mutual exclusion, without having to resort to untying the join of two branches that surround an area of mutual exclusion.

The idea of higher dimensional automata above had to some extent been contemplated before in [26] and applied in [21]. However, [22] offered the first formalisation of the idea. Pratt's formalisation, based on n-categories, takes a *globular* or *hemispherical* approach, in which an n-cell has only two boundaries of dimension $n-1$. Desperate attempts by me to fully grok the globular approach led to an exchange with Pratt in December 1990 that gave rise to the joint conclusion that a *cubical approach*, in which an n-cell has $2n$ boundaries of dimension $n-1$, would be preferable (and easier to understand). A *higher dimensional automaton* (HDA) was henceforth defined as a presheaf over $\square$, the category of *cubical complexes.*[3] At the occasion of proposing notions of *bisimulation equivalence*, *homotopy* and *unfolding* for higher dimensional automata in [8], I reworded this definition in set-theoretical terms as follows.

Definition 2.1 A *cubical set* consists of a family of sets $(Q_n)_{n\geq 0}$ and for every $n\in\mathbb{N}$ a family of maps $s_i, t_i : Q_n \rightarrow Q_{n-1}$ for $1 \leq i \leq n$, such that

$$\alpha_i \circ \beta_j = \beta_{j-1} \circ \alpha_i \qquad \text{for all } 1 \leq i < j \leq n \text{ and } \alpha, \beta \in \{s,t\}. \qquad (*)$$

A *higher dimensional automaton*, labelled over an alphabet A, is a tuple (Q, s, t, I, F, l) with (Q, s, t) a cubical set, $I \in Q_0$, $F \subseteq Q_0$ and $l : Q_1 \rightarrow A$, such that $l(s_i(q)) = l(t_i(q))$ for all $q \in Q_2$ and $i = 1, 2$.

The elements of Q_0 are called *nodes* and those of Q_1, Q_2 and Q_3 are *edges*, *squares* and *cubes*, respectively. In general, the elements of Q_n are called *n-dimensional hypercubes*, or *n-cells*. An n-dimensional hypercube represents a state of a concurrent system in which n transitions are firing concurrently. Because the dimensions of the hypercube are numbered $1,\ldots,n$, these n transitions are *de facto* stored as a list.

An edge q connects 2 nodes: its *source* $s_1(q)$ and its *target* $t_1(q)$. Likewise, an n-dimensional hypercube q has n sources $s_i(q)$ and n targets $t_i(q)$, one in each dimension $i = 1, ..., n$, and each being an $(n-1)$-dimensional hypercube. The source $s_i(q)$ represents the possible state prior to q in which $n-1$ out of n transitions are already firing, but the i^{th} one has not yet started. Likewise, $t_i(q)$ represents the possible state past q in which $n-1$ out of n transitions are still firing, but the i^{th} one has terminated.

When removing the i^{th} transition out of a list of n transitions, an implicit renumbering takes place, and what was formerly the j^{th} transition, for $j > i$, is now called the $(j-1)^{th}$ transition. Hence, first leaving out the j^{th} and then the i^{th}, with $i < j$, leaves us with the same list as first removing the i^{th} and

[3] The objects of $\square$ are the symbols 0,1,2,... denoting the hypercubes of each dimension, and its morphisms from k to n are the embeddings of the k-dimensional hypercube as a k-dimensional face of the n-dimensional hypercube. All dimensions are directed and the morphisms preserve this direction. Thus, there are for example 6 morphisms from 2 to 3, corresponding to the fact that a cube has 6 sides. A *presheaf* over a category $\mathbf{C}$ is a functor $F : \mathbf{C}^{\rm op} \rightarrow \mathbf{Set}$. Thus a presheaf F over $\square$ associates a set $F(n)$ to every object n in $\square$. $F(n)$ is thought of as the set of n-dimensional hypercubes in the HDA. Also, for every morphism $m : k \rightarrow n$, recognising the k-dimensional hypercube as a face of the n-dimensional one, there exists a function $F(m) : F(n) \rightarrow F(k)$, giving for every n-dimensional hypercube its k-dimensional face on the side indicated by m. These functions must satisfy exactly those composition laws that hold for the morphisms in $\square$. The advantage of the categorical approach is that the concept of a HDA is thus completely defined without the need for figuring out these laws.

then the $(j-1)^{th}$ transition. This is the content of the *cubical laws* (*).

I is the initial state, and F the final states of the represented system. These states are required to have dimension 0, meaning that no transition is currently firing. An edge $q \in Q_1$ represents a state where exactly 1 transition is firing, and l indicates the label of that transition. It is required that opposite sides of a square have the same label. This because they represent the same transition, scheduled before and after the firing of another one. The labelling function can trivially be extended to a labelling of arbitrary n-dimensional hypercubes by lists of n actions (cf. Section 6.2).

Based on this definition, and the computational motivation of Pratt [22], numerous papers on higher dimensional automata have emerged [31–64].

A semantics of CCS in terms of higher dimensional automata is provided in Goubault & Jensen [54], and also studied by Lanzmann [57]. In Goubault [47], higher dimensional automata are used as a semantic domain for richer languages, and to compute local invariants which can decide a few computational properties of interest. Cridlig & Goubault [36] provide a semantics of Linda in terms of higher dimensional automata. Applications of higher dimensional automata to scheduling problems and wait-free protocols for distributed systems are studied by Goubault [48,49,51], and model checking applications by Cridlig [34,35]. Algorithms for deadlock detection in terms of higher dimensional automata appear in Fajstrup et al. [38,37]. Takayama [62,63,64] studies parallelisation algorithms by means of higher dimensional automata. Extensions of higher dimensional automata with time are investigated by Goubault [48,50].

In Goubault [53] the relations between 1-dimensional automata, asynchronous transition systems [26,2], and higher dimensional automata are cast in a categorical framework, following the work of Winskel & Nielsen [30]. To this end it appears to be fruitful to equip the cubical sets of Definition 2.1 with degeneracy mappings, so that they become exactly the cubical sets studied before in algebraic topology [25].

Homotopy for higher dimensional automata is defined in Pratt [22] and Van Glabbeek [8], and further investigated by Goubault [48] and Raussen [60]. Gunawardena [56] uses homotopy theory to show that 2-phase locking is safe. Goubault & Raussen [55] show, using homotopy theory, how geometric models of concurrency like higher dimensional automata are suitable for attacking the state-explosion problem.

Homology theories for higher dimensional automata are proposed by Goubault and Gaucher [54,48,39–46]. Gaucher [39–44], like Pratt [22], takes a globular approach to higher dimensional automata, and relates it to the cubical approach. Variations and generalisations of this approach are studied by

Gaucher & Goubault [46,45].

Buckland et al. [31,32] study a simplified version of globular higher dimensional automata, and equip it with process algebraic operations. A generalisation of higher dimensional automata suitable for modelling continuous rather than discrete processes has been proposed by Sokolowsky [61].

In Cattani & Sassone [33] a simplified form of higher dimensional automata is proposed, called *higher dimensional transition systems*. The simplification is payed for in expressive power though: not all higher dimensional automata that arise as the image of a Petri net under the self-concurrent collective token interpretation (see Section 5) exist as higher dimensional transition systems.

In [58,59], Pratt argues that acyclic higher dimensional automata can be seen as subsets of a single infinite dimensional cube, which in turn can be modelled as Chu spaces over 3.

An overview of work related to higher dimensional automata and the use of other geometric methods in computer science is presented in Goubault [52].

3 Embedding ordinary automata in HDA

A (1-dimensional) automaton is a tuple $(Q_0, Q_1, s_1, t_1, I, F, l)$, the special case of higher dimensional automaton in which $Q_n = \emptyset$ for $n > 1$. Often automata are required to be *extensional*, meaning that a transition is completely determined by its source, target and label:

$$s_1(q) = s_1(q') \wedge t_1(q) = t_1(q') \wedge l(q) = l(q') \Rightarrow q = q' \qquad \text{for } q, q' \in Q_1.$$

In that case, a transition $q \in Q_1$ can be named after the triple $(s_1(q), l(q), t_1(q))$, and, writing S for the set of states Q_0, the quadruple (Q_1, s_1, t_1, l) can be conveniently represented by a relation $T \subseteq S \times A \times S$, thereby contracting the definition of an automaton to a quadruple (S, T, I, F).

A straightforward embedding of ordinary automata in higher dimensional automata is given by recognising them as HDA in which $Q_n = \emptyset$ for $n > 1$. However, the concurrent interpretation of automata from [6], elaborated in [9], yields a more expressive model of concurrency. Here an extensional 1-dimensional automaton $\mathrm{G} = (S, T, I, F)$ is, in essence, seen as an abbreviation of a higher dimensional automaton $\mathcal{A}(\mathrm{G})$ by assuming that any nondegenerate n-dimensional hypercube that can be recognised in G is "filled in", i.e. constitutes an n-dimensional cell in $\mathcal{A}(\mathrm{G})$. In [6,9] this was merely a computational interpretation of automata; here I use it to formally define $\mathcal{A}(\mathrm{G})$. An n-dimensional hypercube in G consists of a string $\ell = a_1 \cdots a_n \in A^n$ of n action labels and 2^n corners $p_\xi \in S$, with $\xi \in \{0,1\}^n$ ranging over the strings of n 0s and 1s, such that $(p_\xi, a_i, p_\chi) \in T$ whenever ξ and χ differ only on their i^{th} bit and that bit is 0 in ξ and 1 in χ. The source $s_i(q)$ (resp. target $t_i(q)$) in dimension i of such an n-dimensional hypercube q in G is the $(n-1)$-dimensional hypercube in G obtained by deleting a_i from ℓ and restricting the set of corners p_ξ of q to those in which the i^{th} bit of ξ is 0 (resp. 1).

A hypercube $q = (\ell, p_\xi\ (\xi \in \{0,1\}^n))$ in G is *degenerate* iff there are indices $1 \leq i < j \leq n$ such that $a_i = a_j$ in ℓ and, for certain bits $b_k \in \{0,1\}$ $(k \neq i,j)$,

$$p_{b_1 \cdots b_{i-1} 0 b_{i+1} \cdots b_{j-1} 1 b_{j+1} \cdots b_n} = p_{b_1 \cdots b_{i-1} 1 b_{i+1} \cdots b_{j-1} 0 b_{j+1} \cdots b_n}.$$

In particular, a degenerate square consists of two transitions (p,a,q) and (q,a,r). It is a square by taking $\ell = aa$ and the corners $p_{00} = p$, $p_{01} = p_{10} = q$ and $p_{11} = r$. The reason for not assuming this square to be filled in, is that if I do, I loose the expressive power to specify a sequence of two identical actions that have to occur in sequential order. Hence the property that at least all synchronisation trees are representable as automata under the concurrent interpretation would be lost. In general, a hypercube in G is nondegenerate iff all its 2-dimensional faces are nondegenerate. So the definition of degeneracy above is the most stringent one I could get away with.

4 Embedding event oriented models in HDA

Winskel [28,29] introduced six kinds of event structures: the *prime*, *stable* and [general] *event structures*, each optionally with the restriction of *binary conflict*. The behaviour of these event structures is fully specified by the *families of configurations* that can be associated to them; moreover, the family of all configurations of an event structure is fully determined by the finite configurations in the family. Hence, event structures embed faithfully in the model of *configuration structures* of [11], which generalises the families of configurations of event structures.

In Figure 5, taken from [11], the six models of event structures from [28,29] are ordered with respect to their expressive power as measured by the class of configuration structures they can denote. In addition, the figure includes the *flow event structures* of Boudol [5], and the *bundle event structures* of Langerak [16]. The *synchronisation trees* of Milner [18], which are just tree shaped (1-dimensional) automata, can be seen as special kinds of prime event structures with binary conflict, and, naturally, the maximal expressive power is obtained by the model of all configuration structures.

In [9], the configuration structures are faithfully embedded in the model of ordinary automata under the concurrent interpretation of [6]. To this end, the concurrent interpretation of [6] was extended to cover *all* automata—in [6] it applied merely to automata of a certain shape: the ones arising as the

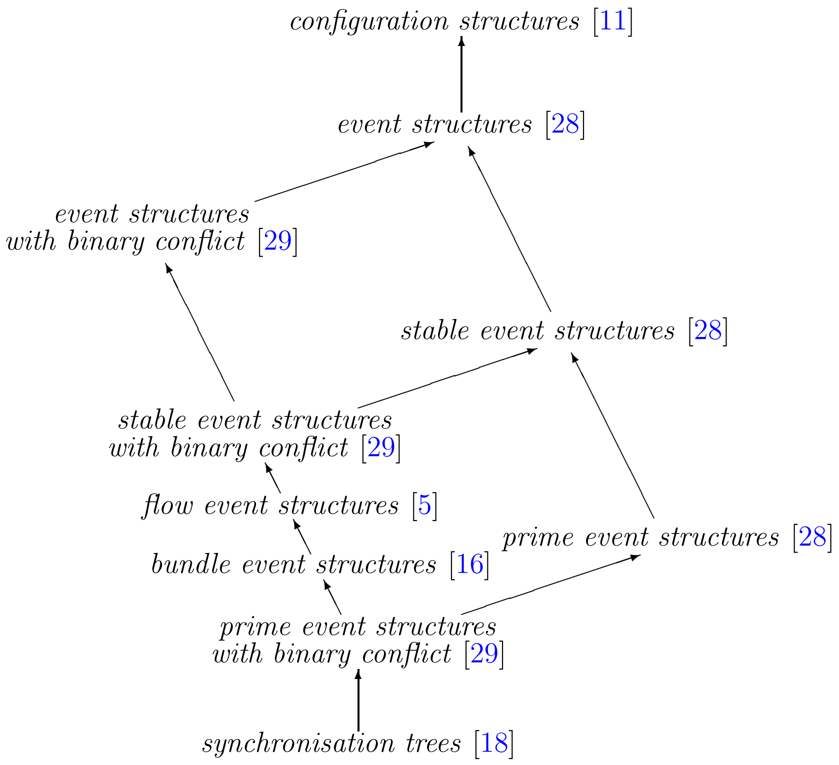

configuration structures [11]
event structures [28]
event structures with binary conflict [29]
stable event structures [28]
stable event structures with binary conflict [29]
flow event structures [5]
prime event structures [28]
bundle event structures [16]
prime event structures with binary conflict [29]
synchronisation trees [18]

Fig. 5. A hierarchy of event oriented models of concurrency

images of (labelled) prime event structures.

By composing the mappings of [11], embedding the models of Figure 5 in configuration structures, of [9], embedding configuration structures in automata under the concurrent interpretation, and of Section 3 above, embedding automata in HDA, all models of Figure 5 embed faithfully in HDA.

5 Embedding Petri nets in HDA

In this section I show that the model of higher dimensional automata is at least at expressive as the model of Petri nets, by giving translations from Petri nets to higher dimensional automata that capture exactly the dynamic behaviour of nets, as expressed by the *firing rule*, informally described below.

Definition 5.1 A (labelled) *Petri net* is a tuple (S, T, F, I, l)[4] with

- S and T two disjoint sets of *places* (*Stellen* in German) and *transitions*,

[4] The components of a net N are called S^{N}, T^{N}, F^{N}, I^{N} and l^{N}, a convention that also applies to other structures given as tuples. When clear from context, the index N is omitted.

- $F : (S \times T \cup T \times S) \rightarrow \mathbb{N}$, the *flow relation*,
- $I : S \rightarrow \mathbb{N}$, the *initial marking*,
- and $l : T \rightarrow A$, for A a set of *actions*, the *labelling function*.

Petri nets are pictured by drawing the places as circles and the transitions as boxes, containing their label. For $x,y \in S \cup T$ there are $F(s,t)$ *arcs* from x to y.

When a Petri net represents a concurrent system, a global state of this system is given as a *marking*, a function $M : S \rightarrow \mathbb{N}$. Such a state is depicted by placing $M(s)$ dots (*tokens*) in each place s. The initial state is given by the marking I. A transition t can *fire* (occur) by taking $F(s,t)$ tokens out of each place s, one for each arc from s to t. If this is possible in the state given by the marking $M : S \rightarrow \mathbb{N}$, i.e. if $F(s,t) < M(s)$ for all $s \in S$, one says that t is *enabled* under M. After t has fired, $F(t,s)$ tokens are added to each place s, one of each arc from t to s. For t a transition in a Petri net N, let ${}^\bullet t : S^{\mathrm{N}} \rightarrow \mathbb{N}$ be given by ${}^\bullet t(s) = F(s,t)$ and $t^\bullet : S^{\mathrm{N}} \rightarrow \mathbb{N}$ by $t^\bullet(s) = F(t.s)$. The multiset ${}^\bullet t$ describes which tokens are consumed by firing t, and $t^\bullet$ which tokens are produced.

In Van Glabbeek & Vaandrager [13], Petri nets were studied from the point of view that transitions may take time, and we introduced global states in which any number of transitions may be currently firing. The states represented by markings as defined above are those in which no transition is currently firing. In general, a state is given by a multiset of places *and transitions*. In order to precisely keep track of causal relationships between transition firings, we found it convenient to represent the (multi)set of transitions in a state as a list. This made it possible to distinguish different occurrences of currently firing transitions (i.e. the third and the fifth). Hence an *ST-marking* is defined as a pair $(M, \sigma) \in \mathcal{M}(S) \times T^*$ of a multiset of places and a list of transitions. This is exactly the notion of state that I need below in my translations of Petri nets to higher dimensional automata.

As pointed out in [11], there are $2 \times 2 = 4$ computational interpretations of Petri nets, called the *individual token* and the *collective token* interpretation, and, orthogonally, the *self-sequential* and the *self-concurrent* interpretation. In the individual token interpretation one distinguishes different tokens residing in the same place, keeping track of where they come from. If a transition fires by using a token that has been produced by another transition, there is a causal link between the two. Consequently, the causal relations between the transitions in a run of a net can always be described by means of a partial order. In the collective interpretation, on the other hand, tokens cannot be distinguished: if there are two tokens in a place, all that is present there is the number 2. This gives rise to more subtle causal relationships between transitions in a run of a net, which cannot be expressed by partial orders.

In the self-concurrent interpretation, a transition may fire concurrently with itself. This is not allowed in the self-sequential interpretation.

The below can be understood as a way of formally pinpointing the differences between these computational interpretations, by giving four translations from Petri nets into higher dimensional automata, one for each interpretation. In some sense this amounts to giving four different semantics of Petri nets.

5.1 The self-concurrent collective token interpretation

First I define the higher dimensional automaton $\mathcal{A}^{\rm CT}(\rm N)$ associated to a Petri net N according to the self-concurrent collective token interpretation of nets. As cells of $\mathcal{A}^{\rm CT}(\rm N)$ I take the ST-markings of N, each being a pair (M,σ) of a multiset M of places and a list σ of transitions in N. The number of transitions in the list is the dimension of the cell. The source $s_i(q)$ (resp. the target $t_i(q)$) in dimension i of a cell $q=(M,\sigma)$ is obtained by omitting the i^{th} transition t from σ, and adding the multiset of places ${}^\bullet t$ (resp. $t^\bullet$) to M. Below, $+$ denotes the union of multisets, given by $(M+X)(s)=M(s)+X(s)$. The empty list is denoted ε. As there is no standard definition of successful termination in Petri nets, I map them to HDA without final states; however, for any definition of successful termination of Petri nets found in the literature, most likely a corresponding definition of the final states of the associated HDA can be obtained.

Definition 5.2 Let N be a Petri net. The higher dimensional automaton $\mathcal{A}^{\rm CT}(\rm N)=(Q,s,t,I,F,l)$ is given by
- $Q_n=\mathcal{M}(S^{\rm N})\times(T^{\rm N})^n$ for $n\in\mathbb{N}$,
- $s_i(M,t_1\cdots t_n)=(M+{}^\bullet t_i,\ t_1\cdots t_{i-1}t_{i+1}\cdots t_n)$ for $1\leq i\leq n$,
- $t_i(M,t_1\cdots t_n)=(M+t_i^\bullet,\ t_1\cdots t_{i-1}t_{i+1}\cdots t_n)$ for $1\leq i\leq n$,
- $I=(I^{\rm N},\varepsilon)$ and $F=\emptyset$,
- $l(M,t)=l^{\rm N}(t)$ for $(M,t)\in Q_1$.

The source and target functions defined above correspond exactly to the start and termination of a transition firing as defined in clauses 2 and 4 of Definition 7.1.1 in [13]. They are also consistent with the informal description of the firing rule given above.

5.2 The self-concurrent individual token interpretation

Below I will define the notion of a token as it could occur in a Petri net, in such a way that all possible token occurrences have a different name. A token will be a triple (t',k,s), with s the place where the token occurs, and t' the transition firing that brought it there. For tokens that are in s initially, I take $t'=*$. When the number of tokens that t' deposits in s in n, I distinguish these tokens by giving them ordinal numbers $k = 0, 1, 2, ..., n-1$. In order to define tokens as announced above I need to define transition firings simultaneously. These will be pairs (X, t) with t the transition that fires, and X the set of tokens that is consumed in the firing. Transitions t that can fire without consuming tokens can fire multiple times on the same (empty) input; these firings will be called (k, t) with $k \in \mathbb{N}$ instead of $(\emptyset, t)$. I define the functions β from tokens to the places where they occur by $\beta(t', k, s) = s$, and η from transition firings to the transition that fires by $\eta(X, t) = t$. The function β extends to a function from sets of tokens X to multisets of places $\beta(X) : S \to \mathbb{N}$, by $\beta(X)(s) = |\{s' \in X \mid \beta(s') = s\}|$.

Definition 5.3 Given a Petri net $\mathrm{N} = (S, T, F, I, l)$, the sets of *tokens* $S_\bullet$ and *transition firings* $T_\bullet$ of N are recursively defined by

- $(*, k, s) \in S_\bullet$ for $s \in S$ and $k < I(s)$;
- $(t', k, s) \in S_\bullet$ for $s \in S$, $t' \in T_\bullet$ and $k < F(\eta(t'), s)$;
- $(X, t) \in T_\bullet$ for $t \in T$ and $X \subseteq S_\bullet$ such that $\beta(X) = {}^\bullet t \neq \emptyset$;
- $(k, t) \in T_\bullet$ for $k \in \mathbb{N}$ and $t \in T$ such that ${}^\bullet t = \emptyset$.

Now I define the higher dimensional automaton $\mathcal{A}^{\mathrm{IT}}(\mathrm{N})$ associated to a Petri net N according to the self-concurrent individual token interpretation of nets. As cells of $\mathcal{A}^{\mathrm{IT}}(\mathrm{N})$ I take the *ST-markings with individual tokens* of N, each being a pair (M, σ) of a multiset M of tokens of N (each token allocated to a place in N) and a list σ of transition firings of N. The number of transition firings in the list is the dimension of the cell. The source $s_i(q)$ in dimension i of a cell $q = (M, \sigma)$ is obtained by omitting the i^{th} transition firing (X, t) from σ, and adding the set of tokens X (or $\emptyset$ in case X is a number k) to M. Likewise, the target $t_i(q)$ of q in dimension i is obtained by omitting the i^{th} transition firing (X, t) from σ and upgrading M by adding $F(t, s)$ tokens to each place s. Below, in applying the multiset union $+$, sets X are identified with multisets by taking $X(s) = 1$ if $s \in X$ and $X(s) = 0$ otherwise, and numbers k and treated as the empty (multi)set.

Definition 5.4 Let N be a Petri net. The HDA $\mathcal{A}^{\mathrm{IT}}(\mathrm{N}) = (Q, s, t, I, F, l)$ is given by

- $Q_n = \mathcal{M}(S_\bullet^{\mathrm{N}}) \times (T_\bullet^{\mathrm{N}})^n$ for $n \in \mathbb{N}$,
- $s_i(M, t'_1 \cdots t'_n) = (M + X, t'_1 \cdots t'_{i-1} t'_{i+1} \cdots t'_n)$ for $1 \leq i \leq n$ and $t'_i = (X, t)$,
- $t_i(M, t'_1 \cdots t'_n) = (M + \{(t'_i, k, s) \mid k < F^{\mathrm{N}}(\eta(t'_i), s)\}, t'_1 \cdots t'_{i-1} t'_{i+1} \cdots t'_n)$,
- $I = (\{(*, k, s) \mid k < I^{\mathrm{N}}(s)\}, \varepsilon)$ and $F = \emptyset$,
- $l(M, t') = l^{\mathrm{N}}(\eta(t'))$ for $(M, t') \in Q_1$.

It may be helpful to observe that $\mathcal{A}^{\mathrm{CT}}(\mathrm{N})$ can be obtained from $\mathcal{A}^{\mathrm{IT}}(\mathrm{N})$ by applying β and η to the tokens and transition firings that make up a cell in

$\mathcal{A}^{\rm IT}({\rm N})$; in particular one has $s_i((\beta,\eta)(q)) = (\beta,\eta)(s_i(q))$, and likewise for t_i.

A cell $q \in Q$ in a HDA A is *reachable* iff it occurs in a path of A as defined in Section 6.3. It is not hard to check that each reachable cell in $\mathcal{A}^{\rm IT}({\rm N})$ is an ST-marking with individual tokens (M,σ) of N in which M is a plain set. The reason to involve multisets of tokens in the definition above is to avoid the problems related to unions $M \cup X$ not being disjoint.

5.3 The self-sequential interpretations

A self-sequential version of the collective token interpretation above is obtained by only allowing cells (M,σ) in which no transition occurs twice in σ. Likewise, a self-sequential version of the individual token interpretation above is obtained by only allowing cells $(M, t'_1 \cdots t'_n)$ such that $\eta(t'_i) = \eta(t'_j) \Rightarrow i = j$.

5.4 The relative expressiveness of the four interpretations

Each of the four computational interpretations above makes a different model of concurrency out of Petri nets. These models can now be compared with respect to their expressive power in denoting higher dimensional automata. A partial result is easily obtained. Let a *standard* Petri net be one in which each transition has at least one incoming arc: $\forall t \in T.\ \exists s \in S.\ F(s,t) > 0$. Now standard nets under the collective token interpretation are at least as expressive as standard nets under the individual token interpretation, in the sense that any higher dimensional automaton that can be denoted by a net under the individual token interpretation can also be a denoted by a net under the collective token interpretation.

Theorem 5.5 *For every standard net* N *there exists a standard net* $\rm N_\bullet$*, such that* $\mathcal{A}^{\rm CT}({\rm N}_\bullet) = \mathcal{A}^{\rm IT}({\rm N})$*.*

Proof. ${\rm N}_\bullet = (S_\bullet, T_\bullet, F_\bullet, I_\bullet, l_\bullet)$ with
- $S_\bullet$ and $T_\bullet$ as in Definition 5.3.
- $F_\bullet(s',t') = 1$ if $t' = (X,t)$ with $s' \in X$; $F_\bullet(s',t') = 0$ otherwise.
- $F_\bullet(t',s') = 1$ if s' has the form (t',k,s); $F_\bullet(t',s') = 0$ otherwise.
- $I_\bullet(*,k,s) = 1$ and $I_\bullet(t',k,s) = 0$ for $t' \in T_\bullet$.
- $l_\bullet(M,t') = l^{\rm N}(\eta(t'))$.

That $\mathcal{A}^{\rm CT}({\rm N}_\bullet) = \mathcal{A}^{\rm IT}({\rm N})$ is straightforward. □

The net $\rm N_\bullet$ constructed above is a close relative of the *unfolding* of a Petri net into an *occurrence net*, as defined in [17]. The difference is that I have not bothered to eliminate unreachable places and transitions.

In general, results as strong as the one above can not be obtained: in order to compare expressiveness in a meaningful way, processes represented by higher dimensional automata, Petri nets, or other models of concurrency should be regarded modulo some semantic equivalence relation. A particularly fine equivalence relation that allows one to totally order the computational interpretations of Petri nets is *isomorphism of reachable parts* of HDA.

Definition 5.6 Two higher dimensional automata A and B are *isomorphic* if there exists a dimension preserving bijection $\mathcal{I}$ between their cells, such that

- $s_i^{\mathrm{B}}(\mathcal{I}(q)) = \mathcal{I}(s_i^{\mathcal{A}}(q))$,
- $t_i^{\mathrm{B}}(\mathcal{I}(q)) = \mathcal{I}(t_i^{\mathrm{A}}(q))$,
- $\mathcal{I}(I^{\mathrm{A}}) = I^{\mathrm{B}}$,
- $\mathcal{I}(q) \in F^{\mathrm{B}} \Leftrightarrow q \in F^{\mathrm{A}}$,
- $l^{\mathrm{B}}(\mathcal{I}(q)) = l^{\mathrm{A}}(q)$ for $q \in Q_1^{\mathrm{A}}$.

The *reachable part* $\mathcal{R}(\mathrm{A})$ of an HDA A is the HDA consisting of its *reachable cells*, those that occur in a path of A as defined in Section 6.3. (Note that $\mathcal{R}(\mathrm{A})$ is closed under s_i and t_i.) Write A $\cong$ B if $\mathcal{R}(\mathrm{A})$ and $\mathcal{R}(\mathrm{B})$ are isomorphic.

By means of a small twist on Theorem 5.5 it can be shown that there is a subclass of Petri nets such that

- for any net N in that subclass, $\mathcal{A}^{\mathrm{CT}}(\mathrm{N}) \cong \mathcal{A}^{\mathrm{IT}}(\mathrm{N})$, and
- for any net N there is a net N′ in the subclass such that $\mathcal{A}^{\mathrm{IT}}(\mathrm{N}') \cong \mathcal{A}^{\mathrm{IT}}(\mathrm{N})$.

In fact, the indented subclass is close to the occurrence nets of Nielsen, Plotkin & Winskel [19], but not necessarily with the requirements that cause the elimination of unreachable parts.

Thus, up to isomorphism of reachable parts of associated HDA, the class of all Petri nets under the individual token interpretation is equally expressive as a subclass of nets on which the two interpretations coincide. The situation with the self-concurrent and self-sequential interpretations is likewise, leading to a hierarchy:
For this reason, the self-concurrent collective token interpretation will be my default; this is the interpretation that comes with the Petri net entries in Figure 1. In order to integrate the hierarchies of Figures 5 and 6, it pays to consider higher dimensional automata up to a semantic equivalence coarser than isomorphism of reachable parts. I will define such equivalences in Section 7, using the material of Section 6. It turns out that up to history preserving bisimulation equivalence the self-sequential and the self-concurrent collective token interpretations of Petri nets coincide.

$$\begin{tikzcd}
\textit{self-concurrent collective token interpretation} \\
\textit{self-sequential collective token interpretation} \arrow[u] \\
\textit{self-concurrent individual token interpretation} \arrow[u] \arrow[d, leftrightarrow] \\
\textit{self-sequential individual token interpretation}
\end{tikzcd}$$

Fig. 6. Relative expressiveness of four computational interpretations of Petri nets

6 Homotopy for Higher Dimensional Automata

6.1 Naming the faces of n-cells

In order to name the faces of a hypercube, following [59], I use the bits 0, $\lrcorner\!\ulcorner$ and 1 to indicate whether a transition has not yet started, is active, or has terminated, respectively. In other work, the bit $\lrcorner\!\ulcorner$ is sometimes called $\frac{1}{2}$, A, or T for *active* or *in transition*, and yet others use the bits 0, 1, 2, or $-$, 0, $+$.

Since an n-cell represents a list of n transitions being active concurrently, each of its 3^n faces can be represented by a list $b_1 \cdots b_n$ of n such bits, where b_i indicates the status of the i^{th} transition in the list. Thus, the dimension of the face $b_1 \cdots b_n$ is given by the number of $\lrcorner\!\ulcorner$s in the list, $\lrcorner\!\ulcorner \cdots \lrcorner\!\ulcorner$ is the identity map, and $\lrcorner\!\ulcorner \cdots \lrcorner\!\ulcorner 0 \lrcorner\!\ulcorner \cdots \lrcorner\!\ulcorner$ (resp. $\lrcorner\!\ulcorner \cdots \lrcorner\!\ulcorner 1 \lrcorner\!\ulcorner \cdots \lrcorner\!\ulcorner$), with only $b_i \neq \lrcorner\!\ulcorner$, denotes the map s_i (resp. t_i). The face $b_1 \cdots b_n$ can also be expressed as $\alpha_1 \circ \ldots \circ \alpha_n$ with $\alpha_i = s_i$ if $b_i = 0$, $\alpha_i = t_i$ if $b_i = 1$, and $\alpha_i = Id$, the identity map, if $b_i = \lrcorner\!\ulcorner$. Using this convention, the 2^n 0-dimensional corners of an n-cell q are named by lists $b_1 \cdots b_n$ with $b_i \in \{0, 1\}$, and the $n \cdot 2^{n-1}$ 1-dimensional edges of q by lists containing exactly one occurrence of $\lrcorner\!\ulcorner$.

6.2 Labelling n-cells

As a HDA is required to satisfy $l(s_i(q)) = l(t_i(q))$ for all $q \in Q_2$ and $i = 1, 2$, it follows that, for each $q \in Q_n$ and each $1 \leq i \leq n$, the 2^{n-1} edges $b_0 \cdots b_{i-1} \lrcorner\!\ulcorner b_{i+1} \cdots b_n$ of q, with $b_j \in \{0, 1\}$ for $j \neq i$, all have the same label. Calling this label $l_i(q)$, the labelling function $l : Q_1 \rightarrow A$ can be extended to $Q := \bigcup_{k=0}^{\infty} Q_k$ by $l(q) = l_1(q) \cdots l_n(q)$ for $q \in Q_n$. Thus, the label of an n-dimensional hypercube q is the list of the labels of the n transitions whose concurrent execution is represented by q.

6.3 Paths and their observable content

Definition 6.1 A *path* in a higher dimensional automaton (Q, s, t, I, F, l) is a sequence of pairs $(\partial_1, q_1)(\partial_2, q_2)\cdots(\partial_m, q_m)$, denoted $I\xrightarrow{\partial_1}q_1\xrightarrow{\partial_2}q_2\xrightarrow{\partial_3}\cdots\xrightarrow{\partial_m}q_m$, with $q_k \in Q$ and $\partial_k \in \{s_i, t_i \mid 1 \leq i \leq d(q_k)\}$ for $1 \leq k \leq m$, such that

$$q_{k-1} = s_i(q_k) \text{ if } \partial_k = s_i \qquad \text{and} \qquad q_k = t_i(q_{k-1}) \text{ if } \partial_k = t_i.$$

Here $q_0 := I$, i.e. I consider only paths starting from the initial state, and $d(q)$, de *dimension* of q, is n if $q \in Q_n$. One writes $end(\pi)$ for q_m.

A path π in a HDA A represents a partial run of the system represented by A in which between every two consecutive states an action starts or terminates. It represents a total run iff $end(\pi) \in F$. If $\partial_k = s_i$, the transition from q_{k-1} to q_k represents the start of the action $l_i(q_k)$, and if $\partial_k = t_i$, it represents the termination of the action $l_i(q_{k-1})$.

Definition 6.2 Write *split-trace*(π) for the sequence $\sigma_1\cdots\sigma_m$, where $\sigma_k = l_i(q_k)^+$ if $\partial_k = s_i$, and $\sigma_k = l_i(q_{k-1})^-$ if $\partial_k = t_i$.

Split-trace(π) approximates the observable content of a path π. It consists of the sequence of starts a^+ and terminations a^- of actions a occurring during the run represented by π. The HDA of Figure 2 in which the square is not filled in, for instance, has only two maximal paths, whose split-traces are $a^+a^-b^+b^-$ and $b^+b^-a^+a^-$. (The prefixes of these sequences are also split-traces of paths.) The HDA of Figure 2 in which the square *is* filled in moreover has paths with split-traces like $a^+b^+b^-a^-$.

If it is possible to keep track of parallel occurrences of the same action, a split-trace falls short of representing the full observable content of a path. The observable behaviour of a real-time execution would consist of a set of action occurrences, with for each action occurrence an interval given by a start time and a termination time, indicating the period during with the action takes place. Abstracting from real-time information, what remains is a split-trace σ—a sequence $\sigma = \sigma_1\cdots\sigma_m$ of *action phases*, each being a start a^+ or a termination a^- of an action a—together with an injective function *start* from the (indices of) termination phases in σ to the (indices of) start phases in σ. This functions tells for every termination of an action occurrence in σ where that action occurrence starts. Naturally, for every termination phase $\sigma_\ell = a^-$ in σ, one has $start(\ell) < \ell$ and $\sigma_{start(\ell)} = a^+$. Such an annotated split-trace is called an *ST-trace*. It can be compactly represented by writing $a^{start(\ell)}$ for σ_ℓ whenever $\sigma_\ell = a^-$. A formal definition of the ST-trace of a path in a higher dimensional automaton will follow in Section 6.5.

A typical ST-trace is depicted as Figure 7. It lists the starts and terminations

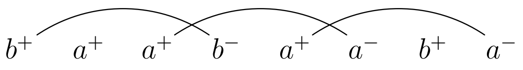


Fig. 7. An ST-trace

of actions occurring in a path, and additionally links the start and termination of the same action. Its compact representation is $b^+a^+a^+b^1a^+a^3b^+a^5$. Note that it contains more information than the underlying split-trace. ST-traces were introduced in [7] in the context of event structures. Arguably, they constitute the best formalisation of the observable content of execution paths.

Paths and their ST-traces lack the possibility to express that action phases happen simultaneously. However, in higher dimensional automata two action phases can occur simultaneously iff they can occur in either order. Therefore, considering only paths in which all action phases are totally ordered does not lead to a decrease in expressive power.

6.4 Homotopy

Two paths in a higher dimensional automaton can be considered equivalent if they differ merely in the timing of causally independent actions. This applies for instance to the path ab and the path ba in the automaton of Figure 4, given that the actions a and b are causally independent. However, it would not apply when the square is not filled in, as this signifies mutual exclusion, and the relative order of a and b would matter. As observed by Pratt [22], this notion of equivalence can be formalised beautifully by means of what he calls "monoidal homotopy". When seeing a higher dimensional automaton as a structure composed of higher dimensional cubes embedded in a higher dimensional Euclidean space, two paths are *homotopic* if one can be obtained out of the other by a continuous transformation, keeping the begin and endpoints the same, and allowing as intermediate stages of the transformation arbitrary paths going through the insides of higher dimensional cells, as long as they are monotonically increasing when projected on the axes of the cells they are going through. The path drawn in Figure 4 for instance could be one of the uncountably many stages in the continuous transformation of ab into ba.

This form of homotopy differs from the standard homotopy used in topology in that the directed nature of the underlying space needs to be preserved during transformations. Therefore it is called *monoidal homotopy* [22], or *directed homotopy* (*dihomotopy*) [55], as opposed to *group homotopy*, although in the context of higher dimensional automata it is simply called *homotopy*.

The following discrete analog of continuous deformation defines the same concept of homotopy without involving the notion of Euclidean space.

Definition 6.3 Two paths π and π' are *adjacent*—denoted $\pi \leftrightarrow \pi'$—if one can be obtained from the other by replacing, for $q, q' \in Q$ and $i < j$,

- a segment $\underline{s_i}q\underline{s_j}$ by $\underline{s_{j-1}}q'\underline{s_i}$,
- a segment $\underline{t_j}q\underline{t_i}$ by $\underline{t_i}q'\underline{t_{j-1}}$,
- a segment $\underline{s_i}q\underline{t_j}$ by $\underline{t_{j-1}}q'\underline{s_i}$,
- or a segment $\underline{s_j}q\underline{t_i}$ by $\underline{t_i}q'\underline{s_{j-1}}$.

Homotopy is the reflexive and transitive closure of adjacency.

The third adjacency replacement above can be motivated as follows: suppose we have a list of n actions, numbered 1 to n, and we first insert an action a at position i (thereby incrementing the slot-numbers $\geq i$ by one) and subsequently delete the j^{th} action ($j > i$) from the list (thereby decrementing the slot-numbers $> j$ by one), then we get the same result as when we first delete the $(j-1)^{th}$ action (thereby decrementing the slot-numbers $\geq j$ by one) and subsequently insert the action a at position i (incrementing the slot-numbers $\geq i$). The other replacements are motivated in a similar way.

The paths with split-traces $a^+a^-b^+b^-$ and $b^+b^-a^+a^-$ in Figure 4 for instance are homotopic, because the first can be transformed into the second through four adjacency replacements, namely (on the level of split-traces)

$$a^+a^-b^+b^- \leftrightarrow a^+b^+a^-b^- \leftrightarrow b^+a^+a^-b^- \leftrightarrow b^+a^+b^-a^- \leftrightarrow b^+b^-a^+a^-.$$

Homotopic paths share their endpoints. A homotopy class of paths in a higher dimensional automaton (with endpoint q) is called a *history* (of q). Histories form the analog of paths, after abstraction from the order or causally independent action occurrences.

6.5 *Matching starts and terminations of action occurrences in paths*

The following proposition illustrates the agreement between Definition 6.3 and the cubical laws of Definition 2.1.

Proposition 6.4 *For every segment $p\underline{s_i}q\underline{t_j}r$ with $i \neq j$, and for every segment $p\underline{s_i}q\underline{s_j}r$ or $p\underline{t_i}q\underline{t_j}r$ in a path π in a HDA* A*, there exists a unique path π' in* A*, adjacent to π, that can be obtained from π by replacing the indicated segment in the manner described in Definition 6.3 (going either right or left).*

Proof. Suppose π contains $p\underline{s_i}q\underline{t_j}r$, with $i < j$. Then $p = s_i(q)$ and $r = t_j(q)$. By the cubical laws in Definition 2.1, $t_{j-1}(p) = t_{j-1}((s_i(q)) = s_i(t_j(q)) = s_i(r)$. Hence the unique replacement is $p\underline{t_{j-1}}q'\underline{s_i}r$ with $q' = t_{j-1}(p) = s_i(r)$.

Likewise, suppose π contains $p\underline{s_i}q\underline{s_j}r$, with $i < j$. Then $p = s_i(q)$ and $q = s_j(r)$. By the cubical laws (*) in Definition 2.1 one has $p = s_{j-1}(s_i(r))$. So the unique replacement is $p\underline{s_{j-1}}q'\underline{s_i}r$ with $q' = s_i(r)$.

The other four cases go similarly. □

The intuition is that when two actions happen concurrently, they can start in either order as well as terminate in either order; moreover, if it is possible for action a to start before b terminates, b could just as well terminate before a starts, provided a and b are distinct action occurrences. Note, however, that Proposition 6.4 does not apply to segments $p\xrightarrow{t_i}q\xrightarrow{s_j}r$. If the termination of one action precedes the start of another, it may be that there is a causal link between the two that prevents this order from being interchanged.

Write $\pi \xleftrightarrow{\ell} \pi'$ if π' can be obtained from $\pi = I\xrightarrow{\partial_1}q_1\xrightarrow{\partial_2}q_2\xrightarrow{\partial_3}\cdots\xrightarrow{\partial_m}q_m$ by an adjacency replacement of the segment $\xrightarrow{\partial_\ell}q_\ell\xrightarrow{\partial_{\ell+1}}$ of π, inducing a swap of the action phases σ_ℓ and $\sigma_{\ell+1}$ in the split-trace $\sigma = \sigma_1\cdots\sigma_m$ of π. Assume $\partial_{\ell+1} = t_i$, i.e. $\sigma_{\ell+1}$ is the termination $l_i(q_\ell)^-$ of an action $l_i(q_\ell)$. In case $\partial_\ell = s_i$ we have that σ_ℓ is the start $l_i(q_\ell)^+$ of the very same occurrence of the action $l_i(q_\ell)$ in π. In this case σ_ℓ and $\sigma_{\ell+1}$ cannot be swapped: there is no path π' such that $\pi \xleftrightarrow{\ell} \pi'$. Proposition 6.4 tells that in all other cases (i.e. when $\partial_\ell \neq s_i$) σ_ℓ and $\sigma_{\ell+1}$ can be swapped: there exists a unique path π' in A with $\pi \xleftrightarrow{\ell} \pi'$. This makes it possible to tell which action phase in *split-trace*(π) is the start of the action occurrence whose termination happens as phase $\sigma_{\ell+1}$: it is the unique phase σ_k such that $\pi \xleftrightarrow{\ell} \pi_\ell \xleftrightarrow{\ell-1} \pi_{\ell-1} \xleftrightarrow{\ell-2} \cdots \xleftrightarrow{k+1} \pi_{k+1} \not\xleftrightarrow{k} \pi_k$.

Definition 6.5 Let $\pi = I\xrightarrow{\partial_1}q_1\xrightarrow{\partial_2}q_2\xrightarrow{\partial_3}\cdots\xrightarrow{\partial_m}q_m$ be a path in a higher dimensional automaton. For $1 \leq \ell \leq m$ such that ∂_ℓ denotes a termination phase, let *start*(ℓ) denote the unique number k such that

$$\pi \xleftrightarrow{\ell-1} \pi_{\ell-1} \xleftrightarrow{\ell-2} \cdots \xleftrightarrow{k+1} \pi_{k+1} \not\xleftrightarrow{k} \pi_k.$$

Now *ST-trace*(π) is the sequence obtained from *split-trace*$(\pi) = \sigma_1\cdots\sigma_k$, by replacing σ_ℓ by $a^{start(\ell)}$ whenever $\sigma_\ell = a^-$.

7 Bisimulation semantics for HDA

Using the material of Section 6, I now extend the main forms of bisimulation equivalence found in the literature that do not involve a special treatment of *hidden* or *internal* actions, to higher dimensional automata. For *interleaving bisimulation equivalence* this is trivial: it is just the standard notion of bisimulation equivalence on ordinary automata found in the literature [18,1], applied to higher dimensional automata by ignoring their higher dimensional cells. *ST-bisimulation equivalence* [13] is a branching time respecting equivalence that takes causality into account to the extent that it is expressible by durational actions overlapping in time. *History preserving bisimulation equivalence* [24,10] is the coarsest equivalence that fully respects branching time, causality and their interplay. *Hereditary history preserving bisimulation* *equivalence* [3] is a variant of the latter that strongly respects the internal structure of processes, while still collapsing choices between indistinguishable courses of action (i.e. satisfying the CCS law $x + x = x$ [18]).

By Definition 6.1, the empty path in a higher dimension automata A starts and ends in the initial state of A and hence is denoted I^{A}. I write $\pi \rightarrow \pi'$ if π is a prefix of a path π', i.e., if π' is an extension of π.

Definition 7.1 Two higher dimensional automata A and B are *history preserving bisimulation equivalent* if there exists a binary relation R between their paths—a *history preserving bisimulation*—such that

(1) the empty paths in A and B are related: $I^{\mathrm{A}} R I^{\mathrm{B}}$,
(2) if $\pi R \rho$ then then $\mathit{ST\text{-}trace}_{\mathrm{A}}(\pi) = \mathit{ST\text{-}trace}_{\mathrm{B}}(\rho)$,
(3) if $\pi R \rho$ and $\pi \leftrightarrow \pi'$ then $\exists \rho'$ with $\rho \leftrightarrow \rho'$ and $\pi' R \rho'$,
(4) if $\pi R \rho$ and $\rho \leftrightarrow \rho'$ then $\exists \pi'$ with $\pi \leftrightarrow \pi'$ and $\pi' R \rho'$,
(5) if $\pi R \rho$ and $\pi \rightarrow \pi'$ then $\exists \rho'$ with $\rho \rightarrow \rho'$ and $\pi' R \rho'$,
(6) if $\pi R \rho$ and $\rho \rightarrow \rho'$ then $\exists \pi'$ with $\pi \rightarrow \pi'$ and $\pi' R \rho'$,
(7) and if $\pi R \rho$ then $\mathit{end}(\pi) \in F^A \Leftrightarrow \mathit{end}(\rho) \in F^B$.

A and B are *hereditary history preserving bisimulation equivalent* if there exists a history preserving bisimulation R between their paths that moreover satisfies

(8) if $\pi R \rho$ and $\pi' \rightarrow \pi$ then $\exists \rho'$ with $\rho' \rightarrow \rho$ and $\pi' R \rho'$, and
(9) if $\pi R \rho$ and $\rho' \rightarrow \rho$ then $\exists \pi'$ with $\pi' \rightarrow \pi$ and $\pi' R \rho'$.

ST-bisimulation equivalence between HDA is defined exactly as history preserving bisimulation equivalence, but dropping clauses (3) and (4).

Note that in the presence of clause (2), related paths have the same length. Hence clauses (8) and (9) are equivalent. I listed them both solely to stress the symmetric nature of the definition. It is not hard to see that the notion of ST-bisimulation equivalence would not change upon adding clauses (8) and (9), but because of clauses (3) and (4), this does not apply to history preserving bisimulation equivalence. Using clause (2) it follows that the effect of clauses (3) and (4) would not change if I wrote $\overset{\ell}{\longleftrightarrow}$ instead of $\leftrightarrow$ throughout these clauses. The clauses (3) and (4) express that the causal relations between action phases in the ST-traces of two related paths are the same, for these relations are determined by the space of all allowed permutations of action phases.

8 A hierarchy of concurrency models

After having defined precisely what the arrows in Figure 1 mean, I will now proceed to argue for their soundness and completeness in describing the relative expressiveness of the models of concurrency under investigation.

8.1 Soundness of the inclusions of Figure 1

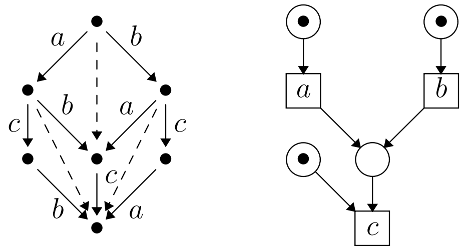
It is well known that stable event structures with binary conflict [29] are more expressive than flow event structures [5], which are more expressive than bundle event structures [16], which are in turn more expressive than prime event structures with binary conflict [29], cf. Figure 5. However, this holds when comparing the families of configurations they can express; the difference disappears when working up to history preserving bisimulation. This follows immediately from the fact that prime and stable event structures with binary conflict specify the same *Scott domains* [28,29], and thus also the same [higher dimensional] automata. Alternatively, a direct proof can be found in [15]. In [15] and [20] it has furthermore been shown that, up to history preserving bisimulation, finitary conflict can be expressed in terms of binary conflict. Thus, by the criteria of this paper, the stable event structures of [28] do not rank higher in expressive power than the ones with binary conflict in [29]. Likewise, the general event structures of [28] do not rank higher in expressive power than the ones with binary conflict in [29]. This shows that up to history preserving bisimulation Figure 5 collapses into the bottom of Figure 1.

A Petri net is *safe* if no reachable marking will every have two tokens in the same place. In [19] the expressive equivalence has been established of the model of safe Petri nets with that of prime event structures with binary conflict. This was done by means of translations between these models that preserve more than history preserving bisimulation equivalence.

Up to history preserving bisimulation equivalence, on safe Petri nets there is no difference between any of the four computational interpretations of Petri nets discussed in Section 5. Moreover, the nets $\mathrm{N}_\bullet$, constructed in the proof of Theorem 5.5, are safe. It follows that under the self-sequential individual token interpretation, as well as under the self-concurrent individual token interpretation, the class of all Petri nets is equally expressive as the class of safe Petri nets.

In [11] two forms of 1-unfolding are defined, one for Petri nets under the self-concurrent collective token interpretation, and one for Petri nets under the self-sequential collective token interpretation. Each of them converts any Petri net into a so-called *1-occurrence net*, a net in which each transition can fire only once. And each of these 1-unfoldings respects history preserving bisimulation equivalence w.r.t. to the computational interpretation of nets that comes with this 1-unfolding. Trivially, on 1-occurrence nets there is no difference between the self-concurrent and the self-sequential interpretation of nets. It follows that, up to history preserving bisimulation equivalence, Petri nets under the

$$E = \{a, b, c\}$$
$$\# = \emptyset$$
$$\begin{aligned} \emptyset &\vdash a \\ \emptyset &\vdash b \\ a &\vdash c \\ b &\vdash c \end{aligned}$$



$$\begin{tikzcd} a \arrow[d] & & b \arrow[d] \\ c \arrow[r, phantom, "\#"] & & c \end{tikzcd}$$

Fig. 8. A system with *disjunctive causality* represented as an event structure of [29], a [higher dimensional] automaton in which the dashed lines indicate that all three squares are filled in, and a Petri net. The last picture is the best representation of the same system as a prime event structure [19,29]. It requires the decomposition of the event c, which is causally dependent on the disjunction of a and b, into two events c_1 and c_2, only one of which may happen: c_1 being causally dependent only on a, and c_2 on b. This prime event structure is ST-bisimulation equivalent to the original one, but not history preserving bisimulation equivalent.

self-sequential collective token interpretation are equally expressive as Petri nets under the self-concurrent collective token interpretation.

A Petri net is *pure* if it has no *self-loops*, i.e. there are no places s and transitions t with $F(s,t) > 0$ and $F(t,s) > 0$. In [11] we showed that configuration structures are equally expressive as pure Petri nets under the collective token interpretation (or precisely, each of the two collective token interpretations). Taking into account that pureness is preserved by the two 1-unfoldings, this was done my means of translations between configuration structures and pure 1-occurrence nets that preserve more than history preserving bisimulation equivalence.

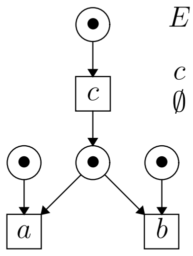
This concludes the justification of the arrows in Figure 1.

8.2 Completeness of the inclusions of Figure 1

That the model of synchronisation trees is less expressive than that of safe Petri nets is witnessed by the process $a\|b$, the parallel composition of two actions a and b. This process can be represented by the safe Petri net P in Figure 2. However, there is no synchronisation tree which is history preserving bisimulation equivalent, or even ST-bisimulation equivalent, to this process.

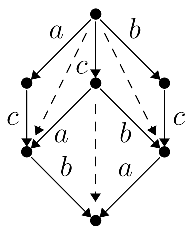
The models of prime and stable event structures are less expressive than that of (general) event structures of [29]. This is witnessed by the process of Figure 8, which is representable as an event structure of [29], but, up to history preserving bisimulation equivalence, not as a prime event structure.

In [12], a generalisation of Winskel's event structures is proposed that (up to history preserving bisimulation equivalence) is equally expressive as Petri nets under the collective token interpretation. Also a subclass of *pure* event structures is defined that matches the expressive power of configuration



$$E = \{a, b, c\}$$

$$\begin{array}{ll} c \vdash \{a,b\} & \\ \emptyset \vdash X & \text{for } X \neq \{a,b\} \end{array}$$



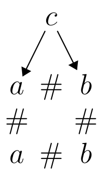
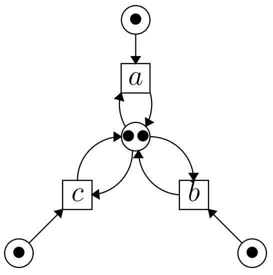


Fig. 9. A system with *resolvable conflict* represented as a pure Petri net, a pure event structure as introduced in [12], and a [higher dimensional] automaton. The events a and b are initially in conflict (only one of them may happen), but as soon as c occurs this conflict is resolved. The last picture is the best representation of the same system as a prime event structure. It yields a system with two maximal runs, in one of which c causes just a, and in the other just b. Again it is ST-bisimulation equivalent to the original, but not history preserving bisimulation equivalent. There is no event structure as in [28,29] that is history preserving bisimulation equivalent to the system above.

structures and pure nets. An example of a pure Petri net and a pure event structure as in [12] that cannot be represented as an event structure of [28,29] appears in Figure 9.

Figure 10 shows a system represented as a Petri net, that cannot be represented as a pure Petri net, or as an automaton under the concurrent interpretation.



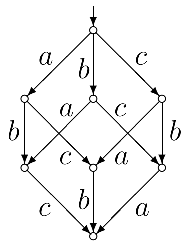


Fig. 10. A *2-out-of-3 semaphore*, represented as a Petri net and as a higher dimensional automaton. In the latter, all six squares are supposed to be filled in, but the interior of the cube is not. Up to history preserving bisimulation equivalence this system cannot be represented as an automaton under the concurrent interpretation, because, due to the filled-in squares, the cube shape is unavoidable, and interior of the cube would by default be understood to be filled in. Hence the system can also not be represented as a pure Petri net.

8.2.1 Beyond Petri nets

The final counterexample witnessing the completeness of the expressiveness hierarchy of Figure 1 concerns the process of Figure 11, that is representable as a higher dimensional automaton, and even as an ordinary automaton under the concurrent interpretation, but not as a Petri net.

The process displayed in Figure 11 was implemented during my presentation

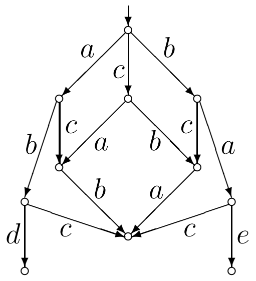


Fig. 11. A process, represented as an ordinary automaton under the concurrent interpretation, that, up to history preserving bisimulation equivalence, cannot be represented as a Petri net.

at EXPRESS 2004. Two computer scientists A and B were travelling from one end of the podium to the other. Their task to was perform the actions a resp. b of crossing a line on the podium. Due to strategic placing of obstacles, the only place were this was possible was at a narrow opening between the obstacles that had room for only one of the scientists A and B at a time. This made the actions a and b mutually exclusive, in the sense that they could not occur simultaneously. A third computer scientist, C, was assigned the task c of removing an obstacle that caused the bottleneck to exists. The action c was executed causally independent of a and b. The actions a and b were mutually exclusive only until c occurred, after which they became causally independent. Finally, a fourth participant was assigned the task of making a statement when a and b had both occurred before the action c started. This statement was going to be d in case A passed the bottleneck before B did, and e in case B passed the bottleneck before A did. Hearing this statement would prevent computer scientist C from carrying out the action c. The resulting process is described by the automaton above, in which all five squares are filled in.

In order to represent the process of Figure 11, up to history preserving bisimulation equivalence, as a Petri net, there must be a single transition representing the action a, regardless of whether it is scheduled before or after b or c. This because of the concurrency inherent in the example. The same holds for b. However, in a Petri net, firing just these two transitions labelled a and b necessarily leads to a unique state, independent of the order in which a and b occur. This is in contradiction with the fact that the process under consideration has two states reachable by doing only a and b, in which different further actions are possible.

8.3 Comparisons modulo other notions of equivalence

If I compare the models of Figure 1 up to hereditary history preserving bisimulation equivalence the same hierarchy results. This because all translations used in Section 8.1 even preserve hereditary history preserving bisimulation equivalence. If I compare them merely up to interleaving bisimulation equivalence, all models turn out to be equally expressive. This because every higher dimensional automaton is trivially interleaving bisimulation equivalent to the 1-dimensional automaton resulting from ignoring its higher dimensional cells, and to the unfolding of that 1-dimensional automaton into a tree.

If I compare the models up to ST-bisimulation equivalence, the model of synchronisation trees is still less expressive than that of event structures, as explained in Section 8.2. I conjecture that all models of Figure 1 other than synchronisation trees are equally expressive, in the sense that any process representable in any of the models can be translated into an ST-bisimulation equivalent prime event structure. For configuration structures this is Theorem 1 in [11]. My conjecture is that this result extends to higher dimensional automata; in other words, that up to ST-bisimulation equivalence the prime event structures have universal expressivity.

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