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The Distance of a Subspace of R^m from Its Axes and n-Widths of Octahedra

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1. INTRODUCTION

It will be evident throughout that this paper was stimulated by the recent work of Pinkus [8] on n-widths of diagonal operators from l_r^m to l_p^m. Here we pick up one of his topics, the case $r = 1$. In addition to extending known results as much as possible, our aim is to unify and emphasize the l_1 flavor. This attempt leads naturally to a consideration of the necessary and sufficient conditions for the existence of an n-dimensional subspace X_n of R^m ($m \leqslant \infty$) with given distances η_j from the principal axes e^j:

$$\eta_j = \min_{x \in X_n} \|e^j - x\|_p. \tag{1.1}$$

In Section 2 precise answers are given for some values of p and n, and estimates for other values; the essence of the case $p = 2$ is taken from Sofman [11]. Section 3 shows how to use these results to obtain the Kolmogorov n-widths

$$d_n^R(D; l_1^m, l_p^m) = \min_{X_n \subset R^m} \max_{\|x\|_1 \leqslant 1} \min_{y \in X_n} \|Dx - y\|_p \tag{1.2}$$

with D a positive diagonal matrix.

Many of these results have appeared before [1, 8, 9, 11, 12]; our main new contributions are a sharp inequality, the identification of optimal subspaces, and the method of derivation. Of special interest is the fact, noted by Pinkus [8] that if the first min in (1.2) is taken over $X_n \subset C^m$ then $d_n^C < d_n^R$ can happen even though D operates on R^m. The results of Sections 2 and 3, however, do not depend on this distinction and therefore we do not mention this point again until we turn to it specifically in Section 4. There we treat the important case $D = I$ primarily with an eye toward obtaining

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exact values for the n-widths. In particular we make the conjecture that the asymptotic behavior for large m and n such that $m^2/n^p \to 0$ is

$$d_n(I; l_1^m, l_p^m) \approx C \frac{m^{1/p}}{\sqrt{n}}, \qquad p > 2. \tag{1.3}$$

Interestingly enough, the substantiation of this conjecture is linked to the combinatorial problem of equiangular lines investigated by Lemmens, Van Lint, and Seidel [4, 5].

2. The Distance of an n-Dimensional Subspace of R^m from Its Axes

Problem 1. Given m reals $\{\eta_i\}_1^m$, $0 \leqslant \eta_i \leqslant 1$, find n-dimensional subspaces X_n of R^m such that

$$E_p(e^i, X_n) = \min_{x \in X_n} \| e^i - x \|_p = \eta_i, \qquad i = 1,..., m.$$

In the solution of this problem we will make frequent use of

$$E_p(e^i, X_n) = \max_{y \perp X_n} \frac{|y_i|}{\|y\|_{p'}}, \qquad \frac{1}{p} + \frac{1}{p'} = 1, \tag{2.1}$$

a familiar consequence of the Hahn–Banach theorem.

Theorem 2.1. *Problem* 1 *has a solution for* $p = 1$, $n < m \leqslant \infty$, *if and only if* $\eta_i < 1$ *for at most n indices i.*

Proof. Suppose that at most n of the η_i do not equal 1, say $i = 1,..., k$, $k \leqslant n$. Using (2.1) it is easy to construct a subspace $\tilde{X}_n$ of R^{n+1} with distances η_i, $i = 1,..., n+1$, namely take $\tilde{X}_n$ orthogonal to the vector $(\eta_1,..., \eta_{n+1})$. Viewing $\tilde{X}_n$ as a subspace of R^m by adding zero coordinates yields the desired subspace since $\eta_{n+i} = 1$.

The converse follows, again on the basis of (2.1), from the statement (cf. Pietsch [9]): given $X_n \subset R^m$ there exists $y \in R^m$, $y \perp X_n$ such that $\max |y_i| = 1$ and $|y_i| < 1$ for at most n indices i. Indeed y may be taken to be any extreme point of the closed convex set $L = \{x \in R^m \mid \|x\|_\infty \leqslant 1, x \perp X_n\}$. For if $\bar{y}$ is such a point but, say, $|\bar{y}_i| < 1$, $i = 1,..., n+1$ then taking $g \in \operatorname{span}\{e^i\}_1^{n+1}$ and $g \perp X_n$ we have $\bar{y} \pm \varepsilon g \in L$ for small enough ε contradicting the extreme point property of $\bar{y}$.

THEOREM 2.2. *Problem* 1 *has a solution for* $p=2$, $n<m\leqslant\infty$ *if and only if*

$$\sum_{i=1}^{m}(1-\eta_i^2)=n. \tag{2.2}$$

Proof. This theorem was essentially proven by Sofman [11]. For completeness we give the proof. Let $\{x^i\}_1^n$ be a orthonormal basis for X_n. Then

$$\eta_k^2=\min_{\alpha}\left\|e^k-\sum_{i=1}^{n}\alpha_i x^i\right\|^2=1-\sum_{i=1}^{n}|(e^k,x^i)|^2; \tag{2.3}$$

the condition is therefore necessary.

The converse is proven by induction on n, the case $n=1$ being immediate from (2.3). Assume then that $n>1$ and for simplicity that $\eta_1=\min\eta_i$ (if $m=\infty$, $\eta_i\to_{i\to\infty}1$). Choose $\{\xi_i\}_1^{m-1}$ such that

$$0\leqslant\eta_i\leqslant\xi_i,\qquad i=2,\dots,m,\qquad \sum_{i=2}^{m}(1-\xi_i^2)=n-1$$

which is possible since $\sum_{i=2}^{m}(1-\eta_i^2)=n-1+\eta_1^2\geqslant n-1$. Now by the induction hypothesis there exists X_{n-1} such that $E_2(e^k,X_{n-1})=\xi_k$, $k=2,\dots,m$, $E_2(e^1,X_{n-1})=1$. Add e^1 to X_{n-1} to obtain X_n, for which $\xi_1=E_2(e^1,X_n)=0$ and $\sum_{i=1}^{m}(1-\xi_i^2)=n$. The proof is completed by showing how to rotate X_n so as to move step by step from $\{\xi_i\}_1^m$ to $\{\eta_i\}_1^m$.

Let $x(\alpha)$ be any of the basis vectors of X_n after a rotation by α in the $1-k$ plane

$$x(\alpha)_1=x_1\cos\alpha-x_k\sin\alpha,\qquad x(\alpha)_k=x_1\sin\alpha+x_k\cos\alpha$$

with all other coordinates unchanged. Thus, in obvious notation (2.3) shows $\xi_i(\alpha)=\xi_i$, $i\neq 1$, k while $\xi_1^2(\alpha)+\xi_k^2(\alpha)=\xi_1^2+\xi_k^2$. Moreover since $\xi_k(0)=\xi_k$, $\xi_k(\pi/2)=\xi_1$, and

$$\xi_1\leqslant\eta_1=\min\eta_i\leqslant\eta_k\leqslant\xi_k,$$

the continuity of $\xi_k(\alpha)$ implies there is a value of α for which $\xi_k(\alpha)=\eta_k$. At the same time $\eta_i\leqslant\xi_i(\alpha)$, $i=2,\dots,m$, and $\sum_{i=1}^{m}(1-\xi_i(\alpha)^2)=n$ continues to hold so that $\xi_1(\alpha)$ is still the smallest and the process may be repeated for all coordinates.

THEOREM 2.3. (a) *Define*

$$f_p(x)=\left[1+\left(\frac{x^p}{1-x^p}\right)^{p'/p}\right]^{-1/p'},\qquad \frac{1}{p}+\frac{1}{p'}=1,\quad 1<p<\infty.$$

Then

$$E_p(e^k; X_n) = f_p(E_{p'}(e^k; X_n^{\perp})). \tag{2.4}$$

(b)

$$E_\infty(e^k; X_n) = \min_{x \in X_n} \left[1 + \frac{|x_k|}{\sup_{i \neq k} |x_i|} \right]^{-1}. \tag{2.5}$$

Proof. (a) Write, e.g., $E_p(e^1; X_n) = \min_{x \in X_n} \min_\alpha \|e^1 - \alpha x\|_p$. Now

$$\min_\alpha \|e^1 - \alpha x\|_p^p = \min_{\beta > 0} \min_\varphi \left[|1 - \beta e^{i\varphi} |x_1||^p + \beta^p \sum_{i=2}^m |x_i|^p \right].$$

Clearly one must have $\varphi = 0$ and $\beta < |x_1|^{-1}$. Noting that $f_p(x)$ is monotone decreasing in x and using (2.1) one easily computes

$$\begin{aligned} E_p(e^1; X_n) &= \min_{x \in X_n} \frac{(\sum_{i=2}^m |x_i|^p)^{1/p}}{[|x_1|^{p'} + (\sum_{i=2}^m |x_i|^p)^{p'/p}]^{1/p'}} \\ &= \min_{x \in X_n} f_p\left(\frac{|x_1|}{\|x\|_p}\right) \\ &= f_p\left(\max_{x \in X_n} \frac{|x_1|}{\|x\|_p}\right) \\ &= f_p(E_{p'}(e^1; X_n^{\perp})) \end{aligned}$$

by (2.1).

(b). Proceeding as in (a), let $|x_2| = \max_{i>1} |x_i|$. Then $\min_\alpha \|e^1 - \alpha x\|_\infty$ is achieved for an α such that $|1 - \alpha x_1| = |\alpha| |x_2|$, i.e., $|\alpha| = [|x_1| + |x_2|]^{-1}$.

COROLLARY 2.4. *The following are necessary and sufficient conditions for the existence of a solution of problem* 1:

(a) $1 < p < \infty$, $n = 1 < m \leqslant \infty$: $\quad \sum_{i=1}^m [f_{p'}(\eta_i)]^p = 1.$ (2.6)

(b) $p = \infty$, $n = 1 < m \leqslant \infty$: $\quad \inf_{i \neq j} [\eta_i + \eta_j] = 1.$ (2.7)

(c) $1 < p \leqslant \infty$, $n = m - 1 < \infty$: $\quad \sum_{i=1}^m (1 - \eta_i^{p'}) = m - 1.$ (2.8)

Proof. (c) Referring to (2.1) we have $\eta_i = \max_{y \perp X_{m-1}} |y_i|/\|y\|_{p'} = |y_i|/\|y\|_{p'}$. Thus (2.8) holds and, conversely, given η_i take $X_{m-1} \perp (\eta_1, \ldots, \eta_m)$.

(a) Note that $f_{p'}(f_p(x)) = x$. Hence (2.6) follows from the theorem and (2.8). Conversely, given η_i satisfying (2.6) choose for X_1 the span of $x = (f_{p'}(\eta_1),\dots,f_{p'}(\eta_m))$.

(b) If $\sup|x_i| = |x_1|$ and $\sup_{i>1}|x_i| = |x_2|$ (e.g., $m < \infty$) then from (2.5) one gets (2.7) with $i = 1, j = 2$. The reasoning for $m = \infty$ is analogous.

Since it seems difficult to obtain any further precise results, we turn to some inequalities.

LEMMA 2.5. *If* $1 \leqslant r \leqslant p \leqslant s \leqslant \infty$ *then*

$$\begin{aligned}\{1 + [E_r(e^k, X_n)^{-r'} - 1]^{p'/r'}(m-1)^{1-(p'/r')}\}^{-1/p'} &\leqslant E_p(e^k, X_n)\\ &\leqslant \{1 + [E_s(e^k, X_n)^{-s'} - 1]^{p'/s'}(m-1)^{1-(p'/s')}\}^{-1/p'}.\end{aligned} \tag{2.9}$$

Proof. Inequality (2.9) is based on the simple inequality

$$\left(\sum_{j\neq k}|x_j|^p\right)^{1/p} \geqslant \left(\sum_{j\neq k}|x_j|^r\right)^{1/r}(m-1)^{(1/p)-(1/r)}, \qquad p \geqslant r.$$

Thus with $1/p' = 1 - (1/p)$,

$$E_p(e^k, X_n)^{p'} = \max_{x\perp X_n}\frac{|x_k|^{p'}}{\|x\|_{p'}^{p'}} \geqslant \max_{x\perp X_n}\frac{|x_k|^{p'}}{|x_k|^{p'} + (\sum_{j\neq k}|x_j|^{r'})^{p'/r'}(m-1)^{1-(p'/r')}}.$$

Inequality (2.9) follows since the right-hand side is monotone increasing in $\alpha = |x_k|\,(\sum_{j\neq k}^m |x_j|^{r'})^{-1/r'}$ and $\max_{x\perp X_n}\alpha = E_r(e^k, X_n)$.

Because an explicit condition is available when $p = 2$ this estimate will be particularly useful when either r or s are 2. A very simple estimate is based directly on (2.1).

LEMMA 2.6. *Let* $1 \leqslant i_1 < \cdots < i_{n+1} \leqslant m \leqslant \infty$. *Then*

$$\min_{\{i_j\}}\sum_{k=1}^{n+1} E_p(e^{i_k}, X_n)^{p'} \geqslant 1. \tag{2.10}$$

Proof. Assume for simplicity $i_k = k$, $k = 1,\dots, n+1$. Clearly there is a $y \perp X_n$ of the form $y = \sum_{k=1}^{n+1}\alpha_k e^k$, $1 = \|y\|_{p'}^{p'} = \sum_{k=1}^{n+1}|\alpha_k|^{p'}$. From (2.1) we have $E_p(e^k, X_n) \geqslant |\alpha_k|$.

3. n-Widths of Diagonal Operators from l_1^m to l_p^m

We are interested in determining the n-widths and the corresponding optimal subspaces for approximating a diagonal operator $D = \operatorname{diag}(D_1, \ldots, D_m)$, where we assume $D_1 \geqslant D_2 \geqslant \cdots \geqslant D_m > 0$, though this could be modified if $m = \infty$. It is well known, and we reprove it shortly, that in the case at hand the Kolmogorov n-width equals the linear n-width defined as

$$\delta_n(D; l_1^m, l_p^m) = \min_{\operatorname{rank} P \leqslant n} \max_{\|x\|_1 \leqslant 1} \|Dx - Px\|_p, \tag{3.1}$$

where P is any matrix of dimensions $m \times m$; i.e., it is sufficient to consider linear approximants rather than best ones. Denote

$$\rho_j = E_p(De^j, X_n). \tag{3.2}$$

Lemma 3.1 (Hutton, Morrell, and Retherford [1]).

$$d_n(D; l_1^m, l_p^m) = \delta_n(D; l_1^m, l_p^m) = \min_{X_n} \max_{j=1,\ldots,m} E_p(De^j, X_n). \tag{3.3}$$

Proof. Since $\pm e^j, j = 1, \ldots, m$, are the extreme points of $\|x\|_1 \leqslant 1$ we have

$$\begin{aligned} d_n(D; l_1^m, l_p^m) &= \min_{X_n} \max_{\|x\|_1 \leqslant 1} \min_{y \in X_n} \|Dx - y\|_p \\ &= \min_{X_n} \max_{j=1,\ldots,m} E_p(De^j, X_n). \end{aligned}$$

Also clearly $d_n \leqslant \delta_n$. On the other hand, if X_n is an optimal subspace, $y^j \in X_n$ a best approximant to De^j, then for any x, $x = \sum_{j=1}^m \alpha_j e^j$, $\sum_{j=1}^m |\alpha_j| \leqslant 1$,

$$\min_{y \in Xn} \|Dx - y\|_p \leqslant \left\| \sum_{j=1}^m \alpha_j (De^j - y^j) \right\|_p \leqslant \sum_{j=1}^m |\alpha_j| \rho_j \leqslant \max_{i=1,\ldots,m} \rho_i,$$

so that the approximation may indeed prodeed linearly.

To connect these notions with the preceding section we observe

$$E_p(De^j, X_n) = \min_{y \in X_n} D_j \|e^j - y\|_p = D_j E_p(e^j, X_n). \tag{3.4}$$

The conditions on $E_p(e^j, X_n)$ formulated in Section 2 may therefore be translated directly into conditions on ρ_j and used to determine n-widths.

Theorem 3.2.

$$d_n(D; l_1^m, l_1^m) = D_{n+1}, \qquad n < m \leqslant \infty, \tag{3.5}$$

and a subspace is optimal if and only if it is spanned by $\{b^i\}_1^n$, $b^i = De^i + \sum_{j \neq i} \alpha_j e^j$ *with* $\sum_{j \neq i} |\alpha_j| \leqslant D_{n+1}$.

Proof. From Theorem 2.1 $\rho_j/D_j < 1$ for at most n indices j and hence $\max_{i=1,\dots,m} \rho_i \geqslant D_{n+1}$. On the other hand clearly $\|De^i - b^i\| \leqslant D_{n+1}$, $i \leqslant n$. For the converse let x be the best approximant to De^1. In particular,

$$D_{n+1} \geqslant \min_{\alpha} \|De^1 - \alpha x\|_1 = \min_{\alpha} \left[|D_1 - \alpha x_1| + |\alpha| \sum_{i>1} |x_i| \right].$$

This minimum cannot be achieved for $\alpha = 0$ hence it must be at $1 = \alpha = D_1/|x_1|$.

Remark. This is but a particular instance of $d_n(D; l_p^m, l_p^m) = D_{n+1}$ (cf. Pinkus [8]).

The remaining precise results are given in the next theorem. Denote $E^k = \operatorname{span}\{e^i\}_1^k$.

THEOREM 3.3. (a) (Sofman [11], Hutton, Morrell, and Rutherford [1]). *For each* $n < m \leqslant \infty$ *there exists a unique* k *such that*

$$D_{k+1} \leqslant d_n(D; l_1^m, l_2^m) = \left(\frac{k-n}{\sum_{i=1}^k D_i^{-2}} \right)^{1/2} < D_k, \tag{3.6}$$

and all optimal subspaces lie in E_k. *In fact* k *is the index* l *which yields*

$$\max_{l \geqslant n+1} \left(\frac{l-n}{\sum_{i=1}^l D_i^{-2}} \right)^{1/2} \tag{3.7}$$

(b) *For each* $p < \infty$, $m \leqslant \infty$ *there exist a unique* ρ *and* k *such that*

$$D_{k+1} \leqslant \rho < D_k, \qquad \sum_{i=1}^k \left[1 + \left(\left(\frac{D_i}{\rho} \right)^{p'} - 1 \right)^{-(p-1)} \right]^{-1} = 1. \tag{3.8}$$

With this ρ *and* k, $d_1(D; l_1^m, l_p^m) = \rho$ *and all optimal subspaces are contained in* E^k.

(c) (Pinkus [8]).

$$d_1(D; l_1^m, l_\infty^m) = (D_1^{-1} + D_2^{-1})^{-1}, \qquad m \leqslant \infty, \tag{3.9}$$

and the span of $x = (D_1, x_2, \dots, x_m)$ *is an optimal subspace if and only if* $x_2 = D_2 \geqslant x_i$, $i \geqslant 2$.

(d) (Pinkus [8]). *For all $p \leqslant \infty$, $m < \infty$,*

$$d_{m-1}(D; l_1^m, l_p^m) = \left(\sum_{i=1}^{m} D_i^{-p'}\right)^{-1/p'} \tag{3.10}$$

and the optimal subspaces are orthogonal to $(\pm D_1^{-1}, ..., \pm D_m^{-1})$ for some choice of signs.

Proof. In the cases $p = 2$; $n = 1$; $n = m - 1$, Theorem 2.2 and Corollary 2.4 yield necessary and sufficient conditions for the existence of X_n with $E_p(De^j, X_n) = \rho_j$, of the form

$$\sum_{i=1}^{m} f_{n,p}\left(\frac{\rho_i}{D_i}\right) = n,$$

where $f_{1,p}(x) = f_p(x)$, $f_{n,2}(x) = f_2(x) = 1 - x^2$, and $f_{m-1,p}(x) = 1 - x^{p'}$. In each case $f_{n,p}(x)$ is monotone decreasing and $f(1) = 0, f(0) = 1$. This characterization suffices to show that if $\rho = \min_{X_n} \max_{i=1,\dots,m} \rho_i$ then $\rho_i = \min(D_i, \rho)$, $i = 1, ..., m$. Indeed, since clearly $\rho_i \leqslant \rho$, $\rho_i \leqslant D_i$, suppose $\rho_j < \min(\rho, D_j)$. Then for $\delta > 0$ such that $\rho_j + \delta < \min(\rho, D_j)$ the monotonicity of f yields the existence of $\varepsilon > 0$ for which

$$f_{n,p}\left(\frac{\rho_j + \delta}{D_j}\right) + \sum_{i \neq j} f_{n,p}\left(\frac{\rho_i - \varepsilon}{D_i}\right) = n.$$

Now Theorem 2.2 and Corollary 2.4 ensure the existence of X_n with these distances from the axes, contradicting the supposed optimality of ρ. Since $f(1) = 0$ it follows that if $D_{k+1} \leqslant \rho < D_k$ then

$$\sum_{i=1}^{k} f_{n,p}\left(\frac{\rho}{D_i}\right) = n.$$

Conversely, because $\sum_{i=1}^{m} f_{n,p}(\min(\rho, D_i)/D_i)$ is a monotone decreasing function of ρ, this equation determines ρ uniquely. Note that one must have $k > n$ because $f(\rho/D_i) \leqslant 1$. These remarks establish (3.8), (3.10). In case $p = 2$ we get the existence of a unique k such that

$$\sum_{i=1}^{k} \left[1 - \left(\frac{\rho}{D_i}\right)^2\right] = n,$$

establishing (3.6). To derive (3.7) we have to show that for all l

$$\frac{l-1}{\sum_{i=1}^{k} D_i^{-2}} \leqslant \rho^2 \qquad \text{or} \qquad \sum_{i=1}^{l} \left[1 - \left(\frac{\rho}{D_i}\right)^2\right] \leqslant n.$$

This follows immediately from $D_1 \geqslant \cdots \geqslant D_k \geqslant \rho \geqslant D_{k+1} \geqslant \cdots \geqslant D_m$ and $\sum_{i=1}^{k} [1 - (\rho/D_i)^2] = n$.

As for the fact that the optimal subspace X_n^* is contained in E^k, recall that we proved $E_p(De^j, X_n^*) = D_j, j > k$. This implies $X_n^* \perp e^j$ since

$$E_p(De^j, X_n^*) = D_j \max_{x \perp X_n^*} \frac{|x_j|}{\|x\|_{p'}}$$

and this can equal D_j only if $e^j \perp X_n^*$.

Turning finally to (c) we have from Theorem 2.3(b) and arguments similar to the previous ones that

$$1 = \rho \min_{i \neq j} (D_i^{-1} + D_j^{-1}) = \rho(D_1^{-1} + D_2^{-1}).$$

Now $\rho_1/D_1 = D_2/(D_1 + D_2)$, $\rho_2/D_2 = D_1/(D_1 + D_2)$ with $x_1 = D_1$ are possible only if $x_2 = D_2 \geqslant x_i$, $i > 2$. Conversely Theorem 2.3(b) shows that if $x_i \leqslant D_2$ then the value ρ_i such that $\rho_i D_i^{-1} = D_1(D_1 + x_i)^{-1} \geqslant D_1(D_1 + D_2)$ is the distance to De^i.

In all these cases if $D_{k+1} \leqslant d_n < D_k$ then there exists an optimal subspace, contained in E^k, equidistant from the first k axes, with distance d_n. It seems reasonable to conjecture that this is true in general. It is clear that in any case the optimal subspace must be equidistant from at least $n + 1$ axes.

It is possible to use Lemmas 2.5, 2.6 to obtain inequalities for the n-widths. We mention

PROPOSITION 3.4. *Denote*

$$d = \max_{\{i_j\}} \left[\sum_{k=1}^{n+1} D_{i_k}^{-p'} \right]^{-1/p'} = \left[\sum_{i=1}^{n+1} D_i^{-p'} \right]^{-1/p'} \tag{3.11}$$

Then

$$d \leqslant d_n(D; l_1^m, l_p^m) \leqslant \max(D_{n+2}, d). \tag{3.12}$$

In particular if $d \geqslant D_{n+2}$ then equality prevails.

Proof. The lower bound follows from Lemma 2.6. To show the upper bound choose $X_n^\perp = \operatorname{span}\{e^{n+2}, \ldots, e^m, u\}$ with $u = (D_1^{-1}, \ldots, D_{n+1}^{-1}, 0, \ldots, 0)$. Then from (2.1) we have

$$\begin{aligned} E_p(De^i, X_n) &= d, & 1 \leqslant i \leqslant n+1, \\ &= D_i, & n+2 \leqslant i \leqslant m, \end{aligned}$$

and therefore $\max E_p(De, X_n) = \max(d, D_{n+2})$. See Pinkus [8] for a generalization of this result.

4. n-Widths of Octahedra

We now specialize to the case $D = I$. Summarizing our previous results we have

$$\begin{aligned}
d_n(I; l_1^m, l_1^m) &= 1, && 1 \leqslant n < m \leqslant \infty,\\
d_n(I; l_1^m, l_2^m) &= \sqrt{1-(n/m)}, && 1 \leqslant n < m \leqslant \infty,\\
d_1(I; l_1^m, l_p^m) &= [1+(m-1)^{-1/(p-1)}]^{-(1-(1/p))}, && 1 < m \leqslant \infty,\quad p \leqslant \infty,\\
d_{m-1}(I; l_1^m, l_p^m) &= m^{-(1-(1/p))}, && p \leqslant \infty. \qquad (4.1)
\end{aligned}$$

On the basis of the explicit result for l_2, Lemma 2.5 yields the estimate

$$d_n(I; l_1^m, l_p^m) \geqslant \left\{1 + \left(\frac{\sqrt{n}}{(m-1)^{1/p}} \frac{\sqrt{1-(1/m)}}{\sqrt{1-(n/m)}}\right)^{p'}\right\}^{-1/p'}, \qquad p \geqslant 2, \qquad (4.2)$$

$1/p + 1/p' = 1$ of which Pinkus [8] gives the case $p = \infty$. This yields asymptotics for large m and large n as follows

(a) $d_n(I; l_1^m, l_p^m) \approx 1$ if $m \to \infty$ and $n^p/m^2 \to 0$, $2 < p < \infty$;

(b) $d_n(I; l_1^m, l_p^m) \geqslant m^{1/p} \sqrt{1-\alpha}/\sqrt{n}$ if $m \to \infty$ and $n^p/m^2 \to \infty$, $2 < p \leqslant \infty$, where $\alpha = \lim_{m\to\infty}(n/m)$, $2 < p \leqslant \infty$.

Note that there is equality in (4.2) for $p = 2$ and all n, and for $n = 1$, $m-1$, and all p. In the remainder of this section we investigate the sharpness of the exact and asymptotic lower bounds.

An examination of the proof of Lemma 2.5, as performed in [7], reveals the following conditions for equality to hold in (4.2).

Theorem 4.1. *The lower bound* (4.2) *is attained for $p > 2$ if and only if there exists a rank n projection P such that*

$$P_{kk} = \frac{n}{m}, \qquad |P_{ik}|^2 = \left(1 - \frac{n}{m}\right) \frac{n}{m(m-1)}, \quad i \neq k;\, i, k = 1,\ldots, m.$$

In that case, $X_n = PR^m$ is an optimal subspace for $d_n(I; l_1^m, l_p^m)$ all $p \geqslant 2$, while $X_n^\perp$ is optimal for $d_{m-n}(I; l_1^m; l_p^m)$, $p \geqslant 2$ again with equality in (4.2).

As a result of this theorem the exact lower bound cannot be attained for all values of n and m. Moreover these restrictions may change when the space R^m is embedded in C^m, e.g., with $n = 2$ there is equality for C^4 but not for R^4, as pointed out by Pinkus [8]. More results on conditions and cases of equality are presented in [7]. Here we conclude by bringing evidence to support the conjecture that the asymptotic lower bound (b) is sharp.

Conjecture. For m and n such that $\lim_{m\to\infty} mn^{-p/2} = 0$,

$$d_n(I; l_1^m, l_p^m) \approx \frac{m^{1/p}}{\sqrt{n}} \sqrt{1-\alpha}, \qquad 2 < p \leqslant \infty,$$

where $\alpha = \lim_{m\to\infty} n/m$.

Evidence. (1) $n \approx m^{2/3}$. Lemmens and Seidel [4, Theorem 3.2] prove that (4.2) is achieved for $n = m/((m-1)^{1/3}+1)$ and $m = k^{3l}+1$, k prime, l arbitrary.

(2) For, $n = \sqrt{m}$, Maiorov [6], see [7], shows

$$d_n(I; l_1^m, l_\infty^m) \approx 1/\sqrt{n}.$$

(3) For $n = \frac{1}{3}m$, $m = 2^{k-1}(2^k - 1)$,

$$d_n(I; l_1^m, l_p^m) \approx \frac{m^{1/p}}{\sqrt{m}} \sqrt{2/3}.$$

Similarly for $n = \frac{2}{3}m$, $m = 2^{k-1}(2^k + 1)$,

$$d_n(I; l_1^m, l_p^m) \approx \frac{m^{1/p}}{\sqrt{n}} \sqrt{1/3},$$

Seidel [10], see [7].

(4) For $n = \frac{1}{2}m$, there is equality already in (4.2) if there exists a symmetric conference matrix, see [7], or, in the complex case, if there exists an $m \times m$ real skew Hadamard matrix S. To prove the latter statement, given such an S (which has entries ± 1, $SS^{\mathrm{T}} = mI$, $S + S^{\mathrm{T}} = 2I$) form $A = 1 - i(m-1)^{-1/2} I + i(m-1)^{-1/2} S$. Then rank $A = n$ since $A + \bar{A} = 2I$ and thus rank A + rank $\bar{A} \geqslant 2n$, while $AA^{\mathrm{T}} = 0$ implies rank $A \leqslant n$. The matrix A has all the properties required by Theorem 4.1.

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