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Fixed Points, Separation, and Induced Topologies for Fuzzy Sets

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1. INTRODUCTION

The theory of fuzzy sets was initiated by L. Zadeh [13] as an attempt to develop a mathematically precise framework in which to treat systems or phenomena which, due to intrinsic indefiniteness—as distinguished from mere statistical variation—cannot themselves be characterized precisely. Interest has been aroused in the application of these ideas to such fields as pattern recognition, artificial intelligence, optimization, and decision theory [1, 3, 5, 6, 10–12].

In this paper, our purpose is three-fold. First, we consider how a topology for a set $\mathscr{X}$ may give rise to an "induced fuzzy topology" for $\mathscr{X}$, thus characterizing the fuzzy subsets of $\mathscr{X}$ which may naturally be considered "open" and at the same time furnishing a concrete class of examples of the "fuzzy topologies" defined by C. L. Chang [4]. Various properties of fuzzy sets in the presence of an induced fuzzy topology are studied.

Next, we use these ideas to prove a generalization to fuzzy sets of the Schauder–Tychonoff Theorem, which asserts that every continuous self-mapping of a compact convex subset of a locally convex linear topological space has a fixed point.

Finally, after noting that the "Fuzzy Separation Theorem" of Zadeh [13] admits of a counterexample, we again employ the notion of "induced fuzzy topology" to prove a revised version of that theorem for appropriate fuzzy subsets of a linear topological space.

2. BASIC DEFINITIONS AND PROPERTIES

Let $\mathscr{X}$ be an arbitrary (nonempty) set. A *fuzzy set* (*in* $\mathscr{X}$) is a function with domain $\mathscr{X}$ and values in [0, 1]. (In particular, if S is an ordinary ("crisp") subset of $\mathscr{X}$, its characteristic function χ_S is a fuzzy set.) If A is a fuzzy set and

$x \in \mathscr{X}$, the function-value $A(x)$ is called the *grade of membership of x in A*; the number $\sup_{x\in\mathscr{X}} A(x)$ is called the *maximal grade in A*. The fuzzy set A' defined by $A'(x) = 1 - A(x)$ is called the *complement of A*. A is called *empty* if $A = \chi_\varnothing$.

Let A and B be fuzzy sets in $\mathscr{X}$. We write $A \subseteq B$ (and say B *includes* A) if $A(x) \leqslant B(x)$ for each $x \in \mathscr{X}$. For any family $\{A_\lambda\}_{\lambda\in\Lambda}$ of fuzzy sets in $\mathscr{X}$, we define the *intersection* $\bigcap_{\lambda\in\Lambda} A_\lambda$ and *union* $\bigcup_{\lambda\in\Lambda} A_\lambda$, respectively, by the formulas

$$\left[\bigcap_{\lambda\in\Lambda} A_\lambda\right](x) = \inf_{\lambda\in\Lambda} A_\lambda(x) \qquad \text{and} \qquad \left[\bigcup_{\lambda\in\Lambda} A_\lambda\right](x) = \sup_{\lambda\in\Lambda} A_\lambda(x).$$

Suppose T is a mapping of a set $\mathscr{X}$ into a set $\mathscr{Y}$ and A is a fuzzy set in $\mathscr{X}$. The fuzzy set TA in $\mathscr{Y}$ is defined by

$$[TA](y) = \begin{cases} \sup\limits_{x\in T^{-1}\{y\}} A(x), & \text{if} \quad T^{-1}\{y\} \neq \varnothing, \\ 0, & \text{if} \quad T^{-1}\{y\} = \varnothing. \end{cases}$$

If B is a fuzzy set in $\mathscr{Y}$, the fuzzy set $T^{-1}B$ in $\mathscr{X}$ is defined by

$$[T^{-1}B](x) = B(T(x)).$$

A family $\mathscr{F}$ of fuzzy sets in $\mathscr{X}$ is called a *fuzzy topology for $\mathscr{X}$* (and the pair $(\mathscr{X}, \mathscr{F})$ a *fuzzy topological space*, or *f.t.s.*) if (1) $\chi_\mathscr{X} \in \mathscr{F}$ and $\chi_\varnothing \in \mathscr{F}$; (2) $\bigcup_{\lambda\in\Lambda} A_\lambda \in \mathscr{F}$ whenever each $A_\lambda \in \mathscr{F}$ $(\lambda \in \Lambda)$; and (3) $A \cap B \in \mathscr{F}$ whenever $A, B \in \mathscr{F}$. The elements of $\mathscr{F}$ are called *open* and their complements *closed.* A mapping T from a f.t.s. $(\mathscr{X}_1, \mathscr{F}_1)$ into a f.t.s. $(\mathscr{X}_2, \mathscr{F}_2)$ is called *F-continuous if* $T^{-1}A \in \mathscr{F}_1$ whenever $A \in \mathscr{F}_2$.

We introduce the following terminology and notation. Let A be a fuzzy set in an arbitrary set $\mathscr{X}$. The *weak* (respectively, *strong*) *r-cut of A*, denoted $\omega_r(A)$ (respectively, $\sigma_r(A)$), is defined (for any $r \in \mathbf{R}$) by

$$\omega_r(A) = \{x \in \mathscr{X} \mid A(x) \geqslant r\}$$

(respectively, $\sigma_r(A) = \{x \in \mathscr{X} \mid A(x) > r\}$). The cut $\sigma_0(A)$ is called the *support* of A and is denoted supp A. A is called *constant* (*with grade c*) if A is a constant function with value c.

Let $\mathscr{X}$ be a linear space (i.e., a vector space over $\mathbf{R}$ or $\mathbf{C}$). A fuzzy set A in $\mathscr{X}$ is called *convex* if $A(tx_1 + (1-t)x_2) \geqslant \min\{A(x_1), A(x_2)\}$ whenever $x_1, x_2 \in \mathscr{X}$ and $t \in [0, 1]$. Equivalently, A is convex if $\omega_r(A)$ is convex for each $r > 0$, or, again, if $\sigma_r(A)$ is convex for each $r > 0$. It is readily verified that $\bigcap_{\lambda\in\Lambda} A_\lambda$ is convex whenever each A_λ $(\lambda \in \Lambda)$ is a convex fuzzy set in $\mathscr{X}$.

If $\mathscr{X}$ is a linear topological space, a fuzzy set A in $\mathscr{X}$ is called *bounded* if, for each $r > 0$, $\omega_r(A)$ is bounded. Equivalently, A is bounded if, for each $r > 0$, $\sigma_r(A)$ is bounded.

For a more detailed account of the concepts outlined above, the reader is referred to [4, 13].

3. Induced Fuzzy Topologies

Throughout this section, $(\mathscr{X}, \mathscr{T})$ will denote a (crisp) topological space.

Let A be a fuzzy set in $\mathscr{X}$. As motivation for our definition of "open" fuzzy set in $\mathscr{X}$, we employ the heuristic principle that A is "closed" if, whenever $\{x_\nu, \nu \in \mathscr{D}\}$ is a net (cf. [8]) in $(\mathscr{X}, \mathscr{T})$ converging to a point $x \in \mathscr{X}$, then x is "at least as much in A as the x_ν ultimately are," i.e., $A(x) \geqslant \limsup_{\nu\in\mathscr{D}} A(x_\nu)$. But this is precisely the condition that the function $A: \mathscr{X} \to \mathbf{R}$ be upper semicontinuous, i.e., that $1 - A$ be lower semicontinuous (cf. [8]). Thus we are led to the following:

Definition 3.1. The *induced fuzzy topology on* $(\mathscr{X}, \mathscr{T})$, denoted $F(\mathscr{T})$, is the collection of all lower semicontinuous fuzzy sets in $\mathscr{X}$.

Proposition 3.2. *$F(\mathscr{T})$ is a fuzzy topology for $\mathscr{X}$, and so $(\mathscr{X}, F(\mathscr{T}))$ is a* f.t.s.

Proof (cf. [8]). Clearly, the constant fuzzy sets $\chi_{\mathscr{X}}$ and $\chi_{\varnothing}$ are lower semicontinuous. Furthermore, since the supremum of an arbitrary family and the infimum of a finite family of lower semicontinuous functions mapping $\mathscr{X}$ into $[0, 1]$ are each lower semicontinuous, it follows that unions of arbitrarily many and intersections of finitely many elements of $F(\mathscr{T})$ are themselves in $F(\mathscr{T})$. Thus $F(\mathscr{T})$ is a fuzzy topology for $\mathscr{X}$.

Henceforth, throughout this paper, topological terms such as "open," when applied to fuzzy sets in a specified topological space, will refer to the induced fuzzy topology unless there is an indication to the contrary.

Since a function $f: \mathscr{X} \to \mathbf{R}$ is lower semicontinuous (respectively, upper semicontinuous) iff, for each $r \in \mathbf{R}$, $\{x \in \mathscr{X} \mid f(x) \leqslant r\}$ is closed (respectively, $\{x \in \mathscr{X} \mid f(x) \geqslant r\}$ is closed), we obtain at once

Proposition 3.3. *A fuzzy set A in $\mathscr{X}$ is open (respectively, closed) iff, for each $r > 0$, $\sigma_r(A)$ is open (respectively, $\omega_r(A)$ is closed).*

Observe that, if S is a $\mathscr{T}$-open set, then χ_S is $F(\mathscr{T})$-open.

F-continuous mappings between induced fuzzy topological spaces may be simply characterized:

PROPOSITION 3.4. *A mapping* $T: (\mathscr{X}, F(\mathscr{T})) \to (\mathscr{Y}, F(\mathscr{U}))$ *is F-continuous iff* $T:(\mathscr{X}, \mathscr{T}) \to (\mathscr{Y}, \mathscr{U})$ *is continuous.*

Proof. Suppose T is F-continuous. If U is a $\mathscr{U}$-open set, then

$$T^{-1}U = \{x \in \mathscr{X} \mid \chi_U(T(x)) = 1\} = \{x \in \mathscr{X} \mid [T^{-1}\chi_U](x) > \tfrac{1}{2}\} = \sigma_{1/2}(T^{-1}\chi_U).$$

But, by (3.3), the latter set is $\mathscr{T}$-open, since χ_U is $F(\mathscr{U})$-open and T is F-continuous. Thus T is continuous.

Conversely, suppose T is continuous and B an open fuzzy set in $\mathscr{Y}$. For any $r > 0$,

$$\sigma_r(T^{-1}B) = \{x \in \mathscr{X} \mid B(T(x)) > r\} = T^{-1}(B^{-1}(r, \infty)).$$

But the latter set is open, since B is lower semicontinuous and T is continuous. Hence, by (3.3), $T^{-1}B$ is $F(\mathscr{T})$-open. Thus T is F-continuous.

Turning our attention now to "compact" fuzzy sets, we must first point out that the approach to compactness initiated in [4] will not meet our needs. There, an arbitrary f.t.s. $(\mathscr{Y}, \mathscr{F})$ is called compact if every open cover of $\chi_{\mathscr{Y}}$ has a finite subcover—an open cover of a fuzzy set A being defined as a collection of $\mathscr{F}$-open fuzzy sets whose union includes A. Thus, compactness is defined, in effect, only for the crisp set $\mathscr{Y}$. But even in this sense, no induced f.t.s. (not even if induced by a compact topological space) is compact. In fact, if $(\mathscr{Y}, \mathscr{F})$ is *any* f.t.s. rich enough to contain a sequence $\{A_n\}_{n=1}^{\infty}$ of open constant fuzzy sets whose grades converge in a strictly increasing manner to 1, then $(\mathscr{Y}, \mathscr{F})$ cannot be compact, since the family $\{A_n\}$ would constitute an open cover of $\chi_{\mathscr{Y}}$ having no finite subcover.

We therefore introduce

DEFINITION 3.5. A fuzzy set A in $\mathscr{X}$ is *compact* if, for each $r > 0$, $\omega_r(A)$ is compact.

PROPOSITION 3.6. *A fuzzy set A in* $\mathbf{R}^n$ *(endowed with the usual topology) is compact iff it is closed and bounded.*

Proof. The result follows at once from the fact that, for each $r > 0$, $\omega_r(A)$ is compact iff it is closed and bounded.

PROPOSITION 3.7. *Suppose T is a continuous mapping of* $(\mathscr{X}, \mathscr{T})$ *into a Hausdorff space* $(\mathscr{Y}, \mathscr{U})$. *Then, for any compact fuzzy set A in* $\mathscr{X}$, *TA is compact.*

Proof. Suppose $r > 0$. For each integer $i \geqslant 2$, put

$$W_i = \{y \in \mathscr{Y} \mid A(x) \geqslant r - r/i \text{ for some } x \in T^{-1}\{y\}\}.$$

Then

$$\omega_r(TA) = \bigcap_{i=2}^{\infty} W_i = \bigcap_{i=2}^{\infty} T\omega_{r-(r/i)}(A).$$

But the sets $T\omega_{r-r/i}(A)$ $(i = 2, 3,...)$ are compact, and $\mathscr{Y}$ is a Hausdorff space; thus $\omega_r(TA)$ is compact. It follows that TA is compact.

DEFINITION 3.8. A fuzzy set A in $\mathscr{X}$ is *connected* if, for each $r > 0$, $\sigma_r(A)$ is connected.

We point out that a fuzzy set in **R** is connected iff it is convex. Also, a fuzzy convex set in a linear topological space is connected. These results follow at once from their counterparts for crisp sets.

PROPOSITION 3.9. *Suppose T is a continuous mapping of $(\mathscr{X}, \mathscr{T})$ into a topological space $(\mathscr{Y}, \mathscr{U})$. Then, for any connected fuzzy set A in $\mathscr{X}$, TA is connected.*

Proof. Suppose $r > 0$. For each positive integer i, put

$$S_i = \{y \in \mathscr{Y} \mid A(x) > r + 1/i \text{ for some } x \in T^{-1}\{y\}\}.$$

Then

$$\sigma_r(TA) = \bigcup_{i=1}^{\infty} S_i = \bigcup_{i=1}^{\infty} T\sigma_{r+(1/i)}(A).$$

Note that the images $T\sigma_{r+1/i}(A)$ $(i = 1, 2, 3,...)$ form a nested nondecreasing sequence of connected sets. And, in particular, those (if any) that are nonempty must have a point in common. It follows (cf. [8]) that $\sigma_r(TA)$ is connected. Thus TA is connected.

We comment briefly on two alternative definitions of connectedness. In [2], it was suggested that A be called connected if each $\omega_r(A)$ is connected. However, one can show (by considering certain fuzzy sets in $\mathbf{R}^2$) that, with this definition, (3.9) is no longer valid. One might also attempt an "intrinsic" definition of connectedness for a fuzzy set A in an arbitrary f.t.s. $(\mathscr{Y}, \mathscr{F})$ by requiring (as in the traditional definition) that there not exist any $\mathscr{F}$-open fuzzy sets U_1, U_2 for which $A \subseteq U_1 \cup U_2$, $A \cap U_1 \neq \chi_\varnothing$, $A \cap U_2 \neq \chi_\varnothing$, and $A \cap U_1 \cap U_2 = \chi_\varnothing$. But, when $(\mathscr{Y}, \mathscr{F})$ is an induced f.t.s., one finds that A is connected in this sense iff supp A is connected. This type of connectedness would not appear to bring into play adequately the fuzzy structure of A. For example, connected fuzzy sets in the induced f.t.s. **R** would no longer necessarily be convex.

4. A Fixed-Point Theorem

In this section, we prove a generalization to fuzzy sets of the Schauder–Tychonoff Theorem.

In preparation for (4.1), we point out that, if A is a fuzzy set in a set $\mathscr{X}$, then the restricted function $A \mid \operatorname{supp} A$ is a fuzzy set in $\operatorname{supp} A$ ("essentially" the same as A).

THEOREM 4.1. *Suppose A is a (nonempty) compact convex fuzzy set in a locally convex linear topological (Hausdorff) space $\mathscr{L}$. Let T be* (1) *a continuous mapping of $\mathscr{L}$ into itself for which $TA \subseteq A$; or, alternatively,* (2) *a continuous mapping of* $\operatorname{supp} A$ *into itself for which* $T(A \mid \operatorname{supp} A) \subseteq A \mid \operatorname{supp} A$. *Then, there exists a point $x \in \operatorname{supp} A$ such that $T(x) = x$ and $A(x)$ is the maximal grade in A.*

Proof. Let M be the maximal grade in A. Put $W = \{t \in \mathscr{L} \mid A(t) = M\}$ (so that $W \subseteq \operatorname{supp} A$), and observe that $W = \bigcap_{i=2}^{\infty} \omega_{M-(M/i)}(A)$. Now, since $M > 0$, each cut $\omega_{M-(M/i)}(A)$ ($i = 2, 3, \ldots$) is compact and convex. Hence, W is compact and convex. Furthermore, each $\omega_{M-(M/i)}(A)$ is nonempty. Since these cuts form a nested sequence, it follows that W is nonempty. We next verify that $TW \subseteq W$. To this end, choose any $w \in W$; then, since either $TA \subseteq A$ or $T(A \mid \operatorname{supp} A) \subseteq A \mid \operatorname{supp} A$, we have $M \geqslant A(T(w)) \geqslant [TA](T(w))$ (or, respectively,

$$\geqslant [T(A \mid \operatorname{supp} A)]\,(T(w))) = \sup_{t \in T^{-1}\{T(w)\}} A(t) \geqslant A(w) = M,$$

so that $A(T(w)) = M$ and $T(w) \in W$. Thus $TW \subseteq W$. By means of the classical Schauder–Tychonoff Theorem [7, p. 456], we now conclude that there is a point $x \in W$ for which $T(x) = x$. This proves our result.

5. Separating Fuzzy Sets

Throughout this section, $\mathscr{L}$ will denote a non-trivial real linear topological space (cf. [9]). All linear functionals referred to will be assumed not identically zero. For brevity, we write "$A \leqslant K$ on S" whenever $A(x) \leqslant K$ for each x in a set S.

DEFINITION 5.1. Let A and B be fuzzy sets in $\mathscr{L}$. For any continuous linear functional f on $\mathscr{L}$ and any $r \in \mathbf{R}$, put

$$M(f, r) = \inf\{K \in \mathbf{R} \mid \text{either } A \leqslant K \text{ on } f^{-1}(-\infty, r] \text{ and } B \leqslant K \text{ on } f^{-1}[r, \infty); \\ \text{or } B \leqslant K \text{ on } f^{-1}(-\infty, r] \text{ and } A \leqslant K \text{ on } f^{-1}[r, \infty)\}.$$

(Observe that $f^{-1}(-\infty, r]$ and $f^{-1}[r, \infty)$ are merely the half-spaces determined by the closed hyperplane $f^{-1}\{r\}$.) We call

$$D(f, r) =_{\text{def.}} 1 - M(f, r)$$

the *degree of separation of* A *and* B *by* $f^{-1}\{r\}$. Put

$$\overline{M} = \inf\{M(f, r) \mid f \text{ is a continuous linear functional on } \mathscr{L} \text{ and } r \in \mathbf{R}\}.$$

We call $D =_{\text{def.}} 1 - \overline{M}$ the *degree of separability of* A *and* B.

In [13], where L. Zadeh introduced essentially the above definition for the case $\mathscr{L} = \mathbf{R}^n$, the following separation theorem was given:

Let A and B be bounded convex fuzzy sets in $\mathbf{R}^n$.

Let M be the maximal grade in $A \cap B$. Then $D = 1 - M$.

However, the following counterexample shows that this result cannot be correct even for (characteristic functions of) crisp sets: Let A be the open unit square $(0, 1) \times (0, 1)$ in $\mathbf{R}^2$ and B the closure of the reflection of A around the X-axis. Then $A \cap B = \varnothing$, so that (for χ_A and χ_B) $1 - M = 1$. However, $D = 0$. The source of the difficulty here (and in Zadeh's proof) can be traced to points in $\sigma_M(A)$ or $\sigma_M(B)$ which lie on an "optimally separating hyperplane" (in this case, the line $y = 0$).

Employing the induced fuzzy topology on $\mathscr{L}$, we recast Zadeh's assertion in the following form:

THEOREM 5.2. *Let* A *and* B *be (nonempty) convex open fuzzy sets in* $\mathscr{L}$ *satisfying at least one of the following conditions:*

(1) *A and B are bounded;*

(2) *neither* supp A *nor* supp B *equals* $\mathscr{L}$;

(3) *neither of A, B includes the other.*

Let M be the maximal grade in $A \cap B$. Then, there is a continuous linear functional f and an $r \in \mathbf{R}$ such that $D = D(f, r) = 1 - M$.

Proof (cf. [9, pp. 50, 118]). It suffices to prove (a) $\overline{M} \geqslant M$; and (b) for some f and r, $M \geqslant M(f, r)$.

To prove (a), let f, r, K be, respectively, any continuous linear functional on $\mathscr{L}$ and any real numbers such that either $A \leqslant K$ on $f^{-1}(-\infty, r]$ and $B \leqslant K$ on $f^{-1}[r, \infty)$, or $B \leqslant K$ on $f^{-1}(-\infty, r]$ and $A \leqslant K$ on $f^{-1}[r, \infty)$. For any $x \in \mathscr{L}$, certainly $x \in f^{-1}(-\infty, r]$ or $x \in f^{-1}[r, \infty)$; thus

$$K \geqslant \min\{A(x), B(x)\} = [A \cap B]\,(x).$$

It follows that $K \geqslant M$, and, in turn, that $\overline{M} \geqslant M$.

Let M_A and M_B be the maximal grades in A and B, respectively. Since $M = \sup_{t \in \mathscr{L}} \min\{A(t), B(t)\}$, we have $M \leqslant \min\{M_A, M_B\}$. To prove (b), we distinguish two cases:

Case I. Assume $M = \min\{M_A, M_B\}$ and (without loss of generality) $M_A \leqslant M_B$, so that $M = M_A$. Since A is nonempty, $M > 0$. If condition (1) holds, then $\sigma_M(B)$ is bounded and hence not equal to $\mathscr{L}$. If (2) holds, then $\sigma_M(B) \subseteq \text{supp } B \neq \mathscr{L}$. And if (3) holds, then again $\sigma_M(B) \neq \mathscr{L}$, for $\sigma_M(B) = \mathscr{L}$ would imply that, for each $x \in \mathscr{L}$, $B(x) > M_A$, whence $B \supseteq A$. Thus, since $\sigma_M(B)$ is open and convex, there exist a continuous linear functional f on $\mathscr{L}$ and an $r \in \mathbf{R}$ such that $\sigma_M(B) \subseteq f^{-1}(-\infty, r]$. However, f is necessarily an open mapping (indeed, let $f(z_0) \neq 0$; then f maps each segment $(x - \epsilon z_0, x + \epsilon z_0)$ onto $(f(x) - \epsilon f(z_0), f(x) + \epsilon f(z_0)))$. Hence $f\sigma_M(B)$ is an open subset of $(-\infty, r]$, and thus of $(-\infty, r)$. Hence, for each $x \in f^{-1}[r, \infty)$, we have $x \notin \sigma_M(B)$, i.e., $B(x) \leqslant M$. Clearly, also, $A \leqslant M$ on $f^{-1}(-\infty, r]$. It follows that $M(f, r) \leqslant M$.

Case II. Assume $M < \min\{M_A, M_B]$. The cuts $\sigma_M(A)$ and $\sigma_M(B)$ are open and convex (even if $M = 0$; for, in that case, each is the union of a nested sequence of open convex r-cuts with $r > 0$). Furthermore, they are nonempty, since $M < \min\{M_A, M_B\}$. Finally, they are disjoint, since otherwise $M \geqslant [A \cap B](x) = \min\{A(x), B(x)\} > M$ for some x in their intersection. It follows that there exist a continuous linear functional f on $\mathscr{L}$ and an $r \in \mathbf{R}$ such that $\sigma_M(A) \subseteq f^{-1}(-\infty, r]$ and $\sigma_M(B) \subseteq f^{-1}[r, \infty)$. As in Case I, we conclude that $\sigma_M(A) \subseteq f^{-1}(-\infty, r)$ and $\sigma_M(B) \subseteq f^{-1}(r, \infty)$ (compare [13, p. 353]). Thus $A \leqslant M$ on $f^{-1}[r, \infty)$ and $B \leqslant M$ on $f^{-1}(-\infty, r]$. It follows that $M(f, r) \leqslant M$, and the theorem is proved.

References

1. R. Bellman, R. Kalaba, and L. Zadeh, Abstraction and pattern classification, *J. Math. Anal. Appl.* **13** (1966), 1–7.
2. J. G. Brown, A note on fuzzy sets, *Inform. Control* **18** (1971), 32–39.
3. R. M. Capocelli and A. De Luca, Fuzzy sets and decision theory, preprint.
4. C. L. Chang, Fuzzy topological spaces, *J. Math. Anal. Appl.* **24** (1968), 182–190.
5. A. De Luca and S. Termini, Algorithmic aspects in complex systems analysis, *Scientia* **106** (1971), 659–671.
6. A. De Luca and S. Termini, A definition of a nonprobabilistic entropy in the setting of fuzzy sets theory, *Inform. Control* **20** (1972), 301–312.
7. N. Dunford and J. Schwartz, "Linear Operators—Part I: General Theory," Interscience, New York, 1958.
8. J. L. Kelley, "General Topology," D. Van Nostrand, Princeton, NJ, 1955.
9. J. L. Kelley, I. Namioka, *et al*, "Linear Topological Spaces," D. Van Nostrand, Princeton, NJ, 1963.

10. E. H. RUSPINI, A fast method for probabilistic and fuzzy cluster analysis using association measures, *in* "Proceedings of the Sixth Hawaii International Conference on System Sciences," Western Periodicals, North Hollywood, CA, 1973.
11. E. S. SANTOS, Fuzzy algorithms, *Inform. Control* **17** (1970), 326–339.
12. L. A. ZADEH, Fuzzy algorithms, *Inform. Control* **12** (1968), 94–102.
13. L. A. ZADEH, Fuzzy sets, *Inform. Control* **8** (1965), 338–353.