Subquadrangles of Generalized Quadrangles of Order \((q^2, q), q \text{ even}\)

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First, we survey the known generalized quadrangles of order \((q^2, q), q \text{ even}\), including a description of their known subquadrangles of order \(q\). Then, in the case of Tits’ generalized quadrangles, we completely classify the subquadrangles of order \(q\), while in the case of the flock quadrangles we classify the subquadrangles of order \(q\) which contain the base point. Finally, we determine the full automorphism groups of the Tits generalized quadrangles \(T_2(\epsilon)\) of order \(q\) and \(T_3(\Omega)\) of order \((q, q^2)\).

Key Words: generalized quadrangles; subquadrangles.

1. INTRODUCTION

A generalized quadrangle of order \((s, t)\) \((GQ(s, t))\) is an incidence structure \((\mathcal{P}, \mathcal{L}, I)\) in which \(\mathcal{P}\) and \(\mathcal{L}\) are disjoint (non-empty) sets of points and lines and for which \(I\) is a symmetric point-line incidence relation satisfying the following axioms:

(i) Each point is incident with \(1 + t\) lines \((t \geq 1)\) and two distinct points are incident with at most one line.

(ii) Each line is incident with \(1 + s\) points \((s \geq 1)\) and two distinct lines are incident with at most one point.

(iii) If \(x\) is a point and \(L\) is a line not incident with \(x\) then there is a unique pair \((y, M) \in \mathcal{P} \times \mathcal{L}\) such that \(x IM I y IL\).
If $s = t$ then the GQ has order $s$ and is denoted by $\text{GQ}(s)$. For an introduction to GQ, see [15].

It is immediate that the dual $L' , \mathcal{P} , I)$ of a GQ $S = (\mathcal{P} , L , I)$ of order $(s , t)$ is a GQ $S^d$ of order $(t , s)$. The GQ $S' = (\mathcal{P}' , L' , I')$ of order $(s' , t')$ is a subquadrangle of the GQ $S = (\mathcal{P} , L , I)$ of order $(s , t)$ if $\mathcal{P}' \subseteq \mathcal{P}$, $L' \subseteq L$ and if $I'$ is the restriction of $I$ to $(\mathcal{P}' \times L') \cup (L' \times \mathcal{P}')$.

Given points $x , y$ of $S$, we write $x \sim y$ if there is a line $L$ such that $x I L I y$. Dually, given lines $L , M$ of $S$ we write $L \sim M$ if there is a point $x$ such that $L I x I M$. Given a point $x$, we write $x^+ = \{ x \} \cup \{ y : y \sim x \}$, and given a line $L$, we write $L^\perp = \{ L \} \cup \{ M : M \sim L \}$. The trace of a pair of points $x , y$ (respectively, lines $L , M$) is the set $\{ x , y \}^\perp = x^\perp \cap y^\perp$ (respectively, the set $\{ L , M \}^\perp = L^\perp \cap M^\perp$). It is immediate that $| \{ x , y \}^\perp | = t + 1$ and $| \{ L , M \}^\perp | = s + 1$. More generally, if $A \subseteq \mathcal{P}$ or $A \subseteq L$ then $A^\perp = \bigcup \{ a^\perp : a \in A \}$, and the span of $x , y$ (respectively, $L , M$) is $\{ x , y \}^{\perp \perp} = \{ w \in \mathcal{P} : \exists z \in \{ x , y \}^\perp \}$ (respectively, $\{ L , M \}^{\perp \perp} = \{ K : K \in N^\perp \cap N \in \{ L , M \}^\perp \}$).

The pair $(x , y)$ of points of $S$ is regular if either $x \sim y$ or $x \not\sim y$ and $| \{ x , y \}^{\perp \perp} | = | \{ L , M \}^{\perp \perp} | = t + 1$. (Dually, the pair $(L , M)$ of lines of $S$ is regular if either $L \sim M$ or $L \not\sim M$ and $| \{ L , M \}^{\perp \perp} | = s + 1$.) The point $x$ is regular if $(x , y)$ is regular for every point $y \neq x$ and the line $L$ is regular if $(L , M)$ is regular for every line $M \neq L$.

In Section 2 we survey the known generalized quadrangles of order $(q^2, q)$, $q$ even, including a description of their known subquadrangles of order $q$. Using some preliminary lemmas proved in Section 3, we completely classify the subquadrangles of order $q$ in the Tits’ generalized quadrangles and the subquadrangles of order $q$ of the flock quadrangles which contain the base point (Section 4). In Section 5 we describe the full automorphism groups of the Tits generalized quadrangles $T_2(\mathcal{C})$ of order $q$ and $T_3(\mathcal{C})$ of order $(q , q^2)$.

We are mainly interested in the generalized quadrangles of order $(s^2 , s)$ (and their duals of order $(s , s^2)$) for $s$ even. However results which are valid also for $s$ odd are stated in that generality.

2. THE KNOWN EXAMPLES

In this section we describe all known families of GQ of order $(s^2 , s)$, $s$ even, including a description of their known subquadrangles of order $s$.

Every known GQ of order $(s^2 , s)$, $s$ even, is of the form either $T_3(\mathcal{C})^\perp$ for some ovoid $\mathcal{C} \subseteq \text{PG}(3 , q)$ (Section 2.2) or $GQ(\mathcal{C})^\perp$ for some $q$-clan $\mathcal{C}$ (Section 2.3). In each case, $s = q$. The classical GQ $H(3 , q^2)$ (Section 2.1) belongs to each of these two classes. We remark that there is a unique GQ of order $(4 , 2)$, see [15, 5.3.2]. Further, a GQ of order $(16 , 4)$ with a subquadrangle of order 4 is isomorphic to $Q(5 , 4)$, see [22, 7.1] and use the
uniqueness of the GQ of order 4 [15, 6.3] and the classification of ovoids in PG(3, 4) [8, 16.1.7].

It is noteworthy that every isomorphism between known subquadrangles \( S_1 \) and \( S_2 \) of order \( q \) of a known GQ \( S \) of order \((q^2, q)\), \( q \) even, extends to an automorphism of \( S \). In particular, \( (\text{Aut } S)_2 \cong \text{Aut } S \). In contrast, there are non-Desarguesian projective planes of square order with full collineation group having more than one orbit on Desarguesian Baer subplanes. The Hall planes are examples of this phenomenon.

2.1. The Classical GQ

Consider a non-singular quadric of Witt index 2, that is, of projective index 1, in PG(4, \( q \)) (respectively, PG(5, \( q \))). The points and lines of the quadric form a GQ \( Q(4, q) \) (respectively, \( Q^-(5, q) \)) of order \( q \) (respectively, \((q, q^2)\)) (see [15, 3.1.1; 21, 3.2]). Let \( H \) be a non-singular Hermitian variety in PG(3, \( q^2 \)). The points and lines of \( H \) form a GQ \( H(3, q^2) \) of order \((q^2, q)\) (see [15, 3.1.1; 21, 3.2]). The points of PG(3, \( q \)) together with the totally isotropic lines with respect to a symplectic polarity form a GQ \( W(q) \) of order \( q \).

**Theorem 1** [15, 3.2.1, 3.2.3; 21, 3.2]. The dual of \( W(q) \) is isomorphic to \( Q(4, q) \) and the dual of \( Q^-(5, q) \) is isomorphic to \( H(3, q^2) \).

Let PG(4, \( q \)) be a non-tangent hyperplane of \( Q^-(5, q) \). The points and lines of PG(4, \( q \)) \( \cap Q^-(5, q) \) form a subquadrangle \( Q(4, q) \) of order \( q \) [15, 3.5(a)].

2.2. Tits’ GQ

For each oval or ovoid \( \Omega \) in PG(\( d, q \)), for \( d = 2 \) or 3, there is a GQ \( T_d(\Omega) \) constructed as follows (see [4; 15, 3.1.2; 21, 4.2]). (For the definition of oval and ovoid, see [7].)

Let \( d = 2 \) (respectively, \( d = 3 \)) and let \( \Omega \) be an oval (respectively, an ovoid) of PG(\( d, q \)). Further, let PG(\( d, q \)) \( \cong \Sigma_\infty \) be embedded as a hyperplane in PG(\( d+1, q \)). Define an incidence structure with points (i) the points of PG(\( d+1, q \)) \( \setminus \Sigma_\infty \), (ii) the hyperplanes \( X \) of PG(\( d+1, q \)) for which \(|X \cap \Omega| = 1\) and (iii) one new symbol (\( \infty \)). The lines are (a) the lines of PG(\( d+1, q \)) which are not contained in \( \Sigma_\infty \) and which meet \( \Omega \) (necessarily in a unique point) and (b) the points of \( \Omega \). Incidence is defined as follows: a point of type (i) is incident only with the lines of type (a) which contain it, a point of type (ii) is incident only with all lines of type (a) contained in it and with the unique line of type (b) on it and the point of type (iii) is incident only with no line of type (a) and all lines of type (b). It is straightforward to verify that \( T_2(\Omega) \) is a GQ(\( q, q^2 \)) and \( T_2(\Omega) \) is a GQ(q).
Theorem 2 [15, 3.2.4; 21, 4.5]. The GQ $T_3(\Omega)$ is isomorphic to $Q(4, q)$ if and only if $\Omega$ is an irreducible conic and $T_3(\Omega)$ is isomorphic to $Q^-(5, q)$ if and only if $\Omega$ is an elliptic quadric.

Since there are two infinite families of ovoids of $\text{PG}(3, q)$ known, namely the elliptic quadrics for all $q$ and the Tits ovoids for $q = 2^h$, $h \geq 3$ odd, there is one infinite family of non-classical $T_3(\Omega)$ known. There are many classes of ovals in $\text{PG}(2, q)$ known, hence many classes of non-classical $T_3(\Omega)$ known.

Let $d = 3$ and let $\pi$ be a plane of $\Sigma_\infty$ secant to $\Omega$. Then $\pi$ meets $\Omega$ in the points of an oval $\mathcal{C}$. Let $\Sigma$ be a hyperplane of $\text{PG}(4, q)$ on $\pi$, distinct from $\Sigma_\infty$, and define a substructure $S(\Sigma)$ of $T_3(\Omega)$ as follows: points are (i) the points of $\Sigma \setminus \Sigma_\infty$, (ii) the hyperplanes $X$ of $\text{PG}(4, q)$ for which $X \cap \Omega$ is a point of $\mathcal{C}$ and (iii) $(\infty)$. The lines are (a) the lines of $\Sigma$ which are not contained in $\Sigma_\infty$ and which meet $\mathcal{C}$ (necessarily in a unique point) and (b) the lines of $\mathcal{C}$. Incidence is inherited from the incidence of $T_3(\Omega)$, Then $S(\Sigma)$ is a subquadrangle on $T_3(\Omega)$, and is isomorphic to $T_2(\mathcal{C})$ (see [15, 3.5(b)]).

2.3. Flock GQ, $q$ Even

We first review Kantor’s construction [9] of a generalized quadrangle from a 4-gonal family, see [15, 8.2]. Let $G$ be a group of order $s^t$ for some integers $s, t$. Let $\mathcal{F} = \{S_i : i = 0, \ldots, t\}$ be a family of $t+1$ subgroups of $G$, each of order $s$, such that for each $i = 0, \ldots, t$ there is a subgroup $S_i^*$ of $G$ of order $st$ containing $S_i$ and with the following properties:

(K1) $S_iS_j \cap S_k = \{1\}$ for distinct $i, j, k$ and

(K2) $S_i^* \cap S_j = \{1\}$ for distinct $i, j$.

Such a family $\mathcal{F}$ of subgroups is called a 4-gonal family for $G$. It is straightforward to verify that the following incidence structure is a generalized quadrangle of order $(s, t)$:

- **points**: (i) elements $g \in G$, (ii) cosets $S_i^*g$ for $g \in G$ and $i = 0, \ldots, t$, and (iii) a symbol $(\infty)$.

- **lines**: (a) cosets $S_i g$ for $g \in G$ and $i = 0, \ldots, t$ and (b) symbols $[S_i]$ for $i = 0, \ldots, t$.

A point $g$ of type (i) is incident with each line $S_i g$ for $i = 0, \ldots, t$. A point $S_i^*g$ of type (ii) is incident with the line $[S_i^*]$ and with each line $S_j h$ contained in $S_i^* g$. The point $(\infty)$ is incident with each line $[S_i]$ for $i = 0, \ldots, t$. There are no further incidences.

Let $\mathcal{G} = \{A_i : t \in GF(q)\}$ be a collection of $2 \times 2$ matrices with entries from $GF(q)$. Following Payne [13], we call $\mathcal{G}$ a $q$-clan if $A_i - A_j$ is anisotropic (that is, the equation $(x, y)(A_i - A_j)(x, y)^T = 0$ has only the trivial solution $(x, y) = (0, 0)$) for all $s, t \in GF(q)$ with $s \neq t$. 
Let \( \mathcal{G} = \{ (x, c, \beta); x, \beta \in GF(q)^2, c \in GF(q) \} \), with multiplication defined as

\[
(x, c, \beta)(x', c', \beta') = (x + x', c + c' + \beta \cdot x', \beta + \beta'),
\]

where, writing \( x = (x_1, x_2) \) and \( \beta = (\beta_1, \beta_2) \), we define \( \beta \cdot x = \beta_1 x_2 + \beta_2 x_1 \).

Let \( \mathcal{G} \) be a normalised \( q \)-clan, and define the following subgroups of \( \mathcal{G} \):

\[
A(x) = \{ (0, 0, \beta); \beta \in GF(q)^2 \}
\]

and

\[
A(t) = \{ (x, \sqrt{xA, x^2}, t^{1/2}x); x \in GF(q)^2 \}, \quad t \in GF(q).
\]

For each \( t \in GF(q) \cup \{ \infty \} \) we define \( A^*(t) = A(t) \cdot Z \) where \( Z = \{ (0, c, 0); c \in GF(q) \} \) is the centre of \( \mathcal{G} \). Then \( \mathcal{F} = \{ A(t); t \in GF(q) \cup \{ \infty \} \} \) is a 4-gonal family for \( \mathcal{G} \) [12; see 15, 10.4].

We have therefore outlined a process by which a \( q \)-clan \( \mathcal{G} \) gives rise to a 4-gonal family for the group \( \mathcal{G} \) above, and hence to a generalized quadrangle \( GQ(\mathcal{G}) \) of order \( (q^2, q) \). In particular, \( (\infty) \) is a regular point of \( GQ(\mathcal{G}) \) and the automorphism group \( Aut GQ(\mathcal{G}) \) of \( GQ(\mathcal{G}) \) is transitive on the set of points not collinear with \( \infty \) (see [15, 8.1, 8.2], since \( GQ(\mathcal{G}) \) is an elation GQ). In 1987 J. A. Thas proved a direct connection between this type of GQ and flocks of quadratic cones in PG(3, q), and for this reason we will in this paper refer to these GQ as Flock GQ.

There are four infinite families of Flock GQs, \( q \) even, known to date, see [20, 10.5]. They are the classical GQ \( H(3, q^2) \), the FTWKB GQ [1, 5, 9, 24], the Payne GQ [12] and the Subiaco GQ [3]. In addition there are six further examples, conjectured to belong to an infinite family [17, 18].

Payne and Maneri [14, Theorem 1] show that a GQ arising from a 4-gonal family for a group \( G \) can have a subquadrangle arising from a certain type of 4-gonal family for a subgroup of \( G \). These results guarantee the existence of certain subquadrangles in \( GQ(\mathcal{G}) \) when \( q \) is even, as in Payne [12].

Let \( \mathcal{G} = \{ A_x; t \in GF(q) \} \) be a \( q \)-clan with \( q \) even. If we write \( g_1(x) = \sqrt{xA, x^2} \) for each \( t \in GF(q) \) and \( x \in GF(q)^2 \), then for \( k \in GF(q) \setminus \{ 0 \} \) we have \( g_1(kx) = kg_1(x) \). It follows immediately that the map \( (x, c, \beta) \mapsto (kx, kc, kb) \) is an automorphism of \( \mathcal{G} \) which fixes setwise each subgroup \( A(t) \) of \( GF(q) \cup \{ \infty \} \). For \( x \in GF(q)^2 \setminus \{ 0 \} \), it is easy to verify that \( \mathcal{G}_x = \{ (x, z, \gamma); x, y, z \in GF(q) \} \) is a subgroup of \( \mathcal{G} \) of order \( q^4 \) with \( Z \leq \mathcal{G}_x \).

For each \( t \in GF(q) \cup \{ \infty \} \), put \( A_x(t) = \mathcal{G}_x \cap A(t) \); so that \( |A_x(t)| = q \). Then \( \{ A_x(t); t \in GF(q) \cup \{ \infty \} \} \) is a 4-gonal family for \( \mathcal{G}_x \). So for each \( x \in GF(q)^2 \setminus \{ 0 \} \), there exists a subquadrangle \( \mathcal{S}_x \) of \( GQ(\mathcal{G}) \) of order \( q \) which contains the points \( (x) \) and \( (0, 0, 0) \) of \( GQ(\mathcal{G}) \). Since \( \mathcal{G}_x = \mathcal{G}_y \) if and only
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if \( \lambda = \lambda \beta \) for some \( \lambda \in \text{GF}(q) \setminus \{0\} \), there are \( q+1 \) distinct subgroups \( \mathcal{G} \) and hence \( q+1 \) distinct subquadrangles \( S_\mathcal{G} \), which we associate with \( \mathcal{G} = (0, 1) \) and \( (1, s^{1/2}) \) for \( s \in \text{GF}(q) \).

We now investigate these subquadrangles further. Let \( \mathcal{G} \in \text{GF}(q)^3 \setminus \{0\} \).

Then

\[
A_\mathcal{G}(\infty) = \{(0, 0, 0); y \in \text{GF}(q)\}
\]

\[
A_\mathcal{G}(t) = \{(x, x, x); x^2 + x^3; x \in \text{GF}(q)\}
\]

Clearly, \( \mathcal{G} \) is a 3-dimensional vector space over \( \text{GF}(q) \), and the isomorphism \( (x, y, z) \rightarrow (x, y, z) \) of \( \mathcal{G} \) onto \( \text{PG}(2, q) \) gives the standard coordinates \( (x, y, z) \) for \( \mathcal{G} \). Under these standard coordinates, \( A_\mathcal{G}(\infty) \) is the point \((0, 1, 0)\) and \( A_\mathcal{G}(t) \) is the point \((1, t^{1/2}, \sqrt{\lambda_A t^3})\). For each \( \mathcal{G} \in \text{GF}(q)^3 \setminus \{0\} \), we therefore have a set of points

\[
\mathcal{E}_\mathcal{G} = \{(1, t^{1/2}, \sqrt{\lambda_A t^3}); t \in \text{GF}(q)\} \cup \{(0, 1, 0)\}.
\]

As in [12] (see [11, Sect. IV, Example 1]), each \( \mathcal{E}_\mathcal{G} \) is an oval in the \( \text{PG}(2, q) \) corresponding via the standard coordinates to \( \mathcal{G} \). Each such oval has nucleus \((0, 0, 1)\), which are the standard coordinates for \( Z \). Further, the standard isomorphism gives a correspondence between the 4-gonal family \( \{A_\mathcal{G}(t); t \in \text{GF}(q) \cup \{\infty\}\} \) for \( \mathcal{G} \) and the 4-gonal family in [11, Sect. IV, Example 1], showing that the generalized quadrangle \( S_\mathcal{G} \) is isomorphic to the generalized quadrangle \( T_2(\mathcal{E}_\mathcal{G}) \).

Thus Payne [12] has constructed \( q+1 \) subquadrangles \( \{S_\mathcal{G}; s \in \text{GF}(q) \cup \{\infty\}\} \) of \( \text{GQ}([\mathcal{G}]) \), where \( S_\mathcal{G} \) is isomorphic to \( T_2(\mathcal{E}_{(0, 1)}) \) and, for \( s \in \text{GF}(q) \), \( S_s \) is isomorphic to \( T_2(\mathcal{E}_{(1, s^{1/2})}) \).

The collection of \( q+1 \) ovals \( \{\mathcal{E}_s; s \in \text{GF}(q) \cup \{\infty\}\} \), where \( \mathcal{E}_\infty = \mathcal{E}_{(0, 1)} \), and, for \( s \in \text{GF}(q) \), \( \mathcal{E}_s = \mathcal{E}_{(1, s^{1/2})} \), has come to be called a herd of ovals. Cherowitzo, Pentinari and Royle [3] have given a self-contained definition of herd and proved directly that a (suitably normalised) \( q \)-clan \( \mathcal{C} \) gives rise to a herd \( \mathcal{H}(\mathcal{C}) \) and conversely, without using the associated GQ([\mathcal{G}]).

3. PRELIMINARY RESULTS

Lemma 3. Let \( S' = (\mathcal{P}', \mathcal{P}', \mathcal{I}') \) and \( S'' = (\mathcal{P}'', \mathcal{P}'', \mathcal{I}'') \) be subquadrangles of order \((s', t') \) of a GQ \( S = (\mathcal{P}, \mathcal{P}, \mathcal{I}) \) of order \((s, t), s > 1 \). If there exist non-concurrent lines \( L, M \in \mathcal{P}' \cap \mathcal{P}'' \), then \( t' = s, t = s^2 \). \( (L, M) \) is a regular pair of lines in each of \( S, S' \) and \( S'' \) and \( S' \cap S'' \) is the subquadrangle of order \((s, 1) \) with lines \( \{L, M\} \cap \{L, M\} \).

Proof. Consider \( \hat{S} = (\mathcal{P}' \cap \mathcal{P}'', \mathcal{P}' \cap \mathcal{P}'', \hat{I}) \) where \( \hat{I} \) is the restriction of \( I \) to \((\mathcal{P}' \cap \mathcal{P}'') \times (\mathcal{P}' \cap \mathcal{P}'') \). Then \( \hat{S} = S' \cap S'' \).
and is a substructure of $S$. If $x, y \in \mathcal{P} \cap \mathcal{P}'$ and $L \in \mathcal{L}$ satisfy $x I L I y$ then $L \in \mathcal{L}' \cap \mathcal{L}''$. Further, each line in $\mathcal{L}' \cap \mathcal{L}''$ is incident with $s+1$ points in $\mathcal{P} \cap \mathcal{P}'$; so we can apply [15, 2.3.1]. Since $s > 1$ and $\hat{S}$ contains a non-concurrent pair of lines, it follows that $\hat{S}$ is a subquadrangle of order $(s, \hat{t})$ of $S$. But then $\hat{S}$ is also a subquadrangle of $S'$, and by [15, 2.2.2] we have $\hat{t} = 1$, $t' = s$ and $t = s^2$. Thus $\hat{S}' \cap \mathcal{P}''$ is a subquadrangle of order $(s, 1)$, with lines $\{L, M\} \perp \{L, M\} \perp \hat{L}$. Since $|\{L, M\} \perp \hat{L}| = s+1$, it follows that $(L, M)$ is a regular pair of lines in each of $S'$, $S''$ and $S$.

**Lemma 4.** Let $L, M$ be non-concurrent lines of a generalized quadrangle $S$ of order $(s, t)$, $s > 1$. If $t' \neq s^2$ then $L, M$ lie together in at most one subquadrangle of order $s$, while if $t = s^2$ then $(L, M)$ is regular in $S$ and $L, M$ lie together in at most $s+1$ subquadrangles of order $s$.

**Proof.** If $L, M$ lie in at least two subquadrangles of order $s$ then by Lemma 3 we have $t = s^2$, $(L, M)$ is regular in $S$ and the subquadrangles of order $s$ on $L, M$ meet pairwise in the subquadrangle $\hat{S} = (\hat{\mathcal{P}}, \mathcal{P}, \hat{I})$ of order $(s, 1)$ with lines $\{L, M\} \perp \{L, M\} \perp \hat{L}$. Now each of the $(s+1)(s^3-s)$ points of $\mathcal{P} \setminus \hat{\mathcal{P}}$ lies in at most one subquadrangle of order $s$ on $L, M$, and such a subquadrangle contains $(s+1)(s^2-s)$ points of $\mathcal{P} \setminus \hat{\mathcal{P}}$. The result follows.

**Lemma 5 (The Dual).** Let $x, y$ be non-collinear points of a generalized quadrangle $S$ of order $(s, t)$, $t > 1$. If $s \neq t^2$ then $x, y$ lie together in at most one subquadrangle of order $t$, while if $s = t^2$ then $(x, y)$ is regular in $S$ and $x, y$ lie together in at most $t+1$ subquadrangles of order $t$.

We remark that essentially these arguments were used by Payne and Maneri [14] in the special case of the FTWKB GQ in an important early paper studying subquadrangles of order $q$ in $GQ$ of order $(q^2, q)$. The related problem of covering the points of a generalized quadrangle by the smallest possible number of subquadrangles was investigated in [23].

4. CLASSIFICATIONS

4.1. Classical GQ

We include the next result for the sake of completeness.

**Theorem 6.** Every subquadrangle of order $q$ of $Q^{-}(5, q)$ is isomorphic to $Q(4, q)$ and arises as in Subsection 2.1.

**Proof.** Let $L, M$ be non-concurrent lines of $Q^{-}(5, q)$ and consider the 3-dimensional space $\langle L, M' \rangle$. Each of the 4-dimensional spaces on $\langle L, M' \rangle$ meets $Q^{-}(5, q)$ in a $Q(4, q)$; which is a subquadrangle of order $q$ of $Q^{-}(5, q)$.
arising as in Section 2.1. By Lemma 4, these are the only subquadrangles of order \( q \) of \( Q(5, q) \) on \( L, M \). Since every subquadrangle of order \( q \) contains a pair of non-concurrent lines, the result follows.

4.2. Non-classical Tits’ \( GQ \)

In Section 2.2 we showed that a generalized quadrangle of the form \( T(3, 0) \) has subquadrangles \( S(\Sigma) \) for suitable 3-dimensional subspaces \( \Sigma \). We now show that these are the only subquadrangles of order \( q \) of a non-classical \( T(3, 0) \).

**Theorem 7.** Let \( q \) be even and let \( \Omega \) be an ovoid of \( PG(3, q) \), which is not an elliptic quadric, with associated generalized quadrangle \( T(3, 0) \). Let \( S \) be a subquadrangle of order \( q \) of \( T(3, 0) \). Then \( S = S(\Sigma) \) for some 3-dimensional space \( \Sigma \) meeting \( \Sigma_{\infty} \) in a secant plane \( \pi \) to \( \Omega \).

**Proof.** We show first that \((\infty) \in S\), arguing by contradiction. Suppose \((\infty) \notin S\). Then no line on \((\infty)\) lies in \( S \); so all lines of \( S \) are lines of \( T(3, 0) \) of type (a). Let \( L \) be a fixed line of \( S \). For a line \( M \) of \( S \) not meeting \( L \), let \( \Sigma_{M} = \langle L, M \rangle \). Then \( \Sigma_{M} \cap \Sigma_{\infty} \) is a plane \( \pi \) containing the points \( L \cap \Omega \) and \( M \cap \Omega \); so \( \pi \cap \Omega \) is an oval. It follows that \( L, M \) are also non-concurrent lines of the subquadrangle \( S(\Sigma_{M}) \) of \( T(3, 0) \). Since \((\infty) \in S(\Sigma_{M})\), it follows that \( S \) and \( S(\Sigma_{M}) \) are distinct. By Lemma 4, \( (L, M) \) is regular in \( T(3, 0) \) and \( S \cap S(\Sigma_{M}) \) is the set of points and lines of the hyperbolic quadric \( \mathcal{H}_{M} \) with lines \( \{L, M\} \cup \{L, M\}^{\perp} \). Since \( S \) has \( q^{3} \) lines not meeting \( L \), and \( \mathcal{H}_{M} \) has \( q \) lines not meeting \( L \), there are \( q^{2} \) such hyperbolic quadrics \( \mathcal{H}_{M} \); each meeting \( \Omega \) in a conic on the point \( x = L \cap \Omega \). Now \( \Omega \) has \( q + 1 \) pencils of ovals on \( x \), each with \( q \) ovals, so there is at least one pencil on \( x \) all of whose ovals are conics. By [6], \( \Omega \) is an elliptic quadric, contrary to hypothesis.

Thus \((\infty) \in S\). Let \( L \) be a line of \( S \) on \((\infty)\) and let \( M \) be a line of \( S \) not concurrent with \( L \). Thus, in \( T(3, 0) \), \( L \) is of type (a) and \( M \) is of type (b). It follows that in \( PG(4, q) \), \( \langle L, M \rangle \) is a plane meeting \( \Sigma_{\infty} \) in a line \( \ell \) secant to \( \Omega \). Now \( \ell \) lies on \( q + 1 \) secant planes \( \pi_{1}, \ldots, \pi_{q+1} \) to \( \Omega \) in \( \Sigma_{\infty} \); so for \( i = 1, \ldots, q+1 \) the 3-dimensional space \( \Sigma_{i} = \langle \pi_{i}, M \rangle \) determines the subquadrangle \( S(\Sigma_{i}) \) of order \( q \). Each of \( S(\Sigma_{1}), \ldots, S(\Sigma_{q+1}) \) contains \( L \) and \( M \), so by Lemma 4 we have \( S = S(\Sigma_{i}) \) for some \( i \in \{1, \ldots, q+1\} \) as required.

4.3. Flock \( GQ \), \( q \) Even

Recall that in Subsection 2.3 we showed that a flock quadrangle \( GQ(\ell) \) has a family of \( q + 1 \) subquadrangles \( \{S_{\ell} : s \in GF(q) \cup \{\infty\}\} \), each containing the points \((\infty)\) and \((0, 0, 0)\). We now show that these, and images of them under the automorphism group of \( GQ(\ell) \), are the only subquadrangles of order \( q \) on \((\infty)\).
Theorem 8. Let \( q \) be even and let \( C \) be a \( q \)-clan with associated generalized quadrangle \( \text{GQ}(C) \) and herd \( \mathcal{H}(C) \). If \( S \) is a subquadrangle of order \( q \) of \( \text{GQ}(C) \) which contains the point \((\infty)\) then there exists \( g \in \text{Aut GQ}(C) \) such that \( gS = S_s \) for some \( s \in \text{GF}(q) \cup \{\infty\} \).

Proof. Since \( S \) has order \( q \), it is immediate that it contains a point \( x \) not collinear with \((\infty)\). Since \( \text{Aut GQ}(C) \) is transitive on the points not collinear with \((\infty)\), there exists \( g \) in \( \text{Aut GQ}(C) \) such that \((0, 0, 0) \neq gS\).

By Lemma 5, there are at most \( q + 1 \) subquadrangles of order \( q \) on \((\infty)\) and \((0, 0, 0)\). But the \( q + 1 \) subquadrangles \( \{S_s : s \in \text{GF}(q) \cup \{\infty\}\} \) all contain \((\infty)\) and \((0, 0, 0)\); so \( gS = S_s \) for some \( s \in \text{GF}(q) \cup \{\infty\} \).

Corollary 9. Let \( q \) be even and let \( C, C' \) be \( q \)-clans with associated generalized quadrangles \( \text{GQ}(C) \), \( \text{GQ}(C') \) and herds \( \mathcal{H}(C) = \{C_s : s \in \text{GF}(q) \cup \{\infty\}\} \), \( \mathcal{H}(C') = \{C'_s : s \in \text{GF}(q) \cup \{\infty\}\} \). If \( \text{GQ}(C) \) is isomorphic to \( \text{GQ}(C') \) then there exists a permutation \( \pi \) of \( \text{GF}(q) \cup \{\infty\} \) such that \( C_s \) is equivalent under \( \pi \) to \( C'_{\pi(s)} \) for each \( s \in \text{GF}(q) \cup \{\infty\} \). (In other words, the herds comprise the same multiset of isomorphism types of ovals.)

Proof. Suppose that \( \text{GQ}(C) \) is not classical and let \( \psi \) be an isomorphism from \( \text{GQ}(C) \) to \( \text{GQ}(C') \). It follows [16] that \( \psi \) maps the base point \((\infty)\) of \( \text{GQ}(C) \) to the base point \((\infty)'\) of \( \text{GQ}(C') \). Further, as \( \text{Aut GQ}(C) \) is transitive on points not collinear with \((\infty)\), we can assume that \( \psi \) maps the origin \((0, 0, 0)\) of \( \text{GQ}(C) \) to the origin \((0, 0, 0)\) of \( \text{GQ}(C') \). Let \( \{S_s : s \in \text{GF}(q) \cup \{\infty\}\} \) be the family of \( q + 1 \) subquadrangles of \( \text{GQ}(C) \), and let \( \{S'_s : s \in \text{GF}(q) \cup \{\infty\}\} \) be the family of \( q + 1 \) subquadrangles of \( \text{GQ}(C') \), constructed in Subsection 2.3. Since these are the only subquadrangles on \((\infty)\) and \((0, 0, 0)\) (respectively \((\infty)'\) and \((0, 0, 0)\)'), and \( \psi \) maps subquadrangles of order \( q \) to subquadrangles of order \( q \), there is a permutation \( \pi \) of \( \text{GF}(q) \cup \{\infty\} \) such that, for each \( s \in \text{GF}(q) \cup \{\infty\} \), \( \psi \) maps \( S_s \) to \( S'_{\pi(s)} \) and hence \( C_s \) is equivalent under \( \pi \) to \( C'_{\pi(s)} \) (applying Theorem 11 below). If \( \text{GQ}(C) \) is classical then so is \( \text{GQ}(C) \), and the subquadrangles constructed as in Section 2.3 are all isomorphic to \( Q(4, q) \). In this case the result is immediate.

Corollary 10. Let \( q \) be even and let \( C \) be a \( q \)-clan with associated generalized quadrangle \( \text{GQ}(C) \) and herd \( \mathcal{H}(C) \). Then the herd \( \mathcal{H}(C) \) is an invariant of the generalized quadrangle \( \text{GQ}(C) \).

5. AUTOMORPHISMS OF \( T_2(4) \) AND \( T_3(0) \)

For the purpose of this section, we recall that each of \( T_2(4) \) and \( T_3(0) \) can be constructed by means of a 4-gonal family \([15, 10.3.1; 10.3.2]\). Then \( T_2(4) \) (respectively \( T_3(0) \)) is in fact a translation generalized quadrangle
(TGO) with base point \((\infty)\) and translation group the group of all elations of \(\PG(3, q)\) (respectively \(\PG(4, q)\)) with axis \(\Sigma_{\infty}\) (see [15, 8.2]).

We have had occasion to need the following theorem; see [10]. An alternative proof of Part (ii) is found in [9, Ex. 2].

**Theorem 11.** (i) Let \(\ell\) be an oval of \(\PG(2, q)\) which is not a conic. Then \(\mbox{Aut} \, T_2(\ell)\) is the stabiliser \(\PGL(4, q)_{\ell}\) of \(\ell\) in \(\PGL(4, q)\).

(ii) Let \(\Omega\) be an ovoid of \(\PG(3, q)\) which is not an elliptic quadric. Then \(\mbox{Aut} \, T_3(\Omega)\) is the stabiliser \(\PGL(5, q)_{\Omega}\) of \(\Omega\) in \(\PGL(5, q)\).

**Proof.** (i) Let \(G = \mbox{Aut} \, T_2(\ell)\). Since \(\ell\) is not a conic, \(q\) is even (see [7, 8.14]) and \((\infty)\) is the unique coregular point of \(T_2(\ell)\) [15, 3.3.2, 3.3.3]; so \((\infty)\) is fixed by \(G\).

Let \(T \subseteq \PGL(4, q)\) be the translation group of \(T_2(\ell)\), that is, the group of all elations of \(\PG(3, q)\) with axis the plane \(\Sigma_{\infty}\). In fact \(T\) is the group of all elations of \(T_2(\ell)\) about \((\infty)\) [15, 8.6.4].

We first show that \(T\) is a normal subgroup of \(G\). For \(t \in T\) and \(g \in G\), the element \(gtg^{-1}\) is an elation of \(T_2(\ell)\) about the point \(g(\infty)\). But \(g(\infty) = \infty\) and so \(gtg^{-1} \in T\), as required.

Next we show that \(T = C_G(T)\). First, since \(T\) is abelian, we have \(T \subseteq C_G(T)\). For the reverse inclusion, let \(x\) be a point not collinear with \((\infty)\). For \(g \in C_G(T)_x\) and for all \(t \in T\), we have \(gt = tg\) and hence \(g(tx) = t(gx) = tx\). Thus \(g\) fixes every point not collinear with \((\infty)\) (as \(T\) is transitive on the set of such points) and we have \(C_G(T)_x = 1\) by [15, 2.4.1]. Let \(g \in C_G(T)_x\). Since \(T\) is transitive on the points not collinear with \((\infty)\), there exists \(t \in T\) such that \(tx = gx\). But then \(t^{-1}g \in C_G(T)_x = 1\); so \(g = t \in T\) as required.

Let \(x\) be a point not collinear with \((\infty)\). Taking account of the above, we see that

\[
G_x \cong G/T = N_G(T)/C_G(T) \leq \mbox{Aut} \, T \cong GL(3h, 2),
\]

where \(q = 2^h\), \(\leq\) means “isomorphic to a subgroup of” and the last isomorphism follows because \(T\) is elementary abelian.

We study \(G_x\) further, letting \(W\) be the group of all whorls about \((\infty)\) fixing \(x\). In fact, by [15, 8.6.5], \(W\) is the group of all homologies of \(\PG(3, q)\) with centre \(x\) and axis \(\Sigma_{\infty}\).

We observe that \(W\) is a normal subgroup of \(G_x\). For \(w \in W\) and \(g \in G_x\), we see that \(gwg^{-1}\) is a whorl about \((\infty)\) fixing \(gx = x\); so \(gwg^{-1} \in W\), as required. Let \(N = N_{GL(3h, 2)}(W)\) and \(C = C_{GL(3h, 2)}(W)\), where \(q = 2^h\). Thus \(G_x \subseteq N\), which we now show is \(\tau L(3, q)\). First we investigate \(C\). We have \(\GL(3, q) \leq \GL(3h, p)\) and \(W = Z(\GL(3, q))\). If \(T \in C\) then \(T(\lambda T) = (\lambda T)T\); so \(T(\lambda x) = \lambda (Tx)\) and \(T\) is \(GF(q)\)-linear. Thus we have \(C \subseteq \GL(3, q)\) (where
again \( \leq \) denotes isomorphic to a subgroup of). Hence \( C = \text{GL}(3, q) \) and thus \( N \leq \Gamma \text{L}(3, q) \). But also \( \Gamma \text{L}(3, q) \leq N \); as required.

Thus, we have \( G = T \rtimes G_e \cong T \rtimes \Gamma \text{L}(3, q) = \Lambda \Gamma \text{L}(4, q) \Gamma_e \). Since \( G \) fixes the point \( (\infty) \) of \( T_3(\ell) \), it also fixes the set of all lines on \( (\infty) \), that is, the set of points \( \ell \). Thus \( G \cong \Gamma \text{L}(4, q) \), as required.

(ii) Once we notice that \( q \) is even (see [8, 16.1.7]) and \( (O) \) is the unique 3-regular point of \( T_3(0) \) [15, 5.3.3], the proof in this case is entirely analogous to the proof in case (i), hence we omit it.

The following is a corollary of the proof.

**Corollary 12.**

(i) Let \( T_3(\ell_1) \) and \( T_3(\ell_2) \) be non-classical Tits GQ for ovals \( \ell_1 \) and \( \ell_2 \) of \( \text{PG}(2, q) \). Every isomorphism between \( T_3(\ell_1) \) and \( T_3(\ell_2) \) is induced by an automorphism of \( \text{PG}(2, q) \) which maps \( \ell_1 \) to \( \ell_2 \). Conversely, an automorphism of \( \text{PG}(2, q) \) which maps \( \ell_1 \) to \( \ell_2 \), when extended to an automorphism of \( \text{PG}(3, q) \), induces an isomorphism mapping \( T_3(\ell_1) \) to \( T_3(\ell_2) \).

(ii) Let \( T_3(\Omega_1) \) and \( T_3(\Omega_2) \) be non-classical Tits GQ for ovoids \( \Omega_1 \) and \( \Omega_2 \) of \( \text{PG}(3, q) \). Every isomorphism between \( T_3(\Omega_1) \) and \( T_3(\Omega_2) \) is induced by an automorphism of \( \text{PG}(3, q) \) which maps \( \Omega_1 \) to \( \Omega_2 \). Conversely, an automorphism of \( \text{PG}(3, q) \) which maps \( \Omega_1 \) to \( \Omega_2 \), when extended to an automorphism of \( \text{PG}(4, q) \), induces an isomorphism mapping \( T_3(\Omega_1) \) to \( T_3(\Omega_2) \).

A similar theorem for another class \( T^*(\ell) \) of GQ of order \((q - 1, q + 1)\) was proved by Bichara et al. [2].

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