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Optimal Error Bounds for Quadratic Spline Interpolation*

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Four types of quadratic spline interpolants are considered for which we obtain error bounds of the form

$$\|(f-s)^{(k)}\|_\infty \le C_k h^{3-k} \|f^{(3)}\|_\infty \qquad (k = 0, 1, 2).$$

Moreover, the optimality of the constant C_0 is proved. © 1996 Academic Press, Inc.

1. INTRODUCTION

In [1] and [4] error bounds were derived for cubic spline interpolation over arbitrary partitions, and in [5] the optimality of these bounds was investigated. In [3] the results were extended to the periodic case. The purpose of this paper is to obtain error bounds for four types of quadratic spline interpolation problems using a technique similar to the technique developed in [1, 4, 5, and 3]. Bounds for periodic cubic and quadratic splines are also presented in [8] for uniform partition.

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Throughout this paper we use the following notation. Let $N \geq 2$ and

$$\Delta\colon a = x_0 < x_1 < \cdots < x_i < \cdots < x_{N-1} < x_N = b$$

be any partition of the interval $[a, b]$. A function s is said to be a spline of degree n over Δ if s is a polynomial of degree at most n on each interval $[x_{i-1}, x_i]$ and $s \in C^{n-1}([a, b], \mathbb{R})$. It is said to be periodic if $s^{(k)}(a) = s^{(k)}(b)$ for $k = 0, \ldots, n-1$. Let

$$h = \max\{h_i = x_i - x_{i-1} \mid i = 1, \ldots, N\}$$

be the mesh size of the partition and

$$\beta = h/\min\{h_i \mid i = 1, \ldots, n\}$$

be the mesh ratio of the partition. Finally, let us consider the following function spaces

$$L^1([a, b]; \mathbb{R}) = \left\{ f\colon [a, b] \to \mathbb{R} \;\middle|\; \int_a^b |f(x)|\, dx < \infty \right\},$$

$$L^\infty([a, b]; \mathbb{R}) = \{ f\colon [a, b] \to \mathbb{R} \mid \|f\|_\infty = \operatorname{ess\,sup}\{|f(x)|\colon x \in [a, b]\} < \infty \},$$

and

$$AC^{n+1}([a, b]; \mathbb{R}) = \left\{ f \in C^n([a, b]; \mathbb{R}) \;\middle|\; \begin{array}{l} \text{(i)}\;\; f^{(n+1)} \in L^\infty([a, b]; \mathbb{R}) \\ \text{(ii)}\;\; f^{(n)}(x)\big|_a^b = \int_a^b f^{(n+1)}(x)\, dx \end{array} \right\}.$$

A function $f \in AC^{n+1}([a, b]; \mathbb{R})$ is said to be periodic if $f^{(k)}(a) = f^{(k)}(b)$ for $k = 0, \ldots, n$.

2. QUADRATIC SPLINE INTERPOLANT

Let Δ' be the partition of $[a, b]$ such that

$$\Delta'\colon a = z_0 < z_1 < \cdots < z_{i-1} < z_i < z_{i+1} < \cdots < z_N < z_{N+1} = b$$

where $z_i = (x_{i-1} + x_i)/2$ $(i = 1, \ldots, N)$. Given any function $f \in AC^3([a, b]; \mathbb{R})$, we consider the following four types of quadratic spline interpolant s of f defined on Δ:

Type I (Marsden [7]). f and s are periodic and

$$s(z_i) = f(z_i) \qquad (i = 1, \ldots, N), \tag{2.1a}$$

$$s^{(k)}(a) = s^{(k)}(b) \qquad (k = 0, 1). \tag{2.1b}$$

Type II (Demko [2], Kammerer *et al*. [6], and Xie [10]).

$$s(z_i) = f(z_i) \qquad (i = 0, \dots, N+1). \tag{2.2}$$

Type III (Xie [10]).

$$s(z_i) = f(z_i) \qquad (i = 1, \dots, N), \tag{2.3a}$$

$$s^{(1)}(z_i) = f^{(1)}(z_i) \qquad (i = 0, N+1). \tag{2.3b}$$

Type IV (Xie [10]).

$$s(z_i) = f(z_i) \qquad (i = 2, \dots, N-1), \tag{2.4a}$$

$$s^{(k)}(z_i) = f^{(k)}(z_i) \qquad (i = 0, N+1; k = 0, 1). \tag{2.4b}$$

The existence and uniqueness results are presented in the references given here. Our goal is to obtain, for each type of interpolant, error bounds of the form

$$\| f^{(k)} - s^{(k)} \|_\infty \le C_k h^{3-k} M$$

for $k = 0, 1, 2$, where $M = \| f^{(3)} \|_\infty$.

3. FUNDAMENTAL SPLINES

Let v and w be two quadratic splines defined on Δ' and such that

$$v(x_i) = 0 = w(x_i) \qquad (i = 0, \dots, N) \tag{3.1}$$

and for

$$\text{Types I and II} \qquad \begin{cases} v^{(1)}(a) = 0 = w^{(1)}(b); \\ v^{(2)}(a) = 1 = (-1)^N w^{(2)}(b). \end{cases} \tag{3.2}$$

$$\text{Type III} \qquad \begin{cases} v^{(1)}(a) = 1 = (-1)^N w^{(1)}(b); \\ v^{(2)}(a) = 0 = w^{(2)}(b). \end{cases} \tag{3.3}$$

$$\text{Type IV} \qquad \begin{cases} v(z_1) = 1 = (-1)^N w(z_N); \\ v^{(2)} \text{ is continuous at } z_1; \\ w^{(2)} \text{ is continuous at } z_N. \end{cases} \tag{3.4}$$

There is no difficulty to show that v and w are well defined by the given conditions. Some of the properties of v and w are stated in the next lemmas.

LEMMA 1. *The splines v and w have the following sign properties*:

$$v^{(1)}(x_i) = (-1)^i |v^{(1)}(x_i)| \qquad (i = 0, \ldots, N), \tag{3.5a}$$

$$v^{(2)}(x_i) = (-1)^i |v^{(2)}(x_i)| \qquad (i = 1, \ldots, N); \tag{3.5b}$$

$$w^{(1)}(x_i) = (-1)^{i+1} |w^{(1)}(x_i)| \qquad (i = 0, \ldots, N), \tag{3.6a}$$

$$w^{(2)}(x_i) = (-1)^i |w^{(2)}(x_i)| \qquad (i = 0 \ldots, N-1). \tag{3.6b}$$

The splines v and w satisfy the following identities:

$$|v^{(1)}(x_i)| = 3|v^{(1)}(x_{i-1})| + |v^{(2)}(x_{i-1})|h_i, \tag{3.7a}$$

$$|v^{(2)}(x_i)|h_i = 8|v^{(1)}(x_{i-1})| + 3|v^{(2)}(x_{i-1})|h_i; \tag{3.7b}$$

$$|w^{(1)}(x_{i-1})| = 3|w^{(1)}(x_i)| + |w^{(2)}(x_i)|h_i, \tag{3.8a}$$

$$|w^{(2)}(x_{i-1})|h_i = 8|w^{(1)}(x_i)| + 3|w^{(2)}(x_i)|h_i; \tag{3.8b}$$

for $i = 1, \ldots, N$ for types I, II, *and* III, *and for type* IV (3.7) *holds for $i = 2, \ldots, N$ and* (3.8) *holds for $i = 1, \ldots, N-1$.*

Furthermore, v and w have no zeros in each interval (x_{i-1}, x_i),and

$$v(x) = (-1)^{i-1}|v(x)| \qquad \text{and} \qquad w(x) = (-1)^i |w(x)| \tag{3.9}$$

for $i = 1, \ldots, N$.

Proof. We prove only these results for v since the proofs for w are similar. We use the following representation of v on $[x_{i-1}, x_i]$

$$v(x) = \sum_{j=1}^{2} v^{(j)}(x_{i-1}) \frac{(x - x_{i-1})^j}{j!} + J \frac{(x - z_i)_+^2}{2!}$$

where $J = v^{(2)}(z_i^+) - v^{(2)}(z_i^-)$. If we evaluate $v^{(1)}(x_i)$ and $v^{(2)}(x_i)$, we obtain

$$-v^{(1)}(x_i) = 3v^{(1)}(x_{i-1}) + v^{(2)}(x_{i-1})h_i \tag{3.10a}$$

$$-v^{(2)}(x_i)h_i = 8v^{(1)}(x_{i-1}) + 3v^{(2)}(x_{i-1})h_i. \tag{3.10b}$$

For v of type I, II, or III, we have from the definition that $v^{(1)}(a)$ and $v^{(2)}(a)$ are nonnegative. Therefore we obtain (3.5) form (3.10) in these cases. If v is of type IV then v is a polynomial on $[a, x_1]$ such that $v^{(1)}(a) = 4/h_1$ and $v^{(2)}(a) = -4/h_1^2$. It follows from (3.10) that $-v^{(1)}(x_1)$ and $-v^{(2)}(x_1)$ are nonnegative and (3.5) follows. As a direct consequence of (3.5) and (3.10) we obtain (3.7). Since v is a spline with a single knot at

z_i on (x_{i-1}, x_i), it follows from the Budan–Fourier Theorem for polynomial splines [9, Theorem 4.58] that v has no zeros in (x_{i-1}, x_i). Therefore (3.9) is a consequence of (3.5). ∎

LEMMA 2. *The quadratic spline v has the properties*

$$\int_{x_{i-1}}^{x_i} |v(x)|\,dx = (-1)^i \big(v^{(1)}(x_i) - v^{(1)}(x_{i-1})\big)\frac{h_i^2}{12}, \tag{3.11a}$$

$$\int_a^{x_i} |v(x)|\,dx \le \frac{h^2}{6}\big|v^{(1)}(x_i)\big|, \tag{3.11b}$$

$$\int_{x_{i-1}}^{x_i} |v(x)|\,dx = (-1)^i \big(v^{(2)}(x_i) + v^{(2)}(x_{i-1})\big)\frac{h_i^3}{24}, \tag{3.11c}$$

$$\int_a^{x_i} |v(x)|\,dx \le \frac{h^3}{24}\big((-1)^i v^{(2)}(x_i) - v^{(2)}(a)\big), \tag{3.11d}$$

for $i = 1, \ldots, N$. Moreover, if the partition is uniform then (d) *is an equality.*

Proof. Let $\omega(u) = u(u - \frac{1}{2})(u-1)/6$ and define $\Omega(x) = h_i^3\omega((x - x_{i-1})/h_i)$ for $x \in [x_{i-1}, x_i]$. Then $\Omega^{(3)}(x) = 1$ and, from (3.9), we have

$$(-1)^{i-1}\int_{x_{i-1}}^{x_i} |v(x)|\,dx = \int_{x_{i-1}}^{x_i} v(x)\,\Omega^{(3)}(x)\,dx.$$

Integrating by parts, using (3.1) and the continuity of $v^{(2)}(x)\Omega(x)$, we obtain

$$\int_{x_{i-1}}^{x_i} v(x)\,\Omega^{(3)}(x)\,dx = -\big(v^{(1)}(x_i) - v^{(1)}(x_{i-1})\big)\frac{h_i^2}{12}$$

and (3.11a) follows. To obtain (3.11b) we have from (3.11a)

$$\int_a^{x_i} |v(x)|\,dx = \sum_{j=1}^{i} (-1)^j \big(v^{(1)}(x_j) - v^{(1)}(x_{j-1})\big)\frac{h_j^2}{12}$$

and, using (3.5a), we obtain

$$\int_a^{x_i} |v(x)|\,dx \le \frac{h^2}{12}\left\{\big|v^{(1)}(x_i)\big| + 2\sum_{j=1}^{i-1}\big|v^{(1)}(x_j)\big| + \big|v^{(1)}(a)\big|\right\}.$$

But from (3.7a) we have $|v^{(1)}(x_i)| \geq 3|v^{(1)}(x_{i-1})|$ (for type IV we have $v^{(1)}(x_1) = -v^{(1)}(x_0)$). It follows that

$$\int_a^{x_i} |v(x)|\,dx \leq \frac{h^2}{12}\left|v^{(1)}(x_i)\right|\left\{1 + 2\sum_{j=1}^{i-1}\frac{1}{3^j} + \frac{1}{3^{i-1}}\right\}$$
$$\leq \frac{h^2}{6}\left|v^{(1)}(x_i)\right|.$$

If we observe that

$$v^{(1)}(x_i) - v^{(1)}(x_{i-1}) = v^{(1)}(x_i) - v^{(1)}(z_i) + v^{(1)}(z_i) - v^{(1)}(x_{i-1})$$
$$= \left(v^{(2)}(x_i) + v^{(2)}(x_{i-1})\right)\frac{h_i}{2}$$

then (3.11a) becomes (3.11c). Also,

$$\int_a^{x_i} |v(x)|\,dx = \sum_{j=1}^{i}\int_{x_{j-1}}^{x_j} |v(x)|\,dx$$
$$\leq \frac{h^3}{24}\sum_{j=1}^{i}(-1)^j\left(v^{(2)}(x_j) + v^{(2)}(x_{j-1})\right)$$
$$\leq \frac{h^3}{24}\left\{(-1)^i v^{(2)}(x_i) - v^{(2)}(a)\right\}$$

and we get (3.11d). ∎

From (3.11d) and (3.5b) we obtain

$$\int_a^{x_i} |v(x)|\,dx \leq \frac{h^3}{24}\begin{cases}\left|v^{(2)}(x_i)\right| - \left|v^{(2)}(a)\right| & \text{for types I and II,}\\ \left|v^{(2)}(x_i)\right| & \text{for type III,}\\ \left|v^{(2)}(x_i)\right| + \left|v^{(2)}(a)\right| & \text{for type IV.}\end{cases}$$

Using the same method we obtain the following result for w.

LEMMA 3. *The quadratic spline w has the properties*

$$\int_{x_i}^{x_{i+1}} |w(x)|\,dx = (-1)^{i+1}\left(w^{(1)}(x_i) - w^{(1)}(x_{i+1})\right)\frac{h_{i+1}^2}{12}, \quad (3.12a)$$

$$\int_{x_i}^{b} |w(x)|\,dx \leq \frac{h^2}{6}\left|w^{(1)}(x_i)\right|, \quad (3.12b)$$

$$\int_{x_i}^{x_{i+1}} |w(x)|\,dx = (-1)^i\big(w^{(2)}(x_i) + w^{(2)}(x_{i+1})\big)\frac{h_{i+1}^3}{24}, \quad (3.12c)$$

$$\int_{x_i}^{b} |w(x)|\,dx \le \frac{h^3}{24}\big((-1)^i w^{(2)}(x_i) - (-1)^N w^{(2)}(b)\big), \quad (3.12d)$$

for $i = 0, \dots, N-1$. *Moreover, if the partition is uniform then* (3.12d) *is an equality.*

From (3.12d) and (3.6b) we get

$$\int_{x_i}^{b} |w(x)|\,dx \le \frac{h^3}{24}\begin{cases} |w^{(2)}(x_i)| - |w^{(2)}(b)| & \text{for types I and II,} \\ |w^{(2)}(x_i)| & \text{for type III,} \\ |w^{(2)}(x_i)| + |w^{(2)}(b)| & \text{for type IV.} \end{cases}$$

4. KERNELS

For the quadratic spline interpolant s of $f \in AC^3([a,b];\mathbb{R})$ we define the error $e(x) = f(x) - s(x)$. Let us observe that for the four types of interpolant considered here we have

$$e(z_i) = 0 \qquad (i = 2, \dots, N-1),$$

$$e^{(1)} \text{ and } e^{(2)} \text{ are continuous at } z_i \qquad (i = 1, \dots, N).$$

Moreover v, $v^{(1)}$, w, and $w^{(1)}$ are continuous at z_i ($i = 0, \dots, N+1$), and $v(x_i) = 0 = w(x_i)$ ($i = 0, \dots, N$). Using these properties we obtain

$$\begin{aligned}\int_a^{x_i} v(x)e^{(3)}(x)\,dx = {} & v^{(1)}(a)e^{(1)}(a) - v^{(1)}(x_i)e^{(1)}(x_i) \\ & + v^{(2)}(x_i)e(x_i) - v^{(2)}(a)e(a) \\ & + \big[v^{(2)}(z_1^-) - v^{(2)}(z_1^+)\big]e(z_1) \\ & + \delta_{iN}\big[v^{(2)}(z_N^-)\text{-}v^{(2)}(z_N^+)\big]e(z_N). \end{aligned} \quad (4.1)$$

Similarly, for w we have

$$\begin{aligned}\int_{x_i}^{b} w(x)e^{(3)}(x)\,dx = {} & w^{(1)}(x_i)e^{(1)}(x_i) - w^{(1)}(b)e^{(1)}(b) \\ & + w^{(2)}(b)e(b) - w^{(2)}(x_i)e(x_i) \\ & + \big[w^{(2)}(z_N^-) - w^{(2)}(z_N^+)\big]e(z_N) \\ & + \delta_{i0}\big[w^{(2)}(z_1^-) - w^{(2)}(z_1^+)\big]e(z_1). \end{aligned} \quad (4.2)$$

Using the supplementary conditions for each type of interpolant we obtain the following results.

Type I. From (2.1) and (3.2) we have

$$\int_a^b v(x)e^{(3)}(x)\,dx = \left(v^{(2)}(b) - v^{(2)}(a)\right)e(a) - v^{(1)}(b)e^{(1)}(a),$$

$$\int_a^b w(x)e^{(3)}(x)\,dx = \left(w^{(2)}(b) - w^{(2)}(a)\right)e(a) + w^{(1)}(a)e^{(1)}(a).$$

It is a linear system for $e(a)$ and $e^{(1)}(a)$ for which the determinant of the coefficient matrix is

$$D(a,b) = w^{(1)}(a)\left(v^{(2)}(b) - v^{(2)}(a)\right) + v^{(1)}(b)\left(w^{(2)}(b) - w^{(2)}(a)\right).$$

But from (3.11d), (3.12d), (3.5), and (3.6), we have

$$\operatorname{sgn}\left(v^{(2)}(b) - v^{(2)}(a)\right) = \operatorname{sgn}\left(v^{(2)}(b)\right) = (-1)^N,$$
$$\operatorname{sgn}\left(w^{(2)}(b) - w^{(2)}(a)\right) = -\operatorname{sgn}\left(w^{(2)}(a)\right) = -1.$$

Hence,

$$D(a,b) = \left[\left|w^{(1)}(a)\right|\left|v^{(2)}(b) - v^{(2)}(a)\right| + \left|v^{(1)}(b)\right|\left|w^{(2)}(b) - w^{(2)}(a)\right|\right](-1)^{N+1}.$$

It follows that

$$e^{(k)}(a) = \int_a^b K_k(x)e^{(3)}(x)\,dx$$

where $e^{(3)}(x) = f^{(3)}(x)$, and

$$K_0(x) = \left[w^{(1)}(a)v(x) + v^{(1)}(b)w(x)\right]/D(a,b),$$

$$K_1(x) = \left[\left(v^{(2)}(b) - v^{(2)}(a)\right)w(x) - \left(w^{(2)}(b) - w^{(2)}(a)\right)v(x)\right]/D(a,b).$$

Therefore, using Lemmas 2 and 3 we have

$$\|K_0\|_1 \le \frac{h^3}{24} \qquad \text{and} \qquad \|K_1\|_1 \le \frac{h^2}{6}.$$

Finally, using the periodicity, and considering any interval of the form $[x_i, x_i + b - a]$, we obtain

$$\max\left\{\left|e^{(k)}(x_i)\right| : i = 0, \ldots, N\right\} \le \|K_k\|_1 M$$

for $k = 0, 1$.

Type II. From (2.2) and (3.2) we have

$$\int_a^{x_i} v(x) e^{(3)}(x)\, dx = v^{(2)}(x_i) e(x_i) - v^{(1)}(x_i) e^{(1)}(x_i),$$

$$\int_{x_i}^b w(x) e^{(3)}(x)\, dx = -w^{(2)}(x_i) e(x_i) + w^{(1)}(x_i) e^{(1)}(x_i),$$

for $i = 0, \dots, N$.

Type III. From (2.3) and (3.3) we have

$$\int_a^{x_i} v(x) e^{(3)}(x)\, dx = v^{(2)}(x_i) e(x_i) - v^{(1)}(x_i) e^{(1)}(x_i),$$

$$\int_{x_i}^b w(x) e^{(3)}(x)\, dx = -w^{(2)}(x_i) e(x_i) + w^{(1)}(x_i) e^{(1)}(x_i),$$

for $i = 0, \dots, N$.

Type IV. From (2.4) and (3.4) we have

$$\int_a^{x_i} v(x) e^{(3)}(x)\, dx = v^{(2)}(x_i) e(x_i) - v^{(1)}(x_i) e^{(1)}(x_i),$$

$$\int_{x_i}^b w(x) e^{(3)}(x)\, dx = -w^{(2)}(x_i) e(x_i) + w^{(1)}(x_i) e^{(1)}(x_i),$$

for $i = 1, \dots, N - 1$.

For these three cases we have to solve a linear system for $e(x_i)$ and $e^{(1)}(x_i)$ for which the determinant of the coefficient matrix is

$$D(x_i) = v^{(2)}(x_i) w^{(1)}(x_i) - v^{(1)}(x_i) w^{(2)}(x_i).$$

Using (3.5) and (3.6) we have

$$D(x_i) = -\left\{ \left| v^{(2)}(x_i) \right| \left| w^{(1)}(x_i) \right| + \left| v^{(1)}(x_i) \right| \left| w^{(2)}(x_i) \right| \right\} \neq 0$$

for all i (depending on the type of the interpolant).

Solving for $e(x_i)$ and $e^{(1)}(x_i)$, we have

$$e^{(k)}(x_i) = \int_a^b K_k(x_i; x) e^{(3)}(x)\, dx$$

where

$$K_k(x_i; x) = \begin{cases} \dfrac{w^{(k+1)}(x_i) v(x)}{D(x_i)} & a \le x \le x_i; \\[2ex] \dfrac{v^{(k+1)}(x_i) w(x)}{D(x_i)} & x_i < x \le b, \end{cases}$$

for $k = 0, 1$. It follows from Lemmas 2 and 3 that

$$\|K_0\|_1 \leq \begin{cases} h^3/24 & \text{for types II and III,} \\ h^3/12 & \text{for type IV,} \end{cases}$$

and $\|K_1\|_1 \leq h^2/6$. Finally,

$$\max\left\{\left|e^{(k)}(x_i)\right| : i = 0, \ldots, N\right\} \leq \|K_k\|_1 M$$

for $k = 0, 1$. In summary we have the following result.

PROPOSITION 4. *Let $f \in AC^3([a, b]; \mathbb{R})$ and s be the quadratic spline interpolant of type* I, II, III, *or* IV. *Then for $e(x) = f(x) - s(x)$ we have*

$$\max\left\{\left|e^{(k)}(x_i)\right| : i = 0, \ldots, N\right\} \leq \|K_k\|_1 M$$

for $k = 0, 1$, where

$$\|K_0\|_1 \leq \begin{cases} h^3/24 & \textit{for types } \text{I, II, } \textit{and } \text{III,} \\ h^3/12 & \textit{for type } \text{IV,} \end{cases}$$

$$\|K_1\|_1 \leq h^2/6,$$

and $M = \|f^{(3)}\|_\infty$.

5. DERIVATION OF ERROR BOUNDS

For any interval $[x_{i-1}, x_i]$ on which we have the condition $f(z_i) = s(z_i)$ we consider the Lagrange interpolating polynomial $p(x)$ of degree 2 such that

$$p(x_{i-1}) = f(x_{i-1}), \qquad p(z_i) = f(z_i), \qquad p(x_i) = f(x_i).$$

Then

$$|f(x) - p(x)| \leq |(1 + t)t(1 - t)| \frac{h_i^3}{48} M \tag{5.1}$$

for $x = z_i + (h_i/2)t$ and $-1 \leq t \leq 1$. The difference $p - s$ is then the Lagrange interpolant of $e = f - s$ on $[x_{i-1}, x_i]$, and

$$|p(x) - s(x)| \leq |t| \max\left\{|e(x_{i-1})|, |e(x_i)|\right\}. \tag{5.2}$$

Hence, from (5.1), (5.2), and Proposition 4, it follows that

$$\max_{x\in[x_{i-1},x_i]} |f(x) - s(x)| \le \frac{h^3}{24}M$$

for types I, II, and III, and

$$\max_{x\in[x_{i-1},x_i]} |f(x) - s(x)| \le \frac{h^3}{12}M$$

for type IV.

For type IV it remains to consider the interval $[x_0, x_1]$ where $f^{(1)}(z_0) = s^{(1)}(z_0)$ (and similarly the interval $[x_{N-1}, x_N]$ where $f^{(1)}(z_{N+1}) = s^{(1)}(z_{N+1})$). In this case we consider the Hermite interpolating polynomial $p(x)$ of degree 2 such that

$$p(x_0) = f(x_0), \qquad p^{(1)}(x_0) = f^{(1)}(x_0), \qquad p(x_1) = f(x_1).$$

Then

$$|f(x) - p(x)| \le \left|(1+t)^2(1-t)\right|\frac{h_1^3}{48}M. \tag{5.3}$$

Also, $p - s$ is the Hermite interpolant of $e = f - s$ on $[x_0, x_1]$, and

$$|p(x) - s(x)| \le \frac{(1+t)^2}{4}|e(x_1)|. \tag{5.4}$$

Hence, from (5.3), (5.4), and Proposition 4, we have

$$\max_{x\in[x_0,x_1]} |f(x) - s(x)| \le \frac{h^3}{12}M.$$

To obtain the error bounds for $e^{(1)}(x)$ and $e^{(2)}(x)$ we use the following lemma about linear interpolation.

LEMMA 5. *Let* $\phi \in AC^2([a,b];\mathbb{R})$, *then*

$$\|\phi\|_\infty \le \max\{|\phi(a)|, |\phi(b)|\} + \frac{(b-a)^2}{8}\|\phi^{(2)}\|_\infty$$

and

$$\|\phi^{(1)}\|_\infty \le \frac{2}{b-a}\max\{|\phi(a)|, |\phi(b)|\} + \frac{b-a}{2}\|\phi^{(2)}\|_\infty.$$

Using this lemma, if $\phi(x) = e^{(1)}(x)$ then

$$\|e^{(1)}\|_\infty \le \max\{|e^{(1)}(x_i)|: i = 0, \dots, N\} + \frac{h^2}{8}M$$

and

$$\|e^{(2)}\|_\infty \le \max\left\{\frac{h}{3h_i} + \frac{h_i}{2h}: i = 1, \dots, N\right\} hM.$$

Finally we have the following result.

THEOREM 6. *Let* s *be a type* I, II, III, *or* IV *interpolant of* $f \in AC^3([a,b];\mathbb{R})$. *Then*

$$\|(f-s)^{(k)}\|_\infty \le C_k h^{3-k} \|f^{(3)}\|_\infty \qquad (k = 0, 1, 2)$$

where

$$C_0 = \begin{cases} \dfrac{1}{24} & \textit{for types I, II, and III}, \\ \dfrac{1}{12} & \textit{for type IV}, \end{cases}$$

$$C_1 = \frac{7}{24},$$

$$C_2 = \begin{cases} \dfrac{5}{6} & \text{if } 1 \le \beta \le 3/2, \\ \dfrac{\beta}{2} + \dfrac{1}{2\beta} & \beta \ge 3/2. \end{cases}$$

Moreover, C_0 *is optimal in the sense that*

$$C_0 = \sup_{f, \Delta} \frac{\|f - s\|_\infty}{h^3 \|f^{(3)}\|_\infty}$$

where the supremum is taken over all partitions Δ *of* $[a,b]$ *and all* $f \in AC^3([a,b];\mathbb{R})$ *such that* $\|f^{(3)}\|_\infty \ne 0$ (*for type* I, f *is assumed to be periodic*).

Proof. It remains to prove the optimality of C_0. We only have to show that for any $\varepsilon > 0$ there exists an index i and a function $f \in AC^3([a,b];\mathbb{R})$ for which

$$|e(x_i)| > (1-\epsilon)\|f^{(3)}\|_\infty \begin{cases} \dfrac{1}{24} & \text{for types I, II, and III}, \\ \dfrac{1}{12} & \text{for type IV}. \end{cases}$$

We consider the following three situations.

Type I. From (3.5a), (3.5b), (3.8a), and (3.8b) we have

$$K_0(x) = \frac{(-1)^i}{D(a,b)}\left[\left|w^{(1)}(a)\right|\left|v(x)\right| + (-1)^N\left|v^{(1)}(b)\right|\left|w(x)\right|\right]$$

for any $x \in (x_{i-1}, x_i)$ and $i = 1, \dots, N$. Hence for N even

$$\int_a^b |K_0(x)|\,dx = \frac{1}{|D(a,b)|}\left[\left|w^{(1)}(a)\right|\int_a^b |v(x)|\,dx + \left|v^{(1)}(b)\right|\int_a^b |w(x)|\,dx\right].$$

But from (3.11d) and (3.12d) for a uniform partition, we obtain

$$\int_a^b |K_0(x)|\,dx = \frac{h^3}{24}.$$

Finally, let $f \in AC^3([a,b];\mathbb{R})$ be a periodic function such that $f^{(3)}(x) = \operatorname{sgn}(K_0(x)) = (-1)^{i-1}$ for $x \in (x_{i-1}, x_i)$, then

$$e(a) = \int_a^b K_0(x) e^{(3)}(x)\,dx = \int_a^b K_0(x) f^{(3)}(x)\,dx$$

$$= \int_a^b |K_0(x)| \|f^{(3)}\|_\infty\,dx = \frac{h^3}{24}\|f^{(3)}\|_\infty.$$

Types II *and* III. From (3.7b) and (3.8b) we have

$$\left|v^{(2)}(x_i)\right| \ge 3^i\left|v^{(2)}(a)\right| \qquad \text{and} \qquad \left|w^{(2)}(x_i)\right| \ge 3^{N-i}\left|w^{(2)}(b)\right|.$$

Hence, for a uniform partition, we obtain from (3.11d) and (3.12d)

$$\int_a^{x_i} |v(x)|\,dx \ge \left(1 - \frac{1}{3^i}\right)\frac{h^3}{24}\left|v^{(2)}(x_i)\right|$$

$$\int_{x_i}^b |w(x)|\,dx \ge \left(1 - \frac{1}{3^{N-i}}\right)\frac{h^3}{24}\left|w^{(2)}(x_i)\right|$$

then

$$\int_a^b |K(x_i;x)|\,dx \ge \left(1 - \frac{1}{3^{\alpha N}}\right)\frac{h^3}{24}$$

for all $\alpha N < i < (1-\alpha)N$ where $\alpha \in (0, \frac{1}{2})$. It follows that

$$\lim_{N\to\infty}\int_a^b |K(x_i;x)|\,dx = \frac{h^3}{24}.$$

Finally, let $f \in AC^3([a,b];\mathbb{R})$ be such that

$$f^{(3)}(x) = \operatorname{sgn}\big(K_0(x_i;x)\big).$$

Then

$$\begin{aligned} e(x_i) &= \int_a^b K_0(x_i;x)e^{(3)}(x)\,dx \\ &= \int_a^b |K_0(x_i;x)|\,dx\,\|f^{(3)}\|_\infty \\ &> (1-\epsilon)\frac{h^3}{24}\|f^{(3)}\|_\infty \end{aligned}$$

for $\alpha N < i < (1-\alpha)N$ and N such that $1/3^{\alpha N} < \epsilon$.

Type IV. Let $N = 2$ and Δ be the uniform partition of $[a,b]$. In this case $v^{(2)}(a) = -|v^{(2)}(x_1)|$ and $w^{(2)}(b) = -|w^{(2)}(x_1)|$. We obtain from (3.11d) and (3.12d)

$$\int_a^{x_1} |v(x)|\,dx = \frac{h^3}{12}\left|v^{(2)}(x_1)\right| \qquad \text{and} \qquad \int_{x_1}^b |w(x)|\,dx = \frac{h^3}{12}\left|w^{(2)}(x_1)\right|.$$

Then

$$\int_a^b |K_0(x_1;x)|\,dx = \frac{h^3}{12}.$$

Finally, let $f \in AC^3([a,b];\mathbb{R})$ be such that $f^{(3)}(x) = \operatorname{sgn}(K_0(x_1;x))$ for all $x \in [a,b]$. Then

$$\begin{aligned} e(x_1) &= \int_a^b K_0(x_1;x)f^{(3)}(x)\,dx \\ &= \int_a^b |K_0(x_1;x)|\,dx\,\|f^{(3)}\|_\infty \\ &= \frac{h^3}{12}\|f^{(3)}\|_\infty. \quad \blacksquare \end{aligned}$$

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