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Study of singular boundary value problems for second order impulsive differential equations ☆

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Abstract

This paper studies the existence of extremal solutions for a class of singular boundary value problems of second order impulsive differential equations. By using the method of upper and lower solutions and the monotone iterative technique, criteria of the existence of extremal solutions are established.

Keywords: Existence; Extremal solutions; Second order impulsive differential equations; Singular boundary value problems; Method of upper and lower solutions; Monotone iterative method

1. Introduction

In this paper, we study the existence of extremal solutions for the following singular boundary value problem of impulsive differential equations:

$$
\begin{aligned}
&u''(t) + f\big(t, u(t)\big) = 0, \quad t \in (0,1),\ t \neq t_1,\\
&\Delta u|_{t=t_1} = I\big(u(t_1)\big),\\
&\Delta u'|_{t=t_1} = N\big(u(t_1), u'(t_1)\big),\\
&u(0) = a, \quad u(1) = b,
\end{aligned}
\tag{P}
$$

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where $a, b \in \mathbb{R}$, $\Delta u|_{t=t_1} = u(t_1^+) - u(t_1)$, $\Delta u'|_{t=t_1} = u'(t_1^+) - u'(t_1^-)$ and $f : D \subset (0,1) \times \mathbb{R} \to \mathbb{R}$, $I : \mathbb{R} \to \mathbb{R}$, $N : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous. We notice that f may be singular at $t = 0$ and/or $t = 1$.

Impulsive differential equations have been studied extensively in recent years. Such equations arise in many applications such as spacecraft control, impact mechanics, chemical engineering and inspection process in operations research. It is now recognized that the theory of impulsive differential equations is a natural framework for a mathematical modelling of many natural phenomena. There have appeared numerous papers on impulsive differential equations during the last ten years. Many of them are on boundary value problems, see [5,6,8,15–19,21–28,30], and it is interesting to note that some of them are about comparatively new applications like ecological competition, respiratory dynamics, vaccination strategies, see [3,7,9,13,29,31–33].

Up to now, the literature devoted to singular impulsive Dirichlet problems is not too much extensive, e.g. the cases of singularities in the space variable by [1,21,30] and the cases of singularities in $t = 0$ and $t = 1$ by [15,16,22]. The main purpose of this paper, on the contrary to the mentioned papers, is not only to obtain existence results, but to prove the existence of extremal solutions. Moreover the solutions in this paper mean so-called w-solutions, i.e. solutions which need not possess finite derivatives $u'(0)$ and $u'(1)$ at the singular points $t = 0$ and $t = 1$. The monotone iterative method, which we shall employ later is a well-known tool for proving the existence of extremal solutions for nonlinear problems. In many cases, the method can be applied successfully with help of the upper and the lower solutions. There is a vast amount of literature about the method. See [4,14,20,34] and references therein. Specially for the second order impulsive o.d.e., one may refer to [8] for two point boundary value problems and [6,18] for periodic boundary value problems. For higher order impulsive o.d.e., one may refer to [5]. As for the upper and lower solutions method, one may refer to [11] for singular Dirichlet problems with no impulse effect, [15] for singular Dirichlet impulsive problems and [23–25] for regular periodic impulsive problems.

If f is continuous or L^1-Caratheodory, then we get solutions in $PC^1[0,1]$, a Banach space of C^1-functions except at t_1. But if we consider a weaker condition on f such that the solutions may not be of $PC^1[0,1]$, then the analysis should depend on more elementary tools at least concerned with topology of the solution set. For example, consider the problem

$$\begin{aligned}
&u''(t) + \frac{1}{t(1-t)} = 0, \quad t \in (0,1),\ t \neq t_1,\\
&\Delta u|_{t=t_1} = I\big(u(t_1)\big),\\
&\Delta u'|_{t=t_1} = N\big(u(t_1), u'(t_1)\big),\\
&u(0) = 0 = u(1).
\end{aligned}$$

Let $t_1 = \frac{1}{2}$, $I(u) = u$. If $N(u,v) = u - \frac{v}{2}$, then the problem has a unique solution $u(t)$ given by

$$u(t) = \begin{cases} (-\ln 2)t - t\ln t - (1-t)\ln(1-t), & 0 < t \leqslant \frac{1}{2},\\ -t\ln t - (1-t)\ln(1-t), & \frac{1}{2} < t < 1,\\ 0, & t = 0, 1, \end{cases}$$

which is not a PC^1-function. It is interesting to see that if $N(u,v) = -8u + v$ or $N(u,v) = uv$, then the problem has no solution or two solutions respectively, on the other hand, the problem with no impulse effect always has a unique solution. This means that, when we consider impulsive problems, we may have less obvious situation for the uniqueness of solutions for the problem defining monotone approximation.

For our results, we will make use of the following assumptions:

(A_0) $u + I(u)$ is nondecreasing.
(A_1) $N(u, v) + Lu$ is nondecreasing in u, for some $0 < L < 1$, i.e. there exists $0 < L < 1$ such that $N(u_2, v) - N(u_1, v) \geqslant -L(u_2 - u_1)$, whenever $u_2 \geqslant u_1$.
And also, $N(u, v)$ is nondecreasing in v, i.e. $N(u, v_2) - N(u, v_1) \geqslant 0$, whenever $v_2 \geqslant v_1$.
(A_1') $N(u, v)$ is nondecreasing in u and $N(u, v) + Lv$ is nondecreasing in v.
(A_2) α and β are respectively the lower and the upper solution of (P) such that $\alpha(t) \leqslant \beta(t)$, for $t \in [0, 1]$.
(A_3) There exists $h_f \in \mathcal{A}$ such that $|f(t, u)| \leqslant h_f(t)$, for all $(t, u) \in D_\alpha^\beta$.
(A_4) There exists $q \in \mathcal{A}$ with $q > 0$ such that $f(t, u_2) - f(t, u_1) \geqslant -q(t)(u_2 - u_1)$, for all $t \in [0, 1]$ and $\alpha(t) \leqslant u_1 \leqslant u_2 \leqslant \beta(t)$,

where we denote

$$\mathcal{A} = \left\{ h \in C(0, 1) \;\middle|\; \int_0^1 s(1 - s)\big|h(s)\big|\, ds < \infty \right\}$$

and

$$D_\alpha^\beta = \big\{(t, u) \mid t \in (0, 1),\ \alpha(t) \leqslant u \leqslant \beta(t)\big\}.$$

The main results in this paper can be stated as follows:

Theorem 1.1. *Assume* (A_0), (A_2)–(A_4) *and also assume either* (A_1) *or* (A_1'). *Then there are monotone sequences* (α_n), (β_n) *such that* $\alpha_n \to \rho$, $\beta_n \to \gamma$ *monotonically and piecewise uniformly on* J, *where* ρ *and* γ *are the minimal and the maximal solutions of* (P)*, respectively, that is, if* u *is a solution of* (P) *with* $\alpha \leqslant u \leqslant \beta$ *on* $[0, 1]$*, then* $\rho \leqslant u \leqslant \gamma$ *on* $[0, 1]$.

We notice that the monotone property of I and N are somewhat reduced, instead, these in almost all previous works are given as monotone nondecreasing. Recently, Rachůnková and Tvrdý [23–25] also considered weakened monotonicity conditions as well.

The paper is organized as follows: In Section 2, we introduce a theorem for upper and lower solutions method for two point boundary value problems of second order impulsive differential equations. In Section 3, we prove the main theorem when $N(u, v)$ satisfies condition (A_1). In Section 4, we prove the main theorem when $N(u, v)$ satisfies condition (A_1').

2. Upper and lower solutions method

In this section, we introduce the fundamental theorem of the upper and lower solutions method for second order impulsive differential equations of the form

$$\begin{cases} u''(t) + F\big(t, u(t)\big) = 0, & t \neq t_1,\ t \in (0, 1), \\ \Delta u|_{t=t_1} = I\big(u(t_1)\big), \\ u(0) = a, \quad u(1) = b, \end{cases} \tag{1}$$

$$\Delta u'|_{t=t_1} = N\big(u(t_1), u'(t_1)\big), \tag{2}$$

where $a, b \in \mathbf{R}$, $F \in C(D, \mathbf{R})$ with $D \subset (0, 1) \times \mathbf{R}$. We denote $J = [0, 1]$ and $J' = [0, 1] \setminus \{0, 1, t_1\}$. Let $PC[0, 1] = \{u \mid u : [0, 1] \to \mathbf{R}$ is continuous at $t \neq t_1$, left continuous at $t = t_1$, and its right-hand limit at $t = t_1$ exists}. Then we know that $PC[0,1]$ is a Banach space with norm $\|u\| = \sup_{t\in[0,1]} |u(t)|$. By a solution of (1) (or (1) + (2)), we mean a function $u \in PC[0,1] \cap C^2(J')(u \in \Omega)$ which satisfies Eq. (1) ((1) + (2)), where $\Omega = \{u \in PC[0,1] \cap C^2(J') \mid u'(t_1^+), u'(t_1^-)$ exist and $u'(t_1) = u'(t_1^-)\}$.

Definition 2.1. $\alpha \in \Omega$ is called a *lower solution* of (1) + (2) if $(t, \alpha(t)) \subset D$ for all $t \in (0,1)$ and

$$\begin{aligned}
&\alpha''(t) + F\big(t, \alpha(t)\big) \geqslant 0, \quad t \neq t_1,\\
&\Delta\alpha|_{t=t_1} = I\big(\alpha(t_1)\big),\\
&\Delta\alpha'|_{t=t_1} \geqslant N\big(\alpha(t_1), \alpha'(t_1)\big),\\
&\alpha(0) \leqslant a, \quad \alpha(1) \leqslant b.
\end{aligned}$$

We also define an *upper solution* $\beta \in \Omega$ if β satisfies the reverse of the above inequalities.

To establish fundamental theorem of upper and lower solution method for problem (1) + (2), we need to use a corresponding theorem for the following singular boundary value problems with no impulse effects,

$$\begin{aligned}
&u''(t) + F\big(t, u(t)\big) = 0,\\
&u(0) = a, \quad u(1) = b.
\end{aligned} \tag{3}$$

The following lemma is due to Habets and Zanolin [11].

Lemma A. *Suppose that $\alpha, \beta \in C[0,1] \cap C^2(0,1)$ are respectively lower and upper solutions for* (3) *with $\alpha(t) \leqslant \beta(t)$. Assume also that there is a function $h \in \mathcal{A}$ such that $|F(t,u)| \leqslant h(t)$ for all $(t,u) \in D_\alpha^\beta$. Then* (3) *has at least one solution u such that $\alpha(t) \leqslant u(t) \leqslant \beta(t)$ on* $[0,1]$.

Remark. It is not hard to see that if we assume the condition $\int_0^1 sh(s)\,ds < \infty$, instead of the condition $\int_0^1 s(1-s)h(s)\,ds < \infty$, then the solution u, we find, belongs to $C^1((0,1])$. Similarly, if $\int_0^1 (1-s)h(s)\,ds < \infty$, then $u \in C^1([0,1))$.

We now give the theorem of upper and lower solutions method for problem (1) + (2).

Theorem 2.1. *Assume* (A_0) *and also assume*

- (F) *$F(t,u)$ is continuous in $t \neq t_1$ and u.* *Furthermore, $\lim_{t\to t_1^-} F(t,u) = F(t_1,u)$ and $\lim_{t\to t_1^+} F(t,u)$ exists.*
- (a_1) *$N(u,v)$ is nondecreasing in v.*
- (a_2) *α and β are respectively lower and upper solutions of* (1) + (2) *such that $\alpha(t) \leqslant \beta(t)$ for all $t \in [0,1]$.*
- (a_3) *There exists a function $h \in \mathcal{A}$ such that $|F(t,u)| \leqslant h(t)$, for all $(t,u) \in D_\alpha^\beta$.*

Then (1) + (2) *has at least one solution u such that $\alpha \leqslant u \leqslant \beta$ on J.*

Proof. Let us consider a real number $C \in [\alpha(t_1), \beta(t_1)]$. Then α and β are respectively lower and upper solutions of the following singular problem with no impulse effect,

$$u'' + F(t,u) = 0, \quad t \in (0, t_1),$$
$$u(0) = a, \quad (t_1) = C. \tag{BC}$$

Thus by Lemma A and Remark, problem (BC) has a solution $v \in C[0,t_1] \cap C^2(0,t_1]$ satisfying $\alpha(t) \leqslant v(t) \leqslant \beta(t)$ for all $t \in [0,t_1]$. By (A_0), we get

$$\alpha\big(t_1^+\big) = \alpha(t_1) + I\big(\alpha(t_1)\big) \leqslant C + I(C) \leqslant \beta(t_1) + I\big(\beta(t_1)\big) = \beta\big(t_1^+\big).$$

Define

$$\tilde{\alpha}(t) = \begin{cases} \alpha(t), & t \in (t_1, 1], \\ \alpha(t_1^+), & t = t_1, \end{cases}$$

and

$$\tilde{\beta}(t) = \begin{cases} \beta(t), & t \in (t_1, 1], \\ \beta(t_1^+), & t = t_1. \end{cases}$$

Then $\tilde{\alpha}$ and $\tilde{\beta}$ are respectively lower and upper solutions of the following problem:

$$u'' + F(t,u) = 0, \quad t \in (t_1, 1),$$
$$u(t_1) = C + I(C), \quad u(1) = b. \tag{FC}$$

Thus again by Lemma A and Remark, (FC) has a solution $w \in C[t_1, 1] \cap C^2[t_1, 1)$ satisfying $\tilde{\alpha}(t) \leqslant w(t) \leqslant \tilde{\beta}(t)$ for all $t \in [t_1, 1]$. Let us define

$$u(t) = \begin{cases} v(t), & t \in [0, t_1], \\ w(t), & t \in (t_1, 1]. \end{cases}$$

Then $u \in \Omega$ and

$$\Delta u|_{t=t_1} = u\big(t_1^+\big) - u(t_1) = w(t_1) - v(t_1) = I(C) = I\big(v(t_1)\big) = I\big(u(t_1)\big),$$

and thus u is a solution of (1) satisfying $u(t_1) = C$ and $\alpha(t) \leqslant u(t) \leqslant \beta(t)$ for all $t \in [0,1]$. So far, we have shown that for each $C \in [\alpha(t_1), \beta(t_1)]$, there exists a solution u_C of (1) satisfying $u_C(t_1) = C$. We now show that one of the solutions u_C satisfies condition (2). Let $X(C) = \{u \mid u \text{ is solution of (1) such that } u(t_1) = C \text{ and } \alpha(t) \leqslant u(t) \leqslant \beta(t), \text{ for all } t \in [0,1]\}$. Then for each $C \in [\alpha(t_1), \beta(t_1)]$, $X(C) \neq \emptyset$. Let us consider the case $\alpha(t_1) = \beta(t_1)$. We claim that any function $u \in X(\alpha(t_1))$ satisfies condition (2). Indeed, if $u \in X(\alpha(t_1))$, then since $u(t) \geqslant \alpha(t)$, $u(t_1) = \alpha(t_1)$, and $u(t_1^+) = \alpha(t_1^+)$, we get $\alpha'(t_1) \geqslant u'(t_1)$ and $\alpha'(t_1^+) \leqslant u'(t_1^+)$. Therefore by ($a_1$),

$$\begin{aligned}\Delta u'|_{t=t_1} = u'\big(t_1^+\big) - u'\big(t_1^-\big) &\geqslant \alpha'\big(t_1^+\big) - \alpha'\big(t_1^-\big) = \Delta\alpha'|_{t=t_1} \\ &\geqslant N\big(\alpha(t_1), \alpha'(t_1)\big) \geqslant N\big(u(t_1), u'(t_1)\big).\end{aligned}$$

On the other hand, if $u \in X(\beta(t_1))$, then we can similarly show $\Delta u'|_{t=t_1} \leqslant N(u(t_1), u'(t_1))$. Since $X(\alpha(t_1)) = X(\beta(t_1)) \neq \emptyset$, this completes the claim. Now let us consider the case $\alpha(t_1) < \beta(t_1)$. Define $\mathcal{S} = \{\bar{C} \in [\alpha(t_1), \beta(t_1))\colon\ C \in (\bar{C}, \beta(t_1)) \text{ implies } \Delta u'_C|_{t=t_1} < N(u_C(t_1), u'_C(t_1)),$ for all $u_C \in X(C)\}$. We will prove cases $\mathcal{S} \neq \emptyset$ and $\mathcal{S} = \emptyset$ separately. First consider the case $\mathcal{S} \neq \emptyset$. Let $C^* = \inf \mathcal{S}$, then $\alpha(t_1) \leqslant C^* < \beta(t_1)$. Here again, we consider two cases separately.

Case 1. $C^* > \alpha(t_1)$. Then by the definition of C^*, we can choose sequences $C_n \in (\alpha(t_1), C^*)$ and $u_{C_n} \in X(C_n)$ such that $C_n \to C^*$ and

$$\Delta u'_{C_n}\big|_{t=t_1} \geqslant N\big(u_{C_n}(t_1), u'_{C_n}(t_1)\big). \tag{4}$$

We know that the sequence (u_{C_n}) is bounded in $PC[0,1]$ and

$$u_{C_n}(t) = \begin{cases} v_{C_n}(t), & t \in [0, t_1], \\ w_{C_n}(t), & t \in (t_1, 1], \end{cases}$$

where v_{C_n} and w_{C_n} are solutions of (BC_n) and (FC_n), respectively. Thus v_{C_n} satisfies

$$v_{C_n}(t) = T_n v_{C_n}(t) \triangleq a + \frac{C_n - a}{t_1} t + \int_0^{t_1} G_1(t,s) F\big(s, v_{C_n}(s)\big)\, ds,$$

for $t \in [0, t_1]$ and w_{C_n} satisfies

$$w_{C_n}(t) = \bar{T}_n w_{C_n}(t) \triangleq b + \frac{b - I(C_n) - C_n}{1 - t_1}(t-1) + \int_{t_1}^{1} G_2(t,s) F\big(s, w_{C_n}(s)\big)\, ds,$$

for $t \in [t_1, 1]$, where $G_1(t,s)$ and $G_2(t,s)$ are the Green's functions of linear homogeneous problem with Dirichlet boundary condition corresponding to (BC) and (FC). It is well known that each T_n and $\bar{T}_n$ is completely continuous and by Arzela–Ascoli theorem, we can find a subsequence, say (v_{C_n}) again converging to v_1 in $C[0, t_1]$. Since $v_{C_n} = T_n(v_{C_n})$ and $v_{C_n} \to v_1$, by the Lebesgue Dominated Convergence Theorem, v_1 satisfies

$$v_1(t) = a + \frac{C^* - a}{t_1} t + \int_0^{t_1} G_1(t,s) F\big(s, v_1(s)\big)\, ds,$$

for $t \in [0, t_1]$. Thus $v_1 \in C[0, t_1] \cap C^2(0, t_1]$ and a solution of

$$u'' + F(t,u) = 0, \quad t \in (0, t_1),$$
$$u(0) = a, \quad u(t_1) = C^*.$$

Keeping the index of the convergent subsequence (v_{C_n}), we see that (w_{C_n}) also has a convergent subsequence converging to say, w_1 in $C[t_1, 1]$. By the similar argument, we see that $w_1 \in C[t_1, 1] \cap C^2(t_1, 1]$ and a solution of

$$u'' + F(t,u) = 0, \quad t \in (t_1, 1),$$
$$u(t_1) = C^* + I(C^*), \quad u(1) = b.$$

Define

$$u_1(t) = \begin{cases} v_1(t), & t \in [0, t_1], \\ w_1(t), & t \in (t_1, 1]. \end{cases}$$

Then $u_1 \in \Omega$ and a solution of (1) satisfying $u_1(t_1) = C^*$ and $\alpha \leqslant u_1 \leqslant \beta$ on J. Furthermore, differentiating v_{C_n} and w_{C_n}, we get

$$\begin{aligned} \Delta u'_{C_n}\big|_{t=t_1} &= u'_{C_n}\big(t_1^+\big) - u'_{C_n}\big(t_1^-\big) = w'_{C_n}(t_1) - v'_{C_n}(t_1) = \frac{b - I(C_n) - C_n}{1 - t_1} \\ &\quad + \frac{1}{1 - t_1} \int_{t_1}^{1} (1-s) F\big(s, w_{C_n}(s)\big)\, ds - \frac{C_n - a}{t_1} + \frac{1}{t_1} \int_0^{t_1} s F\big(s, v_{C_n}(s)\big)\, ds. \end{aligned}$$

Again by Lebesgue Dominated Convergence Theorem, we have

$$\lim_{n\to\infty} \Delta u'_{C_n}\big|_{t=t_1} = \frac{b - I(C^*) - C^*}{1-t_1} + \frac{1}{1-t_1}\int_{t_1}^{1} (1-s)F\big(s, w_1(s)\big)\,ds$$
$$- \frac{C^* - a}{t_1} + \frac{1}{t_1}\int_0^{t_1} sF\big(s, v_1(s)\big)\,ds = w_1'(t_1) - v_1'(t_1)$$
$$= u_1'\big(t_1^+\big) - u_1'\big(t_1^-\big) = \Delta u_1'\big|_{t=t_1}.$$

Thus by (4) and the continuity of N, we have

$$\Delta u_1'\big|_{t=t_1} \geqslant N\big(u_1(t_1), u_1'(t_1)\big).$$

This implies that u_1 is a lower solution of (1) + (2) satisfying $u_1(t_1) = C^*$. Applying the beginning part of the proof for $u_1(t)$ and $\beta(t)$ as the lower and upper solutions, respectively, we guarantee that for all $D \in [C^*, \beta(t_1)]$, there exists $u_D \in X(D)$ with $u_1(t) \leqslant u_D(t) \leqslant \beta(t)$, $t \in [0,1]$. Thus by the definition of C^*, we can find sequences $D_n \in (C^*, \beta(t_1))$ and $u_{D_n} \in X(D_n)$ with $u_1 \leqslant u_{D_n} \leqslant \beta$, on J such that $D_n \to C^*$, $\Delta u'_{D_n}|_{t=t_1} < N(u_{D_n}(t_1), u'_{D_n}(t_1))$. By the similar argument as we construct u_1, we obtain $u_2 \in \Omega$, a solution of (1) satisfying $u_2(t_1) = C^*$, $u_1 \leqslant u_2 \leqslant \beta$ on J, $u_{D_n} \to u_2$ in $PC[0,1]$ as a subsequence if necessary and $\Delta u_2'|_{t=t_1} \leqslant N(u_2(t_1), u_2'(t_1))$. Thus u_2 is an upper solution of (1) + (2). Consequently, u_1 and u_2 are a lower and an upper solution of (1) + (2) respectively satisfying $u_1(t_1) = C^* = u_2(t_1)$ and $u_1 \leqslant u_2$ on J. Therefore, any function $u \in X(C^*)$ satisfies condition (2) and this completes the proof for the case $C^* > \alpha(t_1)$.

Case 2. $C^* = \alpha(t_1)$. Then by the definition of C^*, we may choose sequences D_n and $u_{D_n} \in X(D_n)$ satisfying $D_n \to \alpha(t_1)$ and

$$\Delta u'_{D_n}\big|_{t=t_1} < N\big(u_{D_n}(t_1), u'_{D_n}(t_1)\big).$$

By the same limit argument as we constructed u_2, we obtain $u_3 \in \Omega$, a solution of (1) satisfying $u_3(t_1) = \alpha(t_1)$, $\alpha \leqslant u_3 \leqslant \beta$ on J, $u_{D_n} \to u_3$ in $PC[0,1]$ as a subsequence if necessary and $\Delta u_3'|_{t=t_1} \leqslant N(u_3(t_1), u_3'(t_1))$. Thus u_3 is an upper solution of (1) + (2) and consequently, α and u_3 are a lower and an upper solution of (1) + (2) respectively satisfying $\alpha(t_1) = u_3(t_1)$ and $\alpha \leqslant u_3$ on J. Therefore, any function $u \in X(\alpha(t_1))$ satisfies condition (2) and this completes the proof for the case $C^* = \alpha(t_1)$.

Finally, we consider the case $\mathcal{S} = \emptyset$. In this case, we may choose sequences $C_n \in (\alpha(t_1), \beta(t_1))$ and $u_{C_n} \in X(C_n)$ satisfying $C_n \to \beta(t_1)$ and

$$\Delta u'_{C_n}\big|_{t=t_1} \geqslant N\big(u_{C_n}(t_1), u_{C_n}(t_1)\big).$$

By the similar limit argument as before, we obtain $u_4 \in \Omega$, a solution of (1) satisfying $u_4(t_1) = \beta(t_1)$, $\alpha(t) \leqslant u_4(t) \leqslant \beta(t)$, $t \in [0,1]$, $u_{D_n} \to u_4$ in $PC[0,1]$ as a subsequence if necessary and $\Delta u_4'|_{t=t_1} \geqslant N(u_4(t_1), u_4'(t_1))$. On the other hand, $u_4 \in X(\beta(t_1))$ implies $\Delta u_4'|_{t=t_1} \leqslant N(u_4(t_1), u_4'(t_1))$. Thus u_4 satisfies condition (2) and the whole proof is complete. □

3. Monotone iterative method 1

In this section, we prove the main result if N satisfies (A_1). To obtain the existence of extremal solutions of (P), we will employ the monotone iterative method with the help of upper and lower solutions. We state the main theorem in this section.

Theorem 3.1. *Assume* (A_0), (A_1), (A_2), (A_3) *and* (A_4). *Then problem* (P) *admits the same conclusion as in Theorem* 1.1.

The following two lemmas are maximal principles for singular and impulsive differential operators respectively.

Lemma 3.1. [12] *Let* $q \in C(\tau_1, \tau_2)$, $q > 0$. *Then*

$$u \leqslant 0 \quad \text{on } [\tau_1, \tau_2]$$

holds for each $u \in C[\tau_1, \tau_2] \cap C^2(\tau_1, \tau_2)$ *fulfilling*

$$u''(t) - q(t)u(t) \geqslant 0, \quad t \in (\tau_1, \tau_2),$$
$$u(\tau_1) \leqslant 0, \quad u(\tau_2) \leqslant 0.$$

Lemma 3.2. *Let* $q \in C(0, 1)$, $q > 0$ *and* $L \in (0, 1)$. *Then*

$$u \leqslant 0 \quad \text{on } [0, 1]$$

holds for each $u \in \Omega$ *fulfilling*

(i) $u''(t) - q(t)u(t) \geqslant 0$, $t \in (0, 1)$, $t \neq t_1$,
(ii) $u(t_1) \geqslant 0$, $u(t_1^+) \geqslant 0$,
(iii) $\Delta u'|_{t=t_1} \geqslant -Lu(t_1)$,
(iv) $u(0) \leqslant 0$, $u(1) \leqslant 0$.

Proof. Let $u \in \Omega$ satisfy (i)–(iv).

Step 1. We notice that u cannot have a positive local maximum in J'. Unless otherwise, u has a positive maximum at some $t_0 \in J'$. Then by concavity of u, $u''(t_0) \leqslant 0$ which contradicts to (i).

Step 2. Let $u(t_1) = \max\{u(t) \mid t \in [0, 1]\} > 0$. Then $u'(t_1) \geqslant 0$. Furthermore, by (i), there is $\tau \in [0, t_1)$ such that $u(t) > 0$ for $t \in (\tau, t_1]$ and $u(\tau) = 0$. In particular, $u''(t) > 0$ on (τ, t_1) and $u'(t_1) > u(t_1)/t_1$. Hence, by (ii) and (iii), we get

$$u'\left(t^+\right) \geqslant u'(t_1) - Lu(t_1) > \left(\frac{1}{t_1} - L\right)u(t_1) \geqslant 0.$$

Since (ii) and (iv) yield

$$u\left(t_1^+\right) \geqslant 0 \quad \text{and} \quad u(1) \leqslant 0,$$

this implies that u has a positive local maximum in $(t_1, 1)$ which is impossible by Step 1.

Step 3. Let $u(t_1^+) = \sup\{u(t) \mid t \in [0, 1]\} > 0$ and $u(t_1^+) > u(t_1)$. Clearly, $u'(t_1^+) \leqslant 0$. Furthermore, by (i), there is $\tau \in (t_1, \tau]$ such that $u(t) > 0$ for $t \in (t_1, \tau)$ and $u(\tau) = 0$. In particular, $u''(t) > 0$ for $t \in (t_1, \tau)$ and $u'(t_1^+) < -u(t_1^+)/(1 - t_1)$. Hence, using (iii), we derive

$$-\frac{1}{1 - t_1}u\left(t_1^+\right) > u'\left(t_1^+\right) > u'(t_1) - Lu\left(t_1^+\right),$$

i.e.

$$u'(t_1) < \left(L - \frac{1}{1 - t_1}\right)u\left(t_1^+\right) < 0.$$

In addition, we have $u(t_1) \geqslant 0$ and $u(0) \leqslant 0$ by (ii) and (iv). Therefore, u has a positive local maximum in $(0, t_1)$ which is impossible by Step 1. $\square$

To define approximations to extremal solutions, we need a modified problem. For each $\eta \in \Omega$ with $\alpha(t) \leqslant \eta(t) \leqslant \beta(t)$, for all $t \in [0, 1]$, let us consider

$$\begin{aligned}
&u'' + F_\eta(t, u) = 0, \quad t \in (0, 1),\ t \neq t_1,\\
&\Delta u|_{t=t_1} = I\big(u(t_1)\big),\\
&\Delta u'|_{t=t_1} = N\big(u(t_1), u'(t_1)\big),\\
&u(0) = a, \quad u(1) = b,
\end{aligned} \tag{MP}$$

where $F_\eta(t, u) = f(t, \eta(t)) - q(t)(u - \eta(t))$ and q is given in (A_4).

The following lemma shows the existence of unique solution for (MP). We note that continuity of f and condition (A_4) are not necessary for the proof.

Lemma 3.3. *Let $f(t, u)$ be continuous in $t \neq t_1$ and u such that*

$$\lim_{t\to t_1^-} f(t, u) = f(t_1, u) \quad \text{and} \quad \lim_{t\to t_1^+} f(t, u) \text{ exists}$$

and let $q \in \mathcal{A}$. Also assume (A_0), (A_1), (A_2) *and* (A_3). *Then problem* (MP) *has a unique solution u such that $\alpha \leqslant u \leqslant \beta$ on J.*

Proof. It is not hard to see that F_η satisfies condition (F) and α, β are respectively the lower and the upper solution of (MP). Thus by Theorem 2.1, (MP) has a solution u such that $\alpha \leqslant u \leqslant \beta$ on J. We prove that the solution is unique. Suppose that (MP) admits two distinct solutions u_1 and u_2. Without loss of generality, let us assume that $m(t) \triangleq (u_1 - u_2)(t) > 0$ on some subintervals of J. Then $m \in \Omega$, $m(0) = 0 = m(1)$ and

$$m''(t) = q(t)\big(u_1(t) - \eta(t)\big) - f\big(t, \eta(t)\big) - q(t)\big(u_2(t) - \eta(t)\big) + f\big(t, \eta(t)\big) = q(t)m(t)$$

for $t \in J'$. It is easy to see that m does not attain its positive maximum at $t \in J'$ and thus m satisfies either $m(t_1) = \max_{t\in J} m(t)$ or $m(t_1^+) = \sup_{t\in J} m(t)$. If $m(t_1) = \max_{t\in J} m(t)$, then obviously $m(t_1) > 0$ and $\Delta m|_{t=t_1} = \Delta u_1|_{t=t_1} - \Delta u_2|_{t=t_1} = I(u_1(t_1)) - I(u_2(t_1))$. Thus by (A_0), $m(t_1^+) = u_1(t_1) + I(u_1(t_1)) - [u_2(t_1) + I(u_2(t_1))] \geqslant 0$. On the other hand, if $m(t_1^+) = \sup_{t\in J} m(t)$, then $m(t_1^+) > 0$, that is, $m(t_1^+) = u_1(t_1) + I(u_1(t_1)) - [u_2(t_1) + I(u_2(t_1))] > 0$. Thus by (A_0), $m(t_1) > 0$. Furthermore, we can show $m'(t_1) \geqslant 0$, indeed, otherwise, m should have a local maximum on $(0, t_1) \subset J'$ by the boundary condition of m which is a contradiction. Thus by (A_1), we have

$$\begin{aligned}
\Delta m'|_{t=t_1} &= \Delta u_1'\big|_{t=t_1} - \Delta u_2'\big|_{t=t_1}\\
&= N\big(u_1(t_1), u_1'(t_1)\big) - N\big(u_2(t_1), u_2'(t_1)\big)\\
&= N\big(u_1(t_1), u_1'(t_1)\big) - N\big(u_2(t_1), u_1'(t_1)\big) + N\big(u_2(t_1), u_1'(t_1)\big) - N\big(u_2(t_1), u_2'(t_1)\big)\\
&\geqslant -Lm(t_1).
\end{aligned}$$

Therefore m satisfies (i)–(iv) in Lemma 3.2 so that $m(t) \leqslant 0$ on J and this is a contradiction. $\square$

For any $\eta \in [\alpha, \beta]$, we know that problem (MP) has a unique solution u. Define an operator $A : [\alpha, \beta] \subset PC[0, 1] \to PC[0, 1]$ by $A\eta = u$.

Lemma 3.4. *Assume* (A_0), (A_1), (A_2)–(A_4). *Then the operator* A *satisfies the following*:

(i) $A\alpha, A\beta \in [\alpha, \beta]$.
(ii) A *is a monotone operator on* $[\alpha, \beta]$.

Proof. Proof of (i) is clear by Lemma 3.3.

Proof of (ii). Let $\eta_1, \eta_2 \in [\alpha, \beta]$, $\eta_1 \leqslant \eta_2$ and let $u_i = A\eta_i$, $i = 1, 2$. We show $u_1 \leqslant u_2$ on J, whenever $\eta_1 \leqslant \eta_2$ on J. If it is not true, so let $\sup_{t\in J}[(u_1 - u_2)(t) \triangleq m(t)] > 0$. Then obviously $m \in \Omega$, $m(0) = 0 = m(1)$ and by (A_4), we get

$$\begin{aligned} m''(t) &= u_1''(t) - u_2''(t) \\ &= q(t)\big(u_1(t) - \eta_1(t)\big) - f\big(t, \eta_1(t)\big) - q(t)\big(u_2(t) - \eta_2(t)\big) + f\big(t, \eta_2(t)\big) \\ &= q(t)m(t) + q(t)\big(\eta_2(t) - \eta_1(t)\big) + f\big(t, \eta_2(t)\big) - f\big(t, \eta_1(t)\big) \\ &\geqslant q(t)m(t) + q(t)\big(\eta_2(t) - \eta_1(t)\big) - q(t)\big(n_2(t) - \eta_1(t)\big) \\ &= q(t)m(t), \end{aligned}$$

on J'. Again following the same lines as in the proof of Lemma 3.3, we can show $m(t_1) \geqslant 0$, $m(t_1^+) \geqslant 0$ and $\Delta m'|_{t=t_1} \geqslant -Lm(t_1)$. By Lemma 3.2, $m(t) \leqslant 0$ on J and this is a contradiction. □

Now we prove Theorem 3.1.

Proof of Theorem 3.1. Let $\alpha_0 = \alpha$ and define $\alpha_n = A\alpha_{n-1}$ for $n \in \mathbb{N}$. Then by Lemmas 3.3 and 3.4, $\alpha \leqslant \alpha_n \leqslant \alpha_{n+1} \leqslant \beta$ for every $n \in \mathbb{N}$ and thus (α_n) is increasing. We know that for each $t \in (0, 1)$, the sequence $(\alpha_n(t))$ of real numbers converges to some finite real number $\rho(t)$. Define $\rho(0) = a$, $\rho(1) = b$. We will show that ρ is the minimal solution of (P). Let $\alpha_n(t_1) = C_n$. Then α_n satisfies

$$\begin{aligned} &\alpha_n''(t) + F_{\alpha_{n-1}}\big(t, \alpha_n(t)\big) = 0, \quad t \in (0, t_1), \\ &\alpha_n(0) = a, \quad \alpha_n(t_1) = C_n \end{aligned}$$

and

$$\begin{aligned} &\alpha_n''(t) + F_{\alpha_{n-1}}\big(t, \alpha_n(t)\big) = 0, \quad t \in (t_1, 1), \\ &\alpha_n(t_1) = C_n + I(C_n), \quad \alpha_n(1) = b. \end{aligned}$$

Therefore α_n can be written as

$$\alpha_n(t) = \begin{cases} a + \frac{C_n - a}{t_1} t + \int_0^{t_1} G_1(t,s) F_{\alpha_{n-1}}(s, \alpha_n(s))\, ds, & \text{on } [0, t_1], \\ b + \frac{b - I(C_n) - C_n}{1 - t_1}(t-1) + \int_{t_1}^1 G_2(t,s) F_{\alpha_{n-1}}(s, \alpha_n(s))\, ds, & \text{on } (t_1, 1], \end{cases}$$

where $G_i(t,s)$ are given in the proof of Theorem 2.1. Since $F_{\alpha_{n-1}}(t, \alpha_n(t)) \to f(t, \rho(t))$ for $t \in (0, 1)$, by Lebesgue Dominated Convergence Theorem, we get

$$\lim_{n\to\infty} \alpha_n(t) = a + \frac{\rho(t_1) - a}{t_1} t + \int_0^{t_1} G_1(t,s) f\big(s, \rho(s)\big)\, ds, \tag{5}$$

on $(0, t_1]$ and

$$\lim_{n\to\infty} \alpha_n(t) = b + \frac{b - I(\rho(t_1)) - \rho(t_1)}{1 - t_1}(t - 1) + \int_{t_1}^{1} G_2(t,s) f\bigl(s, \rho(s)\bigr)\, ds, \tag{6}$$

on $(t_1, 1]$. Let ρ_0 and ρ_1 be functions on the right sides of (5) and (6) respectively. Then $\rho_0 \in C[0, t_1] \cap C^2(0, t_1]$ and it is a solution of

$$u''(t) + f\bigl(t, u(t)\bigr) = 0, \quad t \in (0, t_1),$$
$$u(0) = a, \quad u(t_1) = \rho(t_1)$$

and also $\rho_1 \in C[t_1, 1] \cap C^2[t_1, 1)$ and it is a solution of

$$u''(t) + f\bigl(t, u(t)\bigr) = 0, \quad t \in (t_1, 1),$$
$$u(1) = \rho(t_1) + I\bigl(\rho(t_1)\bigr), \quad u(1) = b.$$

Furthermore, we have

$$\rho(t) = \begin{cases} \rho_0(t) & \text{on } [0, t_1], \\ \rho_1(t) & \text{on } (t_1, 1]. \end{cases}$$

Therefore $\rho \in PC[0, 1] \cap C^2(J')$. Since α_n is a solution of second order o.d.e. mentioned above, by using standard arguments, we can easily check that α_n converges to ρ piecewise uniformly and

$$\alpha_n'(t) \to \rho'(t), \quad \text{for } t \in (0, 1). \tag{7}$$

Now we prove ρ is a solution of (P). We know by Guo [10] that the solution α_n of (MP) with $\eta = \alpha_{n-1}$ can be equivalently written as

$$\begin{aligned} \alpha_n(t) = a + (b - a)t + \int_0^1 G(t,s) F_{\alpha_{n-1}}\bigl(s, \alpha_n(s)\bigr)\, ds \\ - t\bigl[I\bigl(\alpha_n(t_1)\bigr) + (1 - t_1) N\bigl(\alpha_n(t_1), \alpha_n'(t_1)\bigr)\bigr] \\ + \sum_{t_1 < t}\bigl[I\bigl(\alpha_n(t_1)\bigr) + (t - t_1) N\bigl(\alpha_n(t_1), \alpha_n'(t_1)\bigr)\bigr], \end{aligned}$$

where $G(t,s)$ is the Green's function of corresponding linear homogeneous problem with Dirichlet condition on $[0, 1]$. Thus by (7) and Lebesgue Dominated Convergence Theorem, we get

$$\begin{aligned} \rho(t) = \lim_{n\to\infty} \alpha_n(t) = a + (b - a)t + \int_0^1 G(t,s) f\bigl(s, \rho(s)\bigr)\, ds \\ - t\bigl[I\bigl(\rho(t_1)\bigr) + (1 - t_1) N\bigl(\rho(t_1), \rho'(t_1)\bigr)\bigr] \\ + \sum_{t_1 < t}\bigl[I\bigl(\rho(t_1)\bigr) + (t - t_1) N\bigl(\rho(t_1), \rho'(t_1)\bigr)\bigr] \end{aligned}$$

and this implies that ρ is a solution of (P). Finally, to show that ρ is the minimal solution of (P), let u be a solution of (P) with $\alpha \leqslant u \leqslant \beta$. Then $\alpha_0 \leqslant u$ implies $\alpha_1 = A\alpha_0 \leqslant Au = u$ and by induction, $\alpha_n \leqslant u$ for all $n \in \mathbb{N}$. Thus $\rho \leqslant u$. Define $\beta_0 = \beta$ and $\beta_n = A\beta_{n-1}$ for $n \in \mathbb{N}$. Then by the same details provided above, we obtain that (β_n) is decreasing and converges to the maximal solution. This completes the proof. $\Box$

If N does not depend on the derivative of solution at t_1, then we have the following corollary.

Corollary 3.1. *Assume* (A_0), (A_2)–(A_4). *Also assume*

(A_1'') *$Nu + Lu$ is nondecreasing, for some* $0 < L < 1$.

Then there are monotone sequences (α_n), (β_n) such that $\alpha_n \to \rho$, $\beta_n \to \gamma$ monotonically and piecewise uniformly on J, where ρ and γ are the minimal and the maximal solution of the following problem

$$\begin{aligned}
&u''(t) + f\big(t, u(t)\big) = 0, \quad t \in (0,1),\ t \neq t_1,\\
&\Delta u|_{t=t_1} = I\big(u(t_1)\big),\\
&\Delta u'|_{t=t_1} = N\big(u(t_1)\big),\\
&u(0) = a, \quad u(1) = b,
\end{aligned}$$

respectively.

We also apply Theorem 3.1 for singular problems with no impulse effect.

Corollary 3.2. *Assume* (A_2)–(A_4). *Then there are monotone sequences (α_n), (β_n) such that $\alpha_n \to \rho$, $\beta_n \to \gamma$ monotonically and piecewise uniformly on J, where ρ and γ are the minimal and the maximal solutions of the following problem*

$$\begin{aligned}
&u''(t) + f\big(t, u(t)\big) = 0, \quad t \in (0,1),\\
&u(0) = a, \quad u(1) = b,
\end{aligned}$$

respectively.

4. Monotone iteration method 2

In this section, we prove the main result when N satisfies (A_1'). In this case, we cannot apply Theorem 2.1 directly so that we need to investigate modified problem (MP) more thoroughly. We state the main theorem in this section.

Theorem 4.1. *Assume* (A_0), (A_1'), (A_2)–(A_4). *Then problem* (P) *admits the same conclusion as in Theorem* 1.1.

Let

$$\bar{\mathcal{A}} = \left\{ h \in PC(0,1) \;\middle|\; \int_0^1 s(1-s)\big|h(s)\big|\,ds < \infty \right\},$$

where

$$PC(0,1) = \big\{u : (0,1) \to \mathbb{R} \;\big|\; u \text{ is continuous at } t \neq t_1,\, u\big(t_1^-\big) = u(t_1) \text{ and } u\big(t_1^+\big) \text{ exists}\big\}.$$

For $h \in \bar{\mathcal{A}}$, consider

$$\begin{cases} Lu \triangleq -u'' + q(t)u = h(t), & t \in (0,1),\ t \neq t_1, \\ \Delta u|_{t=t_1} = I(u(t_1)), \\ u(0) = a, \quad u(1) = b, \end{cases} \tag{8}$$

$$\Delta u'|_{t=t_1} = N\big(u(t_1), u'(t_1)\big). \tag{9}$$

We first consider solutions of (8). We notice from well-known singular eigenvalue problems [2] that

$$Lu = 0, \quad t \in (0, t_1),$$
$$u(0) = 0 = u(t_1)$$

has only the trivial solution. Thus, for any $c \in \mathbb{R}$, the following problem

$$Lu = h(t), \quad t \in (0, t_1),$$
$$u(0) = a, \quad u(t_1) = c$$

has a unique solution u_0 explicitly written as

$$u_0(t) = \frac{a}{z_1(0)} z_1(t) + \frac{c}{z_0(t_1)} z_0(t) + \int_0^{t_1} K_0(t,s)h(s)\,ds,$$

where $K_0(t,s)$ is the Green's function given as

$$K_0(t,s) = \begin{cases} \frac{z_0(s)z_1(t)}{W_z(s)}, & 0 < s \leqslant t, \\ \frac{z_0(t)z_1(s)}{W_z(s)}, & t \leqslant s < t_1. \end{cases}$$

We recall that z_0 and z_1 are the two linearly independent solutions of $Lu = 0$ on $(0, t_1)$ with $z_0(0) = 0$, $z_0'(0) = 1$, $z_1(t_1) = 0$, $z_1'(t_1) = -1$ and $W_z(t)$ is the corresponding Wronskian. We notice that $z_i \in C^1[0, t_1]$, $z_i(t) > 0$ on $(0, t_1)$, $i = 0, 1$, and $z_0'(t) > 0$, $z_1'(t) < 0$ on $[0, t_1]$. Similarly, the problem

$$Lu = h(t), \quad t \in (t_1, 1),$$
$$u(t_1) = c + I(c), \quad u(1) = b$$

has a unique solution u_1 written as

$$u_1(t) = \frac{c + I(c)}{w_1(t_1)} w_1(t) + \frac{b}{w_0(1)} w_0(t) + \int_{t_1}^{1} K_1(t,s)h(s)\,ds,$$

where $K_1(t,s)$ is the Green's function given as

$$K_1(t,s) = \begin{cases} \frac{w_0(s)w_1(t)}{W_w(s)}, & t_1 < s \leqslant t, \\ \frac{w_0(t)w_1(s)}{W_w(s)}, & t \leqslant s < 1. \end{cases}$$

We also notice that $w_i \in C^1[t_1, 1]$, $w_i(t) > 0$ on $(t_1, 1)$, $i = 0, 1$, and $w_0'(t) > 0$, $w_1'(t) < 0$ on $[t_1, 1]$. Define

$$u_c(t) = \begin{cases} u_0(t) & \text{on } [0, t_1], \\ u_1(t) & \text{on } (t_1, 1]. \end{cases}$$

Then $u_c \in \Omega$ and it is a solution of (8) uniquely determined up to $u_c(t_1) = c$. Differentiating u_0 and u_1, we get

$$\Delta u'|_{t=t_1} = u_1'(t_1) - u_0'(t_1) = \sigma - \sigma_1 c - \sigma_2 I(c),$$

where constants σ_1, σ_2 and σ are given by

$$\sigma_1 = \frac{z_0'(t_1)}{z_0(t_1)} - \frac{w_1'(t_1)}{w_1(t_1)},$$
$$\sigma_2 = -\frac{w_1'(t_1)}{w_1(t_1)},$$
$$\sigma = \frac{a}{z_1(0)} + \frac{b}{w_0(1)} + \int_0^{t_1} \frac{z_0}{W_z}(s)h(s)\,ds + \int_{t_1}^{1} \frac{w_1}{W_w}(s)h(s)\,ds.$$

We notice $\sigma_1 > \sigma_2 > 0$. Since the solution of problem (8) + (9) is one of u_c which satisfies condition (9) and $u_c'(t_1) = u_0'(t_1) = \sigma_3 c + \sigma_4$, where

$$\sigma_3 = \frac{z_0'(t_1)}{z_0(t_1)} > 0,$$
$$\sigma_4 = -\frac{a}{z_1(0)} - \int_0^{t_1} \frac{z_0}{W_z}(s)h(s)\,ds,$$

we see that the solution u_c of (8) satisfying

$$\sigma_1 c + \sigma_2 I(c) + N(c, \sigma_3 c + \sigma_4) = \sigma \tag{10}$$

is a solution of (8) + (9). Furthermore, if problem (10) has a unique solution c^*, then u_{c^*} must be a unique solution of (8) + (9). Now we have the existence of unique solution for (8) + (9) as follows.

Lemma 4.1. *Assume* (A_0) *and* (A_1'). *Then for* $h \in \bar{\mathcal{A}}$, *problem* (8) + (9) *has a unique solution.*

Proof. For any $c \in \mathbb{R}$, let u_c be a solution of (8) + (9) satisfying $u_c(t_1) = c$. By the above argument, it is enough to show that Eq. (10) has a unique real solution. Define $\varphi(x) = \sigma_1 x + \sigma_2 I(x) + N(x, \sigma_3 x + \sigma_4)$ for $x \in \mathbb{R}$. Then $\varphi : \mathbb{R} \to \mathbb{R}$ is a continuous function. By showing that φ is strictly increasing and $\operatorname{Ran}\varphi = \mathbb{R}$, we can complete the proof. First, we show that φ is strictly increasing. Let $l(x) = \sigma_3 x + \sigma_4$ and consider φ as the sum of two functions φ_1 and φ_2, where $\varphi_1(x) = \sigma_1 x - Ll(x) + \sigma_2 I(x)$ and $\varphi_2(x) = Ll(x) + N(x, l(x))$. Then by ($A_1'$), $\varphi_2(x)$ is nondecreasing. Furthermore, it is not hard to see by (A_0) that $Ax + BI(x)$ is strictly increasing, if $A > B > 0$. Since

$$\varphi_1(x) = (\sigma_1 - L\sigma_3)x + \sigma_2 I(x) - L\sigma_4$$

and

$$\sigma_1 - L\sigma_3 = (1 - L)\frac{z_0'(t_1)}{z_0(t_1)} - \frac{w_1'(t_1)}{w_1(t_1)} > \sigma_2 > 0,$$

we get $\varphi_1(x)$ is strictly increasing. Therefore φ is the sum of strictly increasing and nondecreasing functions and this implies that φ is strictly increasing. Finally, if we show $\operatorname{Ran}\varphi_1 = \mathbb{R}$, then

$\operatorname{Ran}\varphi = \mathbb{R}$ and the proof is done. If it is not true, then φ_1 is either bounded above or bounded below. If φ_1 is bounded above, then there is $M > 0$ such that $(\sigma_1 - L\sigma_3)x + \sigma_2 I(x) < M$, for all $x \in \mathbb{R}$. Thus we get

$$I(x) < \frac{M}{\sigma_2} - \frac{\sigma_1 - L\sigma_3}{\sigma_2}x, \quad \text{for all } x \in \mathbb{R}. \tag{11}$$

We notice $\frac{\sigma_1 - L\sigma_3}{\sigma_2} > 1$. On the other hand, from (A_0),

$$I(x) > I(0) - x, \quad \text{for all } x > 0. \tag{12}$$

We also notice $I(0) < \frac{M}{\sigma_2}$. By (11) and (12), we obtain

$$I(0) - x < I(x) < \frac{M}{\sigma_2} - \frac{\sigma_1 - L\sigma_3}{\sigma_2}x, \quad \text{for all } x > 0.$$

This is impossible. Similarly, if φ_1 is bounded below then we obtain

$$-\frac{M}{\sigma_2} - \frac{\sigma_1 - L\sigma_3}{\sigma_2}x < I(x) < I(0) - x, \quad \text{for all } x < 0.$$

This is also impossible and the proof is done. □

Lemma 4.2. *A function $u \in \Omega$ satisfying the following inequalities does not exist: For $q \in C(0,1)$ with $q > 0$,*

$$\begin{aligned} &u''(t) - q(t)u(t) \geqslant 0, \quad t \in (0,1),\ t \neq t_1, \\ &u\big(t_1^+\big) \geqslant 0, \quad u'\big(t_1^+\big) > 0, \\ &u(0) \leqslant 0, \quad u(1) \leqslant 0. \end{aligned} \tag{13}$$

Proof. Let $u \in \Omega$ be a function satisfying the above inequalities. Suppose that $u > 0$ somewhere in J, so let $\sup_{t\in J} u(t) > 0$. Then by (13), we only have the following alternatives: $u(t_1) = \max_{t\in J} u(t)$ or $u(t_1^+) = \sup_{t\in J} u(t)$. We want to get contradictions for both cases. Consider the case $u(t_1) = \max_{t\in J} u(t)$ first. If $u(t_1^+) > 0$, then there is $0 < \delta_2 \leqslant 1 - t_1$ such that $u > 0$ on $(t_1, t_1 + \delta_2)$ and $u(t_1 + \delta_2) = 0$. By (13), we know that u is convex down on $(t_1, t_1 + \delta_2)$. Also we know $u \in C^1(t_1, 1)$ and $u(t_1 + \delta_2) = 0$. Thus $u'(t_1^+) \leqslant 0$ and this contradicts to one of the inequalities in assumptions. If $u(t_1^+) = 0$, then applying Lemma 3.1 on $(t_1, 1)$, we get $u \leqslant 0$ on $[t_1, 1]$. Thus $u'(t_1^+) \leqslant 0$ and this also contradicts to the same assumption. Obviously, it is impossible to be assumed $u'(t_1^+) > 0$ for the case $u(t_1^+) = \sup_{t\in J} u(t)$. Therefore we conclude $u(t) \leqslant 0$ on J and thus $u(t_1^+) = 0$. This implies $u'(t_1^+) \leqslant 0$ which is also a contradiction. □

To apply the monotone iterative method, we also need the modified problem (MP) given in Section 3. We notice that problem (MP) is the same as

$$\begin{aligned} &u''(t) - q(t)u(t) + h_\eta(t) = 0, \quad t \in (0,1),\ t \neq t_1, \\ &\Delta u|_{t=t_1} = I\big(u(t_1)\big), \\ &\Delta u'|_{t=t_1} = N\big(u(t_1), u'(t_1)\big), \\ &u(0) = a, \quad u(1) = b, \end{aligned}$$

where $h_\eta(t) = q(t)\eta(t) + f(t, \eta(t))$ and $\eta \in \Omega$ with $\alpha(t) \leqslant \eta(t) \leqslant \beta(t)$, for all $t \in [0,1]$.

Lemma 4.3. *Assume* (A_0), (A_1'), (A_2)–(A_4). *Then problem* (MP) *has a unique solution u such that $\alpha(t) \leqslant u(t) \leqslant \beta(t)$ on J.*

Proof. Since $h_\eta \in \bar{\mathcal{A}}$, Lemma 4.1 implies that (MP) has a unique solution and thus, it is enough to show that the solution u satisfies $\alpha(t) \leqslant u(t) \leqslant \beta(t)$ on J. We prove $\alpha \leqslant u$ on J. The proof for $u \leqslant \beta$ on J is similar. If it is not true, so let $\sup_{t\in J}[(\alpha - u)(t) \triangleq m(t)] > 0$. Then obviously $m \in \Omega$ and $m(0) \leqslant 0$, $m(1) \leqslant 0$. Furthermore

$$\begin{aligned} m''(t) = \alpha''(t) - u''(t) &\geqslant -f\big(t, \alpha(t)\big) - q(t)u(t) + q(t)\eta(t) + f\big(t, \eta(t)\big) \\ &\geqslant -q(t)\big(\eta(t) - \alpha(t)\big) - q(t)u(t) + q(t)\eta(t) = q(t)\big(\alpha(t) - u(t)\big). \end{aligned}$$

Thus $m''(t) - q(t)m(t) \geqslant 0$, for $t \in (0,1)$, $t \neq t_1$. By the above inequality, we can easily see that function m does not attain its positive maximum at $t \in J'$ and thus we have the following alternatives: $m(t_1) = \max_{t\in J} m(t)$ or $m(t_1^+) = \sup_{t\in J} m(t)$. Consider the case $m(t_1) = \max_{t\in J} m(t)$ first. We know $m(t_1) > 0$ and $m(t_1^+) = \alpha(t_1) + I(\alpha(t_1)) - [u(t_1) + I(u(t_1))] \geqslant 0$, by (A_0) and we assume that there exists $\delta_1 \geqslant 0$ such that $m(t) > 0$ on $(t_1 - \delta_1, t_1)$ and $m(t_1 - \delta_1) = 0$. Then $m'(t_1) > 0$. Indeed, it is obvious to see $m'(t_1) \geqslant 0$, since $m(t_1) > 0$, $m(t_1 - \delta_1) = 0$, $m \in C^1(0, t_1)$ and m is convex down on $(t_1 - \delta_1, t_1)$. If $m'(t_1) = 0$, then $m'(t) < 0$ on $(t_1 - \delta_1, t_1)$, since $m''(t) > 0$ on the interval. This contradicts to $m(t_1) = \max_{t\in J} m(t)$. Now by the facts $m(t_1) > 0$, $m'(t_1) > 0$ and (A_1'), we get

$$\begin{aligned} \Delta m'|_{t=t_1} &\geqslant N\big(\alpha(t_1), \alpha'(t_1)\big) - N\big(u(t_1), u'(t_1)\big) \\ &= N\big(\alpha(t_1), \alpha'(t_1)\big) - N\big(\alpha(t_1), u'(t_1)\big) + N\big(\alpha(t_1), u'(t_1)\big) - N\big(u(t_1), u'(t_1)\big) \\ &\geqslant -Lm'(t_1). \end{aligned}$$

Thus $m'(t_1^+) \geqslant (1 - L)m'(t_1) > 0$. Therefore by Lemma 4.2, we conclude that such a function m does not exist. Next, we consider $m(t_1^+) = \sup_{t\in J} m(t)$ case. Since $m(t_1^+) > 0$ and $m(t_1^+) = \alpha(t_1) + I(\alpha(t_1)) - [u(t_1) + I(u(t_1))]$, we get $m(t_1) > 0$ by (A_0). And then following the same argument as the above, we can show $m'(t_1) > 0$ and $m'(t_1^+) > 0$. Therefore by Lemma 4.2 again, we get the same conclusion and the proof is done. □

Now for given $\eta \in [\alpha, \beta]$, let u be a unique solution of (MP) and define an operator $A : PC[0,1] \to PC[0,1]$ by $A\eta = u$.

Lemma 4.4. *Assume* (A_0), (A_1'), (A_2)–(A_4). *Then the operator A satisfies*

(i) $A\alpha, A\beta \in [\alpha, \beta]$.
(ii) *A is a monotone operator on* $[\alpha, \beta]$.

Proof. Proof of (i) is clear by Lemma 4.3.

Proof of (ii). Let $\eta_1, \eta_2 \in [\alpha, \beta]$ with $\eta_1 \leqslant \eta_2$ and let $u_i = A\eta_i$, $i = 1, 2$. Then we show $u_1 \leqslant u_2$ on J if $\eta_1 \leqslant \eta_2$ on J. If it is not true, so let $\sup_{t\in J}[(u_1 - u_2)(t) \triangleq m(t)] > 0$. Then we get $m \in \Omega$, $m(0) = 0 = m(1)$ as checked in the proof of Lemma 3.4(ii), $m''(t) - q(t)m(t) \geqslant 0$ on J'. Again following the same lines on the proof of Lemma 4.3, we get $m(t_1^+) \geqslant 0$, $m'(t_1^+) > 0$ and the proof is completed by Lemma 4.2. □

Proof of Theorem 4.1. Let $\alpha_0 = \alpha$ and $\beta_0 = \beta$ and define $\alpha_n = A\alpha_{n-1}$ and $\beta_n = A\beta_{n-1}$ for $n \in \mathbb{N}$. Then following exactly the same lines as in the proof of Theorem 3.1 replacing Lemmas 3.3 and 3.4 by Lemmas 4.3 and 4.4, we can complete the proof. □

If N depends only on the derivative of solutions at t_1, then we have the following corollary.

Corollary 4.1. *Assume the same hypotheses in Corollary* 3.1. *Then the problem*

$$\begin{aligned}
&u''(t) + f\big(t, u(t)\big) = 0, \quad t \in (0,1),\ t \neq t_1,\\
&\Delta u|_{t=t_1} = I\big(u(t_1)\big),\\
&\Delta u'|_{t=t_1} = N\big(u'(t_1)\big),\\
&u(0) = a, \quad u(1) = b
\end{aligned}$$

admits the same conclusion as in Theorem 1.1.

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