Almost convergence and a core theorem for double sequences

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Abstract

The idea of almost convergence was introduced by Moricz and Rhoades [Math. Proc. Cambridge Philos. Soc. 104 (1988) 283–294] and they also characterized the four dimensional strong regular matrices. In this paper we define the M-core for double sequences and determine those four dimensional matrices which transform every bounded double sequence \( x = [x_{jk}] \) into one whose core is a subset of the M-core of \( x \).

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1. Introduction

A double sequence \( x = [x_{jk}]_{j,k=0}^{\infty} \) is said to be convergent in the Pringsheim sense or P-convergent if for every \( \varepsilon > 0 \) there exists \( N \in \mathbb{N} \) such that \( |x_{jk} - L| < \varepsilon \) whenever \( j, k > N \) and \( L \) is called the Pringsheim limit (denoted by P-lim \( x = L \)) (cf. [8]). We will denote the space of P-convergent sequences by \( c_2 \).

A double sequence \( x \) is bounded if there exists a positive number \( M \) such that \( |x_{jk}| < M \) for all \( j \) and \( k \), i.e., if

\[ \|x\| = \sup_{j,k} |x_{jk}| < \infty. \]
Note that in contrast to the case for single sequences, a convergent double sequence need not be bounded.

Let \( A = [a_{jk}^{mn}]_{j,k=0}^{\infty} \) be a doubly infinite matrix of real numbers for all \( m, n = 0, 1, \ldots \).

Forming the sums
\[
y_{mn} = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{jk}^{mn} x_{jk},
\]
called the \( A \)-means of the double sequence \( x \), yields a method of summability. We say that a sequence \( x \) is \( A \)-summable to the limit \( s \) if the \( A \)-means exist for all \( m, n = 0, 1, \ldots \) in the sense of Pringsheim’s convergence,
\[
\lim_{p,q \to \infty} \sum_{j=0}^{p} \sum_{k=0}^{q} a_{jk}^{mn} x_{jk} = y_{mn}
\]
and
\[
\lim_{m,n \to \infty} y_{mn} = s.
\]

A two dimensional matrix transformation is said to be regular if it maps every convergent sequence into a convergent sequence with the same limit. In 1926 Robinson [9] presented a four dimensional analogue of regularity for double sequences in which he added an additional assumption of boundedness: A four dimensional matrix \( A \) is said to be bounded-regular or RH-regular if it maps every bounded \( P \)-convergent sequence into a \( P \)-convergent sequence with the same \( P \)-limit.

The following is a four dimensional analogue of the well-known Silverman–Toeplitz theorem [1].

**Theorem 1.1.** The four dimensional matrix \( A \) is bounded-regular or RH-regular if and only if (see Hamilton [2], Robinson [9])

- (RH1) \( \operatorname{P-lim}_{m,n} a_{jk}^{mn} = 0 \) (\( j, k = 0, 1, \ldots \)),
- (RH2) \( \operatorname{P-lim}_{m,n} \sum_{j,k=0}^{\infty} a_{jk}^{mn} = 1 \),
- (RH3) \( \operatorname{P-lim}_{m,n} \sum_{j=0}^{\infty} |a_{jk}^{mn}| = 0 \) (\( k = 0, 1, \ldots \)),
- (RH4) \( \operatorname{P-lim}_{m,n} \sum_{k=0}^{\infty} |a_{jk}^{mn}| = 0 \) (\( j = 0, 1, \ldots \)),
- (RH5) \( \sum_{j,k=0}^{\infty} |a_{jk}^{mn}| \leq C < \infty \) (\( m, n = 0, 1, \ldots \)).

Note that (RH1) is a consequence of each of (RH3) and (RH4).

The core (or K-core) of a real number sequence is defined to be the closed interval \([\liminf x, \limsup x]\). The well-known Knopp core theorem states as follows (see Knopp [3], Maddox [5]).

**Theorem 1.2.** In order that \( L(Ax) \leq L(x) \) for every bounded sequence \( x = (x_k) \), it is necessary and sufficient that \( A = (a_{nk}) \) should be regular and \( \lim_{n} \sum_{k=0}^{\infty} |a_{nk}| = 1 \), where \( L(x) = \lim sup x \).
Recently in [7], Patterson extended this idea for double sequences by defining the Pringshein core as follows.

Let $P-C_n \{x\}$ be the least closed convex set that includes all points $x_{jk}$ for $j, k > n$; then the Pringshein core of the double sequence $x = \{x_{jk}\}$ is the set $P-C \{x\} = \bigcap_{n=1}^{\infty} [P-C_n \{x\}]$.

Note that the Pringshein core of a real-valued bounded double sequence is the closed interval $[\text{P-lim inf } x, \text{P-lim sup } x]$.

In this regard, Patterson proved the following

**Theorem 1.3.** If $A$ is a four dimensional matrix, then for all real-valued double sequences $x$,

$$P\text{-lim sup } Ax \leq P\text{-lim sup } x$$

if and only if

(1) $A$ is RH-regular and

(2) $P\lim_{mn} \sum_{j,k=0}^{\infty} |a_{mn}| = 1$.

In the present paper we define the M-core of a double sequence by using the idea of almost convergence introduced and studied by Moricz and Rhoades [6], and then proved an analogue of Theorem 1.3.

2. Almost convergence and M-core

The notion of almost convergence for single sequences was introduced by Lorentz [4]. Recently Moricz and Rhoades [6] extended this idea for double sequences.

A double sequence $x = \{x_{jk}\}_{j,k=0}^{\infty}$ of real numbers is said to be almost convergent to a limit $s$ if

$$\lim_{p,q,m,n \to \infty} \sup_{m,n \geq 0} \left| \frac{1}{pq} \sum_{j=m}^{m+p-1} \sum_{k=n}^{n+q-1} x_{jk} - s \right| = 0,$$

that is, the average value of $\{x_{jk}\}$ taken over any rectangle $\{(j, k): m \leq j \leq m + p - 1; \ n \leq k \leq n + q - 1\}$ tends to $s$ as both $p$ and $q$ tend to $\infty$, and this convergence is uniform in $m$ and $n$.

Note that a convergent single sequence is also almost convergent but for a double sequence this is not the case, that is, a convergent double sequence need not be almost convergent. However every bounded convergent double sequence is almost convergent.

Using the idea of almost convergence, Lorentz [4] introduced and characterized strongly regular matrices. We say that a matrix $A$ is strongly regular if every almost convergent sequence $x$ is $A$-summable to the same limit, and the $A$-means are also bounded.

If a double sequence $x$ is almost convergent to $s$, then we write $f_2$-lim $x = s$ and $f_2$ for the space of almost convergent double sequences.

In [6], Moricz and Rhoades gave four dimensional analogue of strongly regular matrices as follows.
Theorem 2.1. Necessary and sufficient conditions for a matrix \( A = [a_{jk}^{mn}] \) to be strongly regular are that \( A \) is bounded-regular and satisfies the following two conditions:

(MR1) \( \lim_{m,n \to \infty} \sum_{j,k=0}^{\infty} |\Delta_{10} a_{jk}^{mn}| = 0 \),

(MR2) \( \lim_{m,n \to \infty} \sum_{j,k=0}^{\infty} |\Delta_{01} a_{jk}^{mn}| = 0 \),

where \( \Delta_{10} a_{jk}^{mn} = a_{jk}^{mn} - a_{j+1,k}^{mn} \) and \( \Delta_{01} a_{jk}^{mn} = a_{jk}^{mn} - a_{j,k+1}^{mn} \) \((j, k = 0, 1, \ldots)\).

We define the following. Let us write

\[
L^*(x) = \limsup_{p,q \to \infty} \sup_{m,n \geq 0} \frac{1}{pq} \sum_{j=m}^{m+p-1} \sum_{k=n}^{n+q-1} x_{jk}.
\]

Then we define the M-core of a real-valued bounded double sequence \( x \) to be the closed interval \([-L^*(-x), L^*(x)]\).

Since every bounded convergent double sequence is almost convergent, we have

\[
L^*(x) \leq P-\limsup x,
\]

and hence it follows that M-core\(\{x\} \subseteq P\)-core\(\{x\}\) for a bounded double sequence \( x = [x_{jk}]_{j,k=0}^{\infty} \).

We quote here the following useful lemma.

Lemma 2.2 [7]. If \( A \) is a real or complex-valued four dimensional matrix such that (RH3), (RH4), and

\[
P-\limsup_{m,n \to \infty} \sum_{j,k=0}^{\infty} |a_{jk}^{mn}| = M
\]

hold, then for any bounded double sequence \( x \) we have

\[
P-\limsup |Ax| \leq M(\ P-\limsup |x|).
\]

3. Main result

Here we prove a core theorem for double sequences making use of four dimensional strongly regular matrices due to Moricz and Rhoades [6].

Theorem 3.1. For every bounded double sequence \( x \),

\[
L(Ax) \leq L^*(x)
\]

(or P-core\(\{Ax\} \subseteq M\)-core\(\{x\}\)) if and only if

(i) \( A = [a_{jk}^{mn}] \) is strongly regular and

(ii) \( P-\lim_{m,n \to \infty} \sum_{j,k=0}^{\infty} |a_{jk}^{mn}| = 1 \).
Proof. Necessity. Let us consider a bounded double sequence \( x \) to be almost convergent to \( s \). Then we have \( \mathbb{L}^* (x) = - \mathbb{L}^* (-x) \). By (3.1), we get

\[
s = - \mathbb{L}^* (x) \leq - \mathbb{L}^* (-Ax) \leq \mathbb{L} (Ax) \leq \mathbb{L}^* (x) = s.
\]

Hence \( Ax \) is P-convergent and \( \text{P-lim} Ax = f_2 \text{-lim} x = s \), and so \( A \) is strongly regular, i.e., condition (i) holds.

Since every strongly regular matrix is also bounded-regular, by Lemma 2.2 there exists a bounded double sequence \( x \) such that \( \lim \sup |x| = 1 \) and \( \text{P-lim} \sup Ax = C \), where \( C \) is defined by (RH5). Therefore we have

\[
1 \leq \text{P-lim inf}_{m,n} \sum_{j,k=0}^{\infty} \left| a^m_{jk} \right| \leq \text{P-lim sup}_{m,n} \sum_{j,k=0}^{\infty} \left| a^m_{jk} \right| \leq 1,
\]

i.e., condition (ii) holds.

Sufficiency. Given \( \varepsilon > 0 \), we can find fixed integers \( p, q \geq 2 \) such that

\[
\frac{1}{pq} \sum_{j,m} \sum_{k,n} x_{jk} < \mathbb{L}^* (x) + \varepsilon.
\]

(3.2)

Now as in [6], we can write

\[
y_{MN} = \sum_{j,k=0}^{\infty} a^M_{jk} x_{jk} = \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4 + \Sigma_5 + \Sigma_6 + \Sigma_7 + \Sigma_8,
\]

(3.3)

where

\[
\Sigma_1 = \frac{1}{pq} \sum_{m,n=0}^{\infty} \sum_{j,k=0}^{m+p-1 n+q-1} a^M_{mn} x_{jk},
\]

\[
\Sigma_2 = -\frac{1}{pq} \sum_{j=0}^{p-2} \sum_{k=0}^{q-2} x_{jk} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a^M_{mn},
\]

\[
\Sigma_3 = -\frac{1}{pq} \sum_{j=p-1}^{\infty} \sum_{k=0}^{q-2} x_{jk} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a^M_{mn},
\]

\[
\Sigma_4 = -\frac{1}{pq} \sum_{j=0}^{p-2} \sum_{k=q-1}^{q-1} x_{jk} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a^M_{mn},
\]

\[
\Sigma_5 = -\sum_{j=p-1}^{\infty} \sum_{k=q-1}^{\infty} x_{jk} \left\{ \frac{1}{pq} \sum_{m=j-p+1}^{\infty} \sum_{n=k-q+1}^{\infty} a^M_{mn} - a^M_{jk} \right\},
\]

\[
\Sigma_6 = \sum_{j=0}^{p-2} \sum_{k=0}^{q-2} a^M_{jk} x_{jk},
\]

\[
\Sigma_7 = \sum_{j=p-1}^{\infty} \sum_{k=0}^{q-2} a^M_{jk} x_{jk},
\]

\[
\Sigma_8 = \sum_{j=0}^{p-2} \sum_{k=0}^{q-2} a^M_{jk} x_{jk}.
\]
\[ \Sigma_8 = - \sum_{p=q-1}^{p-2} \sum_{k=0}^{\infty} a_{jk}^{MN} x_{jk}. \]

Using the conditions of strong regularity of \( A \), we observe that

\[ |\Sigma_2| \leq \|x\| \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} |a_{mn}| |a_{MN}| \to 0 \quad (M, N \to \infty), \]

and

\[ |\Sigma_6| \leq \|x\| \sum_{j=0}^{p-2} \sum_{k=0}^{q-2} |a_{jk}^{MN}| \to 0 \quad \text{by (RH}_1\text{)}, \]

\[ |\Sigma_3| \leq \|x\| \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} |a_{mn}| |a_{MN}| \to 0, \]

and

\[ |\Sigma_7| \leq \|x\| \sum_{j=p-1}^{\infty} \sum_{k=0}^{q-2} |a_{jk}^{MN}| \to 0 \quad \text{by (RH}_3\text{)}, \]

\[ |\Sigma_4| \to 0 \quad \text{and} \quad |\Sigma_8| \to 0 \quad \text{by (RH}_4\text{)}. \]

Now

\[ |\Sigma_5| \leq \|x\| \sum_{r=0}^{\infty} \sum_{s=0}^{q-1} \left\{ (p - r - 1) \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \Delta_{10} a_{MN}^{j,k} \right\} \to 0 \quad \text{by (MR}_1\text{) and (MR}_2\text{)}. \]

Therefore we have by (3.3),

\[ L(Ax) \leq \limsup_{M,N} \sum_{m,n=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \left| a_{MN}^{j,k} \right| \frac{m+p-1}{pq} \sum_{j=0}^{m} \sum_{k=0}^{n} x_{jk}. \]

Now conditions (RH}_1\text{), (RH}_3\text{) and (ii) yield

\[ L(Ax) \leq L^*(x) + \varepsilon. \]
Since \( \varepsilon \) is arbitrary we finally have
\[
L(Ax) \leq L^*(x).
\]
This completes the proof of the theorem. \( \square \)

4. Examples

4.1. Almost convergent sequences

(i) Define the double sequence \( x = [x_{jk}] \) by
\[
x_{jk} = \begin{cases} 
1 & \text{if } j \text{ is odd, for all } k, \\
0 & \text{otherwise.}
\end{cases}
\]
Then \( x \) is almost convergent to 1/2.

(ii) Define \( x = [x_{jk}] \) by
\[
x_{jk} = (-1)^j \quad \text{for all } k.
\]
Then \( x \) is almost convergent to 0.

4.2. Strongly regular matrix

Define \( A = [a_{jk}^{mn}] \) by
\[
a_{jk}^{mn} = \begin{cases} 
\frac{1}{m^2} & \text{if } m = n \text{ and } j, k \leq m \text{ (even)}, \\
\frac{1}{m^2 - m} & \text{if } m = n, j \neq k \text{ and } j, k \leq m \text{ (odd)}, \\
0 & \text{otherwise.}
\end{cases}
\]
We can easily verify that \( A \) is strongly regular, that is, conditions (RH1)–(RH3), (MR1) and (MR2) hold. Moreover, for the sequence (i), we have
\[
\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_{jk}^{mn} x_{jk} = a_{11}^{mm} x_{11} + a_{12}^{mm} x_{12} + \cdots + a_{1m}^{mm} x_{1m} \\
+ a_{21}^{mm} x_{21} + a_{22}^{mm} x_{22} + \cdots + a_{2m}^{mm} x_{2m} \\
+ a_{31}^{mm} x_{31} + a_{32}^{mm} x_{32} + a_{33}^{mm} x_{33} + \cdots + a_{3m}^{mm} x_{3m} \\
\vdots \\
+ a_{m-1}^{mm} x_{m-1,1} + \cdots + a_{m-1,m}^{mm} x_{m-1,m} \\
+ a_{m1}^{mm} + \cdots + a_{mm}^{mm}
\]
\[
= \frac{m}{m^2} \cdot \frac{m}{2} \quad \text{(if } m \text{ is even)}
\]
\[
\to \frac{1}{2} \quad \text{as } m, n \to \infty.
\]
Similarly
\[ \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_{jk}^{mn} x_{jk} = \frac{m - 1}{m^2 - m} \cdot \frac{m + 1}{2} \quad \text{(if } m \text{ is odd)} \]
\[ \rightarrow \frac{1}{2} \quad \text{as } m, n \rightarrow \infty. \]

That is
\[ \text{P-lim}_{m,n} Ax = \frac{1}{2} = f_2-\text{lim} x, \]
and so \( A \) transforms almost convergent sequence into convergent (P-convergent) to the same limit.

4.3. Bounded-regular matrix which is not strongly regular

In Section 4.2, \( A \) is strongly regular and so bounded regular. Let us define \( A = [a_{jk}^{mn}] \) as
\[ a_{jk}^{mn} = \begin{cases} \frac{2}{m^2} & \text{if } m = n, j + k = \text{even}, \text{and } j, k \leq m \text{ (even)}, \\ \frac{1}{m^2 - m} & \text{if } m = n, j \neq k \text{ and } j, k \leq m \text{ (odd)}, \\ 0 & \text{otherwise}. \end{cases} \]
Then \( A \) is bounded-regular but not strongly regular. Conditions (RH1)–(RH5) can easily be verified. But
\[ \lim_{m,n} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} |a_{jk}^{mn} - a_{j+1,k}^{mn}| = \begin{cases} 2 & \text{if } m \text{ is even}, \\ 0 & \text{if } m \text{ is odd}, \end{cases} \]
and also
\[ \lim_{m,n} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} |a_{jk}^{mn} - a_{j,k+1}^{mn}| = \begin{cases} 2 & \text{if } m \text{ is even}, \\ 0 & \text{if } m \text{ is odd}. \end{cases} \]
Therefore conditions (MR1) and (MR2) do not hold and so \( A \) is not strongly regular.

4.4. In Theorem 3.1, strong regularity of \( A \) cannot be replaced by bounded-regularity.

Consider the matrix \( A = [a_{jk}^{mn}] \) as defined in Section 4.3. This is bounded-regular but not strongly regular, and also
\[ \text{P-lim}_{m,n} \sum_{j,k=0,0}^{\infty,\infty} |a_{jk}^{mn}| = 1, \]
i.e., condition (ii) of Theorem 3.1 holds. Take the bounded double sequence \( x = [x_{jk}] \) defined by \( x_{jk} = (-1)^{j+k} \), which is almost convergent to zero, that is,
\[ L^*(x) = 0. \]
Now
\[ \sum_{j,k} a_{jk}^{mn} x_{jk} = \begin{cases} \frac{2}{m^2} \cdot \frac{m}{2} & \text{if } m \text{ is even}, \\ \frac{1}{m+1} \cdot m & \text{if } m \text{ is odd}. \end{cases} \]

Therefore
\[ \limsup_{m,n} \sum_{j,k} a_{jk}^{mn} x_{jk} = 1 \]

and
\[ \liminf_{m,n} \sum_{j,k} a_{jk}^{mn} x_{jk} = 0, \]

i.e., \( L(Ax) = 1 \). Hence \( L(Ax) > L^*(x) \), that is (3.1) does not hold.

References