An optimized Runge–Kutta method for the solution of orbital problems

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Abstract

We present a new explicit Runge–Kutta method with algebraic order four, minimum error of the fifth algebraic order (the limit of the error is zero, when the step-size tends to zero), infinite order of dispersion and eighth order of dissipation. The efficiency of the newly constructed method is shown through the numerical results of a wide range of methods when these are applied to well-known periodic orbital problems.

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1. Introduction

Many orbital problems in astronomy, astrophysics, celestial mechanics, etc. are expressed by the second-order differential equation of the form

\[ y''(x) = f(x, y(x)), \quad y(x_0) = y_0, \]
\[ y'(x_0) = y'_0. \]  

(1)
that is, differential equations where \( f \) is independent from the first derivative of \( y \). In order to use first-order numerical methods to solve these problems we set
\[
\begin{align*}
y_1(x) &= y(x), \\
y_2(x) &= y'(x).
\end{align*}
\]
In this way, (1) can be expressed by a system of two first-order ODEs
\[
\begin{align*}
y_1'(x) &= y_2(x), \\
y_2'(x) &= f(x, y_1(x)).
\end{align*}
\]

2. Basic theory

2.1. General form of explicit Runge–Kutta methods

An \( s \)-stage explicit Runge–Kutta method used for the computation of the approximation of \( y_{n+1}(x) \) in problem (2), when \( y_n(x) \) is known, can be expressed by the following relations:
\[
y_{n+1} = y_n + \sum_{i=1}^{s} b_i k_i,
\]
\[
k_i = hf \left( x_n + c_i h, y_n + h \sum_{j=1}^{i-1} a_{ij} k_j \right), \quad i = 1, \ldots, s.
\]

The method mentioned previously can also be presented using the Butcher table below:

\[
\begin{array}{c|ccc}
& c_2 & a_{21} \\
c_3 & a_{31} & a_{32} \\
& \vdots & \vdots & \vdots \\
c_s & a_{s1} & a_{s2} & \cdots & a_{s,s-1} \\
\hline b_1 & b_2 & \cdots & b_{s-1} & b_s
\end{array}
\]

The following equations must always hold:
\[
c_i = \sum_{j=1}^{s} a_{ij}, \quad i = 2, \ldots, s.
\]

**Definition 1** *(Hairer et al. [4])* A Runge–Kutta method has algebraic order \( p \) when the method’s series expansion agrees with the Taylor series expansion in the \( p \) first terms
\[
y^{(n)}(x) = y^{(n)}_n(x), \quad n = 1, 2, \ldots, p.
\]
Equivalently, a Runge–Kutta method must satisfy a number of equations, in order to have a certain algebraic order. These equations will be shown later in this paper.

2.2. Phase-lag analysis of Runge–Kutta methods

The phase-lag analysis of Runge–Kutta methods is based on the test equation

\[ y' = i \omega y, \quad \omega \text{ real}. \] (6)

Application of the Runge–Kutta method described in (3) to the scalar test equation (6) produces the numerical solution

\[ y_{n+1} = a^n_s y_n, \quad a_s = A_s(v^2) + ivB_s(v^2), \] (7)

where \( v = \omega h \) and \( A_s, B_s \) are polynomials in \( v^2 \) completely defined by Runge–Kutta parameters \( a_{i,j}, b_i \) and \( c_i \), as shown in (4).

**Definition 2 (Simos [5]).** In the explicit \( s \)-stage Runge–Kutta method, presented in (4), the quantities

\[ t(v) = v - \arg[a_s(v)], \quad a(v) = 1 - |a_s(v)| \]

are, respectively, called the phase-lag or dispersion error and the dissipative error. If \( t(v) = O(v^{q+1}) \) and \( a(v) = O(v^{r+1}) \), then the method is said to be of dispersive order \( q \) and dissipative order \( r \).

Although dispersion (or phase-lag) was introduced for cyclic orbit, Runge–Kutta methods with high phase-lag order are more efficient in many other problem types than methods with lower phase-lag order and higher algebraic order with the same number of stages. They are even more effective in problems with oscillating solutions.

3. Construction of the new method

We consider a 6-stage explicit Runge–Kutta method as shown in (8):

\[
\begin{array}{c|cccccc}
0 \\
1 & b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \\
\hline
0 \\
c_2 & a_{21} \\
c_3 & a_{31} & a_{32} \\
c_4 & a_{41} & a_{42} & a_{43} \\
c_5 & a_{51} & a_{52} & a_{53} & a_{54} \\
c_6 & a_{61} & a_{62} & a_{63} & a_{64} & a_{65} \\
\end{array}
\] (8)
There are 26 unknowns totally. The number of necessary equations that must hold so that the methods have 4th and 5th algebraic order are eight and seventeen, respectively.

First algebraic order (1 equation):

$$\sum_{i=1}^{6} b_i = 1.$$  

Second algebraic order (2 equations):

$$\sum_{i=1}^{6} b_i c_i = \frac{1}{2},$$

$$\sum_{i,j=1}^{6} b_i a_{ij} c_j = \frac{1}{6}.$$  

Third algebraic order (4 equations):

$$\sum_{i=1}^{6} b_i c_i^2 = \frac{1}{3},$$

$$\sum_{i,j=1}^{6} b_i a_{ij} c_j^2 = \frac{1}{12},$$

$$\sum_{i,j,k=1}^{6} b_i a_{ij} a_{jk} c_k = \frac{1}{24}. \quad (9)$$  

Fourth algebraic order (8 equations):

$$\sum_{i=1}^{6} b_i c_i^3 = \frac{1}{4},$$

$$\sum_{i,j=1}^{6} b_i c_i a_{ij} c_j = \frac{1}{8},$$

$$\sum_{i,j=1}^{6} b_i a_{ij} c_j^2 = \frac{1}{12},$$

$$\sum_{i,j,k=1}^{6} b_i a_{ij} a_{jk} c_k = \frac{1}{24}. \quad \text{(9)}$$

Fifth algebraic order (17):

$$\sum_{i=1}^{6} b_i c_i^4 = \frac{1}{5}, \quad \sum_{i,j=1}^{6} b_i c_i^2 a_{ij} c_j = \frac{1}{10},$$

$$\sum_{i,j=1}^{6} b_i c_i a_{ij} c_j^2 = \frac{1}{15}, \quad \sum_{i,j,k=1}^{6} b_i c_i a_{ij} a_{jk} c_k = \frac{1}{30}. \quad \text{(9)}$$
\[ \sum_{i,j=1}^{6} b_i a_{ij} c_j^3 = \frac{1}{20}, \quad \sum_{i,j,k=1}^{6} b_i a_{ij} c_j a_{jk} c_k = \frac{1}{40}, \]
\[ \sum_{i,j,k=1}^{6} b_i c_i a_{ij} a_{jk} c_k^2 = \frac{1}{60}, \quad \sum_{i,j,k,l=1}^{6} b_i a_{ij} a_{jk} a_{kl} c_l = \frac{1}{120}, \]
\[ \sum_{i,j,k=1}^{6} b_i a_{ij} c_j a_{ik} c_k = \frac{1}{20}. \]  

(10)

We chose \( b_2 = 0 \), \( c_2 = \frac{1}{10} \), \( c_3 = \frac{1}{7} \), \( c_4 = \frac{2}{3} \), \( c_5 = \frac{9}{10} \) and \( c_6 = 1 \) beforehand, in order to simplify the system of equations required to be solved. After satisfying (5) and 13 out of the 17 equations from (9) and (10) (all except (i–iv)), the coefficients now depend on \( a_{65} \) and \( a_{43} \). They are shown below:

\[
\begin{array}{cccccc}
0 & & & & & \\
\frac{1}{10} & & & & & \frac{1}{10} \\
\frac{1}{3} & & \frac{-2}{9} & & \frac{5}{9} & \\
\frac{2}{3} & & \frac{-14}{9} + \frac{7}{3} a_{43} & & \frac{20}{9} - \frac{10}{3} a_{43} & a_{43} \\
\frac{9}{10} & & \frac{52983}{800} - \frac{3213}{800} a_{43} & & \frac{-747}{80} + \frac{459}{80} a_{43} & \frac{3213}{1000} - \frac{1377}{800} a_{43} & \frac{3213}{800} \\
1 & & -\frac{119}{206} a_{65} + 38 - 21 a_{43} & & -55 + 30 a_{43} & 18 + \frac{189}{100} a_{65} - 9 a_{43} & -\frac{459}{200} a_{65} & a_{65} \\
1 & & & & & \frac{23}{216} & 0 & \frac{63}{136} & \frac{9}{56} & \frac{1000}{3213} & -\frac{1}{24} \\
\end{array}
\]

(11)

By nullifying phase-lag, we get

\[
a_{65} = -\frac{8000}{1071} - a_{43} v^5 + 6 \tan(v) v^4 + 24 v^3 - 72 \tan(v) v^2 - 144 v + 144 \tan(v) v^5 (a_{43} \tan(v) v + 12 - 10 a_{43}),
\]

(12)

where \( v \) was defined in (7). After expanding dissipation over the Taylor series towards \( v \) around zero it becomes

\[
\text{dissipation} = \left( \frac{1}{1440} a_{43} - \frac{1}{720} \right) v^6 + \left( \frac{1}{13440} \frac{28 a_{43}^2}{5 a_{43} - 6} - \frac{81 a_{43} + 54}{5 a_{43} - 6} \right) v^8 + \cdots.
\]

(13)

so by setting \( a_{43} = 2 \), we increase the order of dissipation to eight. This way we keep many coefficients constant, opposite to the case where we demand zero dissipation.

After satisfying these equations, we have managed to construct a method with fourth algebraic order, infinite order of dispersion and eighth order of dissipation and that has six stages. However, the actual
order of the method tends to five, when \( v = w \ast h \) tends to zero, where the frequency \( w \) can be found by the law that we will mention later. This is because the limit of the remaining nonsatisfied equations (i–iv) is zero when \( v \) tends to zero:

\[
(i) = -\frac{1}{720} A, \quad (ii) = -\frac{1}{540} A, \quad (iii) = \frac{1}{144} A, \quad (iv) = \frac{1}{180} A, \quad \text{where}
\]

\[
A = \frac{\tan(v)v^6 + 6v^5 - 30\tan(v)v^4 - 120v^3 + 360\tan(v)v^2 + 720v - 720\tan(v)}{v^5(\tan(v)v - 4)}.
\]

These conclusions are very important since by decreasing the step-size, \( v = wh \) also decreases when \( w \) remains constant; the absolute error of the fifth algebraic order decreases and tends to zero.

4. Numerical results

4.1. The methods

We will compare the newly constructed method to a wide range of already known methods, presenting the results of the best five. These methods are constant-step or variable-step Runge–Kutta methods with algebraic order up to six:

- **Variable-step formula Fehlberg 6(5)8** from [2], where a(b)c means that the two embedded methods have algebraic order a, b and c stages. Fehlberg 5(4)6 and England 5(4)6 have also been tested, but they are not presented, since they had much worse results.
- **Constant-step formula Butcher** from [2]. Fehlberg I, Fehlberg 5th, Fehlberg 4th, Kutta–Nyström, England II and England I have also been tested, but had slightly to moderately worse results. Fehlberg II had similar results with Fehlberg 6(5)8.
- **Variable-step formula Dormand–Prince 5(4)7** from [1].
- **Constant-step formula Houwen 3.7** from [7]. 3.3–3.5, 3.8–3.9, 3.13–3.15 and 3.17 behaved similarly to 3.7, but with slightly worse results.
- **Constant-step formula New 4-Inf-8-6** constructed in this paper and shown in (11), where A-B-C-D means that the method has algebraic order A, phase–lag order B, dissipation order C and D stages.

4.2. The problems

The methods were tested using three well-known periodic orbit problems: The two-body problem, the “almost” periodic orbit problem studied by Franco and Palacios [3] and the “almost” periodic orbit problem studied by Stiefel and Bettis [6].

Since the newly constructed method has variable coefficients that depend on the frequency \( w \), we need a way to determine it. *A law in order to have an estimation of the frequency* of each IVP expresses the problem in the form \( y' = A \cdot y + B \), where \( B \) is a matrix in which several forms of \( y^2 \), \( y^3 \), \( \cos(y) \), etc. and constant terms can be involved. The estimated frequency is the spectral radius matrix \( A \).
4.2.1. Two-body problem

The system of coupled differential equations that follows is known as the Two-body problem:

\[ y'' = -y/r^3, \quad y(0) = 1, \quad y'(0) = 0, \]
\[ z'' = -z/r^3, \quad z(0) = 0, \quad z'(0) = 1, \] (14)

where \( r = \sqrt{y^2 + z^2} \).

The theoretical solution of problem (14) is given below:

\[ y(x) = \cos(x), \]
\[ z(x) = \sin(x). \] (15)

The system of equations (14) has been solved for \( x \in [0, 1000 \pi] \). The estimated frequency is \( w = \sqrt{(1/r^3)} \). In Fig. 1 we present the accuracy of the tested methods expressed by the \(-\log_{10} (\text{maximum of the absolute errors})\) of the functions used \( (y(x), y'(x), z(x), z'(x)) \) for all steps performed until the end-point versus the \( \log_{10} (\text{function evaluations}) \).

4.2.2. Orbit problem by Franco and Palacios

The “almost” periodic orbit problem studied by Franco and Palacios [3] can be described by

\[ y'' + y = \varepsilon e^{i\psi x}, \quad y(0) = 1, \quad y'(0) = i, \quad y \in \mathbb{C}, \] (16)
or equivalently by

\[ u'' + u = \varepsilon \cos(\psi x), \quad u(0) = 1, \quad u'(0) = 0, \]
\[ v'' + v = \varepsilon \sin(\psi x), \quad v(0) = 0, \quad v'(0) = 1, \] (17)

where \( \varepsilon = 0.001 \) and \( \psi = 0.01 \).
The theoretical solution of problem (16) is given below:

\[ y(x) = u(x) + i v(x), \quad u, v \in \mathbb{R}, \]
\[ u(x) = \frac{1 - \psi^2}{1 - \psi^2} \cos(x) + \frac{\psi}{1 - \psi^2} \cos(\psi x), \]
\[ v(x) = \frac{1 - \psi^2}{1 - \psi^2} \sin(x) + \frac{\psi}{1 - \psi^2} \sin(\psi x). \] (18)

The solution given in (18) represents the motion of a perturbation of a circular orbit in the complex plane. The system of equations (17) has been solved for \( x \in [0, 1000 \pi] \). The estimated frequency according to the law mentioned in the previous subsection is \( w = 1 \). In Fig. 2 we present the maximum of the absolute errors.

4.2.3. Orbit problem by Stiefel and Bettis

The “almost” periodic orbit problem studied by [6] can be described by

\[ y'' + y = 0.001 e^{ix}, \quad y(0) = 1, \quad y'(0) = 0.9995 i, \quad y \in \mathbb{C}, \] (19)

or equivalently by

\[ u'' + u = 0.001 \cos(x), \quad u(0) = 1, \quad u'(0) = 0, \]
\[ v'' + v = 0.001 \sin(x), \quad v(0) = 0, \quad v'(0) = 0.9995. \] (20)

The theoretical solution of problem (19) is given below:

\[ y(x) = u(x) + i v(x), \quad u, v \in \mathbb{R}, \]
\[ u(x) = \cos(x) + 0.0005x \sin(x), \]
\[ v(x) = \sin(x) - 0.0005x \cos(x). \] (21)

The solution given in (21) represents the motion of a perturbation of a circular orbit in the complex plane. The system of equations (20) has been solved for \( x \in [0, 1000 \pi] \). The estimated frequency is also \( w = 1 \). In Fig. 3, we present the maximum of absolute errors.
5. Conclusions

A new explicit Runge–Kutta method of fourth algebraic order with minimized fifth-order error, infinite order of dispersion and eighth order of dissipation is produced in this paper. The results show the efficiency of the newly constructed method, while it is compared to many well-known methods from the literature. It is remarkable that the new method is compared to methods that have algebraic order up to six and step-size control and yet is more efficient in all problems tested. This also reveals the importance of dispersion and dissipation in orbital problems.

References