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Matrix coefficients of the middle discrete series of $SU(2,2)$ II, the explicit asymptotic expansion

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Abstract

We obtain the explicit asymptotic expansion of the matrix coefficients of the middle discrete series representation of $SU(2,2)$ by means of degenerate Gaussian hypergeometric series.

Keywords: Matrix coefficient; Spherical function; Discrete series representation; Hypergeometric series

1. Introduction

There is a series of important papers on the asymptotic expansion of matrix coefficients of the principal series representations of semisimple Lie groups, firstly initiated by Harish-Chandra and later, say, studied by Casselman and Miličić [1,6].

However for the matrix coefficients of the discrete series, little seems to be known. When G/K is hermitian and we are concerned with the matrix coefficients of the holomorphic discrete series, the asymptotic expansion has the single series and there is nothing to investigate.

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In an attempt to sharpen our previous results on the matrix coefficients of the non-holomorphic *middle* discrete series of $SU(2,2)$ [4], we found that the "asymptotic expansion" in this case is reduced to the recent result of Vidūnas on degenerate Gaussian hypergeometric series [5]. To describe this is the purpose of the present paper.

Let $[r,s;u] \in \mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0} \times \mathbb{Z}$ be the Blattner parameter of the middle discrete series representation π of $SU(2,2)$ with Harish-Chandra parameter belonging to Ξ_{III} in the previous paper [4, §1.4, p. 303]. The radial part of the matrix coefficients $\Phi_{\pi,\tau}(a)$ with the minimal K-type $\tau = \tau_{[r,s;u]}$ belonging to this discrete series is described essentially by hypergeometric series $\Phi(m_1, m_2, u; t)$ defined as follows (cf. [4, Theorem 8.1, p. 339]). Here

$$m_1 = k_1 + l_1 + b(M,i) + j, \qquad m_2 = k_2 + l_2 - b(M,i) - j \tag{1}$$

satisfy the relation

$$m_1 + m_2 = k_1 + k_2 + l_1 + l_2 = r + s, \quad m_1 \geqslant 0,\ m_2 \geqslant 0, \tag{2}$$

because $-l_1 \leqslant b(M,i) + j \leqslant l_2$ by definition [4, Notation 6.1, p. 322]. The function $\Phi(m_1, m_2; u; t)$ is the unique solution, non-vanishing and holomorphic at $t = 1$, with value $\Phi(m_1, m_2; u; 1) = \binom{r+s}{m_1}^{-1}$ whose Riemann's P-scheme is

$$P\begin{bmatrix} 0 & 1 & \infty \\ (2m_2+2-u)/4 & 0 & (2m_2+2+u)/4 \\ (2m_1+2+u)/4 & -r-s-1 & (2m_1+2-u)/4 \end{bmatrix}. \tag{3}$$

Our new claim in this paper is that this function is an elementary function written as a linear combination

$$R_1(t)\log(t) + R_2(t),$$

where $R_1(t)$, $R_2(t)$ are rational functions. In fact $R_1(t)$ is a non-holomorphic solution of the same differential equations, $R_2(t) \in \mathbb{C}[t, \frac{1}{1-t}]$. Thanks to Vidūnas [5], we can write $R_1(t)$ and $R_2(t)$ explicitly, to have the main formula of this paper (Theorem 7).

For the authors, the results of Gon [2, Theorem 14.14 in §14.4] on the explicit formula of generalized Whittaker functions belonging to the middle discrete series seemed to be a bit unexpected, because they are expressed in terms of only the most simple transcendental function; *i.e.*, the exponential function. However, in view of our results on the matrix coefficients of the middle discrete series, the result of Gon looks quite natural.

2. Matrix coefficients of the middle discrete series

2.1. A supplement and a correction to the notation of the previous paper

In [4, §1.1], we carelessly omitted the definition of the constant:

$$h := h(M,i) = \min\{i, l_1, l_2, l_1 + l_2 - i\}$$

(but given in the announcement [3, §5.2]).

In [4, Theorem 8.1], there is a misprint in the case $s \leqslant d(M) \leqslant 2s$: the symbol $\sum_{i=0}^{d(M)}$ should be read $\sum_{i=0}^{d(M^\wedge)}$. In the following we restate the theorem to clarify this.

2.2. *The main theorem of the previous paper*

We briefly review the formula of the radial part of the matrix coefficients of the middle discrete series of Lie group $G = SU(2,2)$ [4, §1.1]. Let $r > 0$, $s \geqslant 0$, u be integers under the condition $r + s + u$ is even and $r > s + 2 + |u|$. Then the symmetric tensor representation $\tau = \mathrm{sym}^r \otimes \mathrm{sym}^s \otimes e^{\sqrt{-1}u}$ becomes a minimal K-type of the middle discrete series representation π [4, §1.2]. Take the $\mathbb{R}$-split torus A of G of rank 2 and consider the specified coordinates $(a_1, a_2) \in A$ as in [4, §2.3]. Then we can define the radial part of matrix coefficient $\Phi_{\pi,\tau}$ by the set of coefficient functions $\{c_M(a_1, a_2)\}_{M \in \mathcal{M}(d)}$ along the Cartan decomposition $G = KAK$ [4, §3]. Here the subscripts $M = (k_1, l_1, k_2, l_2)$ run over $\mathcal{M}(d)$, the set of elements of $\mathbb{Z}^4$ satisfying $0 \leqslant k_1, k_2 \leqslant r$, $0 \leqslant l_1, l_2 \leqslant s$ with $k_1 + l_1 + k_2 + l_2 = r + s$ [4, Notation 3.1]. Recall here that (k_1, l_1) (resp. (k_2, l_2)) represents the pairs of weights in the standard basis of the left (resp. the right) K-type $\mathrm{sym}^r \otimes \mathrm{sym}^s \otimes e^{\sqrt{-1}u}$ of $\Phi_{\pi,\tau}$.

To write down $c_M(a_1, a_2)$ we introduce the change of coordinates (p, t) using hyperbolic functions $\mathrm{sh}(a) = (a - a^{-1})/2$, $\mathrm{ch}(a) = (a + a^{-1})/2$,

$$p = \mathrm{ch}(a_1)\,\mathrm{ch}(a_2), \qquad t = \left(\frac{\mathrm{ch}(a_1)}{\mathrm{ch}(a_2)}\right)^2.$$

Recall constants $l_{\min}(M) = \min\{l_1, l_2\}$, $l_{\max}(M) = \max\{l_1, l_2\}$, $h = h(M, i) = \min\{i, l_1, l_2, l_1 + l_2 - i\}$ and

$$b(M,i) = \begin{cases} 0, & i \leqslant l_{\min}(M), \\ (\mathrm{sgn}(l_2 - l_1)) \cdot (i - l_{\min}(M)), & l_{\min}(M) < i \leqslant l_{\max}(M), \\ l_2 - l_1, & i > l_{\max}(M) \end{cases}$$

[4, Notation 6.1]. For the binomial constants $q_m(M, i, j)$ we refer to [4, Proposition 7.1].

Theorem 1. *(See [4, Theorem 8.1].) Let $\Phi(m_1, m_2; u; t)$ be the hypergeometric function defined in Section 1. Then the matrix coefficients $\{c_M(a_1, a_2)\}_{M \in \mathcal{M}(d)}$ are given by the following.*

Suppose that $d(M) := l_1 + l_2 \leqslant s$. Then

$$\begin{aligned} c_M(a_1, a_2) = {} & c_0 (-1)^{r-k_1-l_1} \binom{r}{k_1}\binom{r}{k_2}\binom{s}{l_1}\binom{s}{l_2} \big(\mathrm{sh}(a_1)\,\mathrm{sh}(a_2)\big)^{s-d(M)} \\ & \cdot \sum_{i=0}^{d(M)} p^{-(r+s+2)/2+d(M)-i} \\ & \cdot \sum_{j=-h}^{h} (-1)^{h-|j|} \sum_{m=0}^{[(h-|j|)/2]} q_m(M, i, j) \big(t^{1/2} + t^{-1/2}\big)^{h-|j|-2m} \\ & \cdot \Phi\big(k_1 + l_1 + b(M,i) + j, k_2 + l_2 - b(M,i) - j; u; t\big) \end{aligned}$$

for some constant multiple c_0.

Suppose that $s \leqslant d(M) \leqslant 2s$. Put $M^\wedge = (r-k_1, s-l_1; r-k_2, s-l_2)$ and $h^\wedge = h(M^\wedge, i)$. Then

$$c_M(a_1,a_2) = c_0(-1)^{r-k_1-l_1}\binom{r}{k_1}\binom{r}{k_2}\binom{s}{l_1}\binom{s}{l_2}\bigl(\mathrm{sh}(a_1)\,\mathrm{sh}(a_2)\bigr)^{d(M)-s}$$
$$\cdot \sum_{i=0}^{d(M^\wedge)} p^{-(r+s+2)/2+2s-d(M)-i}$$
$$\cdot \sum_{j=-h^\wedge}^{h^\wedge} (-1)^{h^\wedge-|j|} \sum_{m=0}^{[(h^\wedge-|j|)/2]} q_m\bigl(M^\wedge, i, j\bigr)\bigl(t^{1/2}+t^{-1/2}\bigr)^{h^\wedge-|j|-2m}$$
$$\cdot\, \Phi\bigl(k_2+l_2+b\bigl(M^\wedge, i\bigr)+j, k_1+l_1-b\bigl(M^\wedge, i\bigr)-j; -u; t\bigr)$$

for the common constant multiple c_0.

3. Reduction to Vidūnas's formula

Let $0 \leqslant i \leqslant l_1+l_2$, $|j| \leqslant h(M,i)$ as in Theorem 1. Define m_1, m_2 by (1), (2). Then we have $k_1 \leqslant m_1 \leqslant r-k_2+s$, $k_2 \leqslant m_2 \leqslant r-k_1+s$.

We proceed, starting with the dichotomy:

$$(\mathrm{A}):\quad m_2-m_1 \geqslant u, \qquad (\mathrm{B}):\quad m_2-m_1 \leqslant u.$$

We introduce a new symbol $\Phi_0(t)$ by

$$\Phi(m_1,m_2;u;t) = \begin{cases} t^{(2m_2-u+2)/4}\Phi_0(t), & (\mathrm{A}),\\ t^{(2m_1+u+2)/4}\Phi_0(t), & (\mathrm{B}).\end{cases} \tag{4}$$

Then the P-scheme for $\Phi_0(t)$ is given by either

$$P\begin{bmatrix} 0 & 1 & \infty \\ 0 & 0 & m_2+1 \\ \frac{m_1-m_2+u}{2} & -r-s-1 & \frac{r+s-u+2}{2}\end{bmatrix} \quad \text{or} \quad P\begin{bmatrix} 0 & 1 & \infty \\ 0 & 0 & m_1+1 \\ \frac{m_2-m_1-u}{2} & -r-s-1 & \frac{r+s+u+2}{2}\end{bmatrix}$$

according to the cases (A) or (B). Now introduce the parameters corresponding to the formula of Vidūnas [5, §9].

Definition 2. Set $n = |\frac{m_2-m_1+u}{2}|$, $m = |\frac{m_1-m_2+u}{2}|$ and set

$$l = \begin{cases} \inf\{\frac{r+s+u}{2}, m_1\}, & (\mathrm{A}),\\ \inf\{\frac{r+s-u}{2}, m_2\}, & (\mathrm{B}).\end{cases} \tag{5}$$

Remark 3. If we replace (m_2, m_1, u) by $(m_1, m_2, -u)$ in (A) then we have the corresponding formula in (B) and vice versa.

Then we set $a = m + l + 1$, $b = m + l + n + 1$ and $c = m + n + 2l + 2$.

Lemma 4. *We have $l, m, n \in \mathbb{Z}_{\geqslant 0}$. We also have $(l, m + l, n + l, m + n + l) = (m_1, \frac{r+s-u}{2}, \frac{r+s+u}{2}, m_2)^\sigma$ by a permutation σ of degree 4. In particular $a + b \geqslant c$ and $c - a - b = -m$. Moreover we have $c = r + s + 2$.*

Proof. The non-negativity holds by definition, one may check other statements from definitions. $\square$

If we put $\mathcal{F}_0(z) = \Phi_0(1 - z)$, then $\mathcal{F}_0(z)$ belongs to the P-scheme

$$P\begin{bmatrix} 0 & 1 & \infty \\ 0 & 0 & a \\ 1-c & c-a-b & b \end{bmatrix} \quad \text{(Gaussian standard form).} \tag{6}$$

By our normalization, $\mathcal{F}_0(0) = \binom{r+s}{m_1}^{-1} = \binom{r+s}{m_2}^{-1}$.

Let us recall the formula of Vidūnas in our context.

Lemma 5. *$\mathcal{F}_0(z)$ always has logarithmic singularities at $z = 1, \infty$.*

Proof. We apply [5, Theorem 2.2(4)]. $\square$

Proposition 6. *The function $\Phi_0(t)$ introduced in (4) is of the form*

$$\Phi_0(t) = (-1)^{m+1}(1-t)^{-(r+s+1)}\big(R_1(t)\log t + C \cdot R_2(t)\big).$$

Here

$$R_1(t) = (r + s + 1)\binom{r+s}{\frac{r+s+u}{2}} \cdot {}_2F_1\left(\begin{matrix} -l, & -n-l \\ -m-n-2l \end{matrix} \,\middle|\, 1-t\right)$$

is the second solution of (6) at $z = 0$, $C = (r + s + 1)!/\binom{r+s}{m_1}$ and $R_2(t)$ is a Laurent polynomial in $\mathbb{C}[t, t^{-1}]$ given by a finite sum

$$\sum_{k=0}^{l} \frac{\psi(n+l-k+1) + \psi(l-k+1) - \psi(m+k+1) - \psi(k+1)}{(m+k)!(n+l-k)!(l-k)!k!} t^k$$
$$+ \sum_{k=0}^{m-1} \frac{(-1)^{k+m+1}(m-1-k)!}{(m+n+l-k)!(m+l-k)!k!} t^{k-m}$$
$$+ \sum_{k=0}^{n-1} \frac{(-1)^{n-k}(n-1-k)!}{(m+n+l-k)!(n+l-k)!k!} t^{n+l-k}$$

where $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$ is the digamma function.

Proof. In order to match with the results of Vidūnas, we have to change the function $\Phi_0(t)$ by $U(t) = (1-t)^{r+s+1}\Phi_0(t)$. Then we apply [5, Theorem 9.1] with help of Lemma 4. □

Theorem 7. *Replace $\Phi(m_1, m_2; u; t)$ in Theorem 1 by (4) and Proposition 6. Then $c_M(a_1, a_2)$ is of the form $c_{M,1}(p,t)\log t + c_{M,2}(p,t)$. Here $c_{M,i}(p,t)$ are rational functions in $(p, t^{1/2})$. This expansion corresponds to the asymptotic expansion of $\Phi_{\pi,\tau}$ at the regular singularity at infinity.*

4. An example of the spherical trace function of middle discrete series

Consider the middle discrete series π with Blattner parameter $[r, s; u] = [4, 0; 0]$. Then it has the trivial infinitesimal character and the dimension of its minimal K-type $\tau = [4, 0; 0]$ is $(4+1)(0+1) = 5$. The condition $\Phi_{\pi,\tau}(mam^{-1}) = \Phi_{\pi,\tau}(a)$ $(a \in A,\ m \in Z_K(A))$ implies that the radial parts of non-diagonal entries of $\Phi_{\pi,\tau}$ vanish [4, Lemma 3.1].

The trace of $\Phi_{\pi,\tau}(a)$ has the expression

$$\operatorname{trace}\bigl(\Phi_{\pi,\tau}(a)\bigr) = \sum_{k=0}^{r}\sum_{l=0}^{s}(-1)^{k-l}\binom{r}{k}^{-1}\binom{s}{l}^{-1} c_{k,l,r-k,s-l}(a)$$

under the normalization of [4, Remark 8.1] (recall here we also employed the dual basis). We want to have an explicit expression of $\operatorname{trace}(\Phi_{\pi,\tau}(a))$.

We can readily see from Theorem 1 that these diagonal entries $c_{k,l,r-k,s-l}(a)$ are simple linear combination of

$$\Phi(4,0) = \Phi(0,4), \qquad \Phi(3,1) = \Phi(1,3) \text{ and } \Phi(2,2)$$

with coefficients in $\mathbb{Q}[p, t^{1/2} + t^{-1/2}]$. Here we can apply

$$\Phi(k, r-k; 0; t) = \binom{r}{k}^{-1} {}_2F_1\left(\begin{matrix} r/2+1,\ k+1 \\ r+2 \end{matrix}\ \middle|\ 1-t\right)$$

for $r/2 \leqslant k \leqslant r$.

Now Proposition 6 says

$$\begin{aligned}
\Phi(4,0) &= \frac{t^{5/2}}{(1-t)^5}\left(-30\log t - \frac{5}{2}\bigl(t^2 - t^{-2}\bigr) + 20\bigl(t - t^{-1}\bigr)\right),\\
\Phi(3,1) &= \frac{t^2}{(1-t)^5}\left(15(1+t)\log t + \frac{5}{2}\left(\frac{1}{t} + 9 - 9t - t^2\right)\right),\\
\Phi(2,2) &= \frac{t^{3/2}}{(1-t)^5}\bigl(-5\bigl(t^2+4t+1\bigr)\log t - 15\bigl(1-t^2\bigr)\bigr).
\end{aligned}$$

Therefore

Example 8.

$$\begin{aligned}\operatorname{trace}\bigl(\Phi_{\pi,\tau}(a)\bigr) &= c_0(-1)^r p^{-\frac{r+2}{2}} \sum_{k=0}^{r} \binom{r}{k} \Phi(k, r-k) \\ &= c_0 p^{-3}(1+\sqrt{t})^{-4}\sqrt{t}\left(5t + 8\sqrt{t} + 5 - 30t\frac{\log t}{1-t}\right).\end{aligned}$$

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