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Contributions to Digit Expansions with Respect to Linear Recurrences

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Extensions and improvements of a recent paper, "On Digit Expansions with Respect to Linear Recurrences" by A. Pethö and R. F. Tichy (*J. Number Theory* 33 (1989), 243–256) are established. Furthermore distribution properties mod 1 of the sequence $(xs_G(n))$ are investigated, where $s_G(n)$ denotes the sum-of-digits function with respect to the linear recurrence G. © 1990 Academic Press, Inc.

1. INTRODUCTION

For every non-negative integer n, $s_G(n)$ denotes the sum of the G-ary digits of n, where G is a suitable linear recurring sequence (for short l.r.s.). In [8] it is proved that

$$\frac{1}{N}\sum_{0\leqslant n<N} s_G(n)=c_G\frac{\log N}{\log\alpha}+F\left(\frac{\log N}{\log\alpha}\right)+O\left(\frac{\log N}{N}\right), \tag{1.1}$$

where c_G is a constant depending on the l.r.s., α is the dominating characteristic root of G, and F is a bounded, Riemann integrable, nowhere differentiable function of period 1. In the following we consider l.r.s. of the type

$$G_{k+d}=a_1G_{k+d-1}+a_2G_{k+d-2}+\cdots+a_dG_k \tag{1.2}$$

with integral coefficients and integral initial values. For $d=1$ we assume $G_0=1$ and $a_1>1$.

For $d\geqslant 2$ we assume that the coefficients $a_1\geqslant a_2\geqslant\cdots\geqslant a_d>0$ are non-increasing and

$$1=G_0,\qquad G_k\geqslant a_1G_{k-1}+\cdots+a_kG_0+1\qquad\text{for}\quad k=1,\ldots,d-1. \tag{1.3}$$

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If equality holds in (1.3) for all $k=1, ..., d-1$, we call the initial values $G_0, ..., G_{d-1}$ canonical. For an arbitrary positive integer n define $L=L(n)$ by

$$G_L \leqslant n < G_{L+1} \tag{1.4}$$

and put $L(0)=0$. Set ($[t]$ denoting the integral part of t) $n_L = n$,

$$\begin{aligned} \varepsilon_j &= \left[\frac{n_j}{G_j}\right] \quad \text{and} \quad n_{j-1} = n_j - G_j \cdot \varepsilon_j \quad (1 \leqslant j \leqslant L) \\ \varepsilon_0 &= n_0. \end{aligned} \tag{1.5}$$

Hence we obtain a well-defined representation of any positive integer n in the form

$$n = \sum_{j=0}^{L(n)} \varepsilon_j \cdot G_j, \tag{1.6}$$

the so-called G-ary representation of n with digits $\varepsilon_j = \varepsilon_j(n)$. Furthermore we define

$$s(n) = s_G(n) = \sum_{j=0}^{L(n)} \varepsilon_j, \tag{1.7}$$

the sum of G-ary digits of n.

In Theorem 1 of [8] (see also Fraenkel [3]) it is proved that a $(t+1)$-tuple $(\varepsilon_0, ..., \varepsilon_t) \in \mathbf{N}_0^{t+1}$ is the sequence of G-ary digits of an integer if and only if

$$\sum_{i=0}^{k} \varepsilon_i G_i < G_{k+1} \quad \text{for all} \quad 0 \leqslant k < d-1 \tag{1.8}$$

and

$$(\varepsilon_k, ..., \varepsilon_{k-l+1}) < (a_1, ..., a_l) \quad \text{for all} \quad d+l-1 \leqslant k \leqslant t, \quad 1 \leqslant l \leqslant d, \tag{1.9}$$

where $<$ in (1.9) denotes the lexicographic order. In the canonical case the two conditions (1.8) and (1.9) can be replaced by

$$(\varepsilon_k, ..., \varepsilon_{k-l+1}) < (a_1, ..., a_l) \quad \text{for all} \quad l-1 \leqslant k \leqslant t, \quad 1 \leqslant l \leqslant d. \tag{1.10}$$

In Section 2 of this article we show that in the case of canonical initial values the remainder function F of (1.1) is periodic with period 1 and continuous. If the initial values are noncanonical, then there is no periodic remainder function of this type. The different results in [8] are due to some computational errors. In Section 3 we prove that for irrational x, the sequence $(xs_G(n))$ is uniformly distributed modulo 1. This result remains true if $x=(x_1, ..., x_m)$ is a vector, where $1, x_1, ..., x_m$ are linearly independent over the rationals.

2. Analytical Properties of the Remainder Function

We set $N=\sum_{j=0}^{L} \varepsilon_j G_j$ in G-ary representation. Then we have

$$S(N)=\sum_{j=0}^{L} \varepsilon_j \left(S(G_j)+G_j \sum_{j<k\leqslant L} \varepsilon_k + \frac{\varepsilon_j - 1}{2} G_j \right); \tag{2.1}$$

this formula is a slightly changed version of (3.5) in [8]. Now by Lemma 4 of [8] we obtain the explicit formulas

$$\begin{aligned} G_j &= A'_1 \alpha_1^j + \cdots + A'_d \alpha_d^j \\ S(G_j) &= (A_1 j + B_1)\, \alpha_1^j + \cdots + (A_d j + B_d)\, \alpha_d^j, \end{aligned} \tag{2.2}$$

where $\alpha=\alpha_1 > a_1 \geqslant 1 > |\alpha_2| \geqslant \cdots \geqslant |\alpha_d|$ are the characteristic roots of the l.r.s. G. Setting $X=\sum_{j=0}^{L} \varepsilon_j \alpha^j$, $A'=A'_1$, $A=A_1$, and $B=B_1$, we define

$$\begin{aligned} S^*(X) &= \sum_{j=0}^{L} \varepsilon_j \alpha^j \left(Aj + B + \sum_{j<k\leqslant L} \varepsilon_k + \frac{\varepsilon_j - 1}{2} \right), \\ \Phi(X) &= S^*(X) - A\,\frac{X \log X}{\log \alpha}. \end{aligned} \tag{2.3}$$

As in [8] we derive

$$S(N)-S^*(X)=O(\log N)$$

and

$$\Phi(\alpha X)-\alpha\Phi(X)=S^*(\alpha X)-\alpha S^*(X)-\alpha A X=0 \tag{2.4}$$

if and only if the initial values are canonical. Hence in the canonical case $X^{-1}\Phi(X)$ is a periodic function of $\log X/\log\alpha$ with period 1.

If we put $y=\alpha^{-L}X$ with $L=L(N)$ it is clear that $1 \leqslant y < \alpha$. Then we have

$$\frac{S^*(X)}{X} - A\,\frac{\log X}{\log \alpha} = \frac{1}{y}\,\psi(y) - A\,\frac{\log y}{\log \alpha} \tag{2.5}$$

with

$$\psi(y)=\sum_{j=0}^{\infty} b_j\alpha^{-j}\left(A'\sum_{k<j} b_k+A'\frac{b_j-1}{2}+B-A_j\right), \tag{2.6}$$

where

$$y=b_0+\frac{b_1}{\alpha}+\frac{b_2}{\alpha^2}+\cdots \tag{2.7}$$

in α-adic representation (cf. Lemma 5 in [8], Theorem 3 in [7]).

In the following we prove the continuity of the remainder function F in the canonical case. It is sufficient to prove the continuity of ψ in the interior of $[1, \alpha)$ and to show that ψ approaches the correct values at the boundary. For simplicity we consider

$$\tilde{\psi}(y)=\sum_{j=0}^{\infty} b_j\alpha^{-j}\left(A'\sum_{k<j} b_k+A'\frac{b_j}{2}+Aj\right)=\psi(y)-\left(B-\frac{A'}{2}\right)y$$

and define the infinite sequence $(c_0, c_1, ...)$,

$$c_{i-1}=\begin{cases} a_{i(\operatorname{mod} d)} & \text{if } i\not\equiv 0 \pmod d \\ a_d-1 & \text{if } i\equiv 0 \pmod d. \end{cases}$$

For the proof of continuity it is enough to consider $\tilde{\psi}$ at the points

$$y=\sum_{j=0}^{L} b_j\alpha^{-j}=\sum_{j=0}^{L-1} b_j\alpha^{-j}+(b_L-1)\,\alpha^{-L}+\alpha^{-L-1}\sum_{k=0}^{\infty} c_k\alpha^{-k} \tag{2.8}$$

(cf. [8, Sect. 4] in the second order case). Thus we have to show

$$\begin{aligned}
&\sum_{j=0}^{L} b_j\alpha^{-j}\left(A'\sum_{k<j} b_k+\frac{A'}{2}b_j+Aj\right)\\
&\quad=\sum_{j=0}^{L-1} b_j\alpha^{-j}\left(A'\sum_{k<j} b_k+\frac{A'}{2}b_j-Aj\right)\\
&\qquad+(b_L-1)\,\alpha^{-L}\left(A'\sum_{k<L} b_k+\frac{A'}{2}(b_L-1)-AL\right)\\
&\qquad+\alpha^{-L-1}\sum_{k=0}^{\infty} c_k\alpha^{-k}\left(A'B_L-A'+A'\sum_{l<k} c_l+\frac{A'}{2}c_k-A(L+k+1)\right),
\end{aligned}$$

where $B_L=b_0+\cdots+b_L$. By elementary but lengthy calculations this is equivalent to

$$0=A'\left(\sum_{l=2}^{d} a_l\alpha^{d-1}(a_1+\cdots+a_{l-1})+\frac{1}{2}\sum_{l=1}^{d} a_l^2\,\alpha^{d-l}-\frac{\alpha^d}{2}\right)-A\alpha p'(\alpha),$$

where p is the characteristic polynomial of G.

Obviously $\tilde{\psi}(1) = A'/2$ and $(1/\alpha)\,\tilde{\psi}(\alpha) = A + A'/2$ by the same computations as above. Hence the function on the right hand side of (2.5) attains the same values for $y = 1$ and $y = \alpha$.

Thus the continuity of ψ is equivalent to

$$A = c_G A'$$

with

$$c_G = \frac{1}{\alpha p'(\alpha)} \left(\sum_{k=2}^{d} a_k \alpha^{d-k} (a_1 + \cdots + a_{k-1}) + \frac{1}{2} \sum_{k=1}^{d} a_k^2 \alpha^{d-k} - \frac{1}{2} \alpha^d \right). \tag{2.9}$$

In order to prove this equation we consider the frequency of an α-adic digit z given by

$$\begin{aligned} Q(z) &= \frac{\alpha^{d-1}}{p'(\alpha)} \int_{z/\alpha}^{(z+1)/\alpha} \left(1 + \frac{1}{\alpha} \chi_{[0, \alpha - a_1]}(x) + \cdots \right. \\ &\quad \left. + \frac{1}{\alpha^{d-1}} \chi_{[0, \alpha^{d-1} - a_1 \alpha^{d-2} - \cdots - a_{d-1}]}(x) \right) dx \\ &= \frac{\alpha^{d-1}}{p'(\alpha)} \left(\sum_{\substack{k=1 \\ a_k > z}}^{d} \frac{1}{\alpha^k} + \sum_{k=1}^{d} \delta_{a_k, z} \left(\frac{a_{k+1}}{\alpha^{k+1}} + \cdots + \frac{a_d}{\alpha^d} \right) \right), \end{aligned}$$

where χ_I denotes the characteristic function of an interval and $\delta_{i,j}$ is the usual Kronecker symbol (cf. Parry [7]).

From this we obtain for the expected value

$$\sum_{z=1}^{a_1} z Q(z) = \frac{\alpha^{d-1}}{p'(\alpha)} \left(\sum_{k=1}^{d} \frac{1}{\alpha^k} \sum_{z < a_k} z + \sum_{k=1}^{d} a_k \left(\frac{a_{k+1}}{\alpha^{k+1}} + \cdots + \frac{a_d}{\alpha^d} \right) \right) = c_G.$$

Since

$$\frac{1}{N} S(N) = \frac{A}{A'} \frac{\log N}{\log \alpha} + O(1), \tag{2.10}$$

this expected value of the digits is also equal to A/A', hence Eq. (2.9) is proved. Thus we have shown

THEOREM 1. *Let G be a l.r.s. with canonical values; then*

$$\frac{1}{N} \sum_{n<N} s_G(n) = c_G \frac{\log N}{\log \alpha} + F\left(\frac{\log N}{\log \alpha} \right) + O\left(\frac{\log N}{N} \right),$$

where c_G is a constant depending on the l.r.s. given by (2.9) and α is the dominating characteristic root of G. The remainder function F is periodic with period 1, continuous, and nowhere differentiable.

Remark 1. If the initial values are non-canonical, then we have seen above that the function F is not periodic of period 1 in $\log N/\log \alpha$. It can easily be seen that it is not periodic in any period. Nevertheless in this non-canonical case we have proved the asymptotic relation (2.10) with the same constant c_G as in the canonical case.

Remark 2. The coefficient A' of the dominating characteristic root can be computed explicitly in any case. For instance in the canonical case we obtain

$$A' = \frac{\alpha^d - 1}{(\alpha - 1)\, p'(\alpha)}.$$

COROLLARY 1. *Let G be a second order l.r.s. $G_{n+2} = a_1 G_{n+1} + a_2 G_n$ with $G_0 = 1$ and $G_1 > a_1$. Then the remainder function F in Theorem 1 is periodic with period 1 and continuous, if and only if $G_1 = a_1 + 1$. In the case $a_1 = a_2 = 1$ this characterizes the sequence of Fibonacci numbers.*

Remark 3. A different proof of Theorem 1 will be given in a subsequent paper by J.-M. Dumont. The special case of Fibonacci numbers was considered by J. Coquet and P. Van Den Bosch [1], see also Dumont [2].

3. DISTRIBUTION PROPERTIES OF $(xs_G(n))$

Let us recall the definition of the discrepancy $D_N(x_n)$ of a sequence of real numbers:

$$D_N(x_n) = \sup_{0 \leqslant a < b < 1} \left| \frac{1}{N} \sum_{n < N} \chi_{[a,b)}(x_n - [x_n]) - (b - a) \right|. \tag{3.1}$$

A sequence is called uniformly distributed mod 1 if

$$\lim_{N \to \infty} D_N(x_n) = 0. \tag{3.2}$$

By Weyl's criterion (cf. [5, 6]) this is equivalent to

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n < N} e^{2\pi i h x_n} = 0 \tag{3.3}$$

for all integers $h \neq 0$. We will prove the following theorem.

THEOREM 2. *Let G be a l.r.s. as in (1.2) satisfying (1.3) and let x be an irrational number. Then the sequence $(xs_G(n))$ is uniformly distributed mod 1.*

For the proof we need three auxiliary results. The first one is a special case of a general method due to Coquet–Rhin–Toffin.

LEMMA 1. *Let $g\colon \mathbf{N}_0 \mapsto \mathbf{C}$ be a function with $g(0)=1$, $|g(n)| \leqslant 1$, and*

$$g(n) = \prod_{k=0}^{L(n)} g(\varepsilon_k(n)\, G_k)$$

for $n = \sum_{k=0}^{L(n)} \varepsilon_k(n)\, G_k$. Assume further that

$$\left| \frac{1}{G_k} \sum_{n=0}^{G_k - 1} g(n) \right| \leqslant \frac{1}{f(G_k)} \qquad \text{for} \quad k = 1, 2, \ldots,$$

where $f\colon [1, \infty) \mapsto (0, \infty)$ is a continuous nondecreasing function with $f(u) \leqslant u$. Then

$$\left| \frac{1}{N} \sum_{n=0}^{N-1} g(n) \right| \leqslant \frac{(C\alpha + 1)}{f(\sqrt{N})},$$

with a suitable constant C satisfying

$$\alpha^n \leqslant G_n \leqslant C\alpha^n.$$

Proof. Let $N = \sum_{k=0}^{L} \varepsilon_k G_k$ be the G-ary representation N ($\varepsilon_L \neq 0$), and set

$$N(j) = \sum_{k=j}^{L} \varepsilon_k G_k \qquad \text{for} \quad j = 0, \ldots, L.$$

We split the sum into two parts:

$$\sum_{n=0}^{N-1} g(n) = \sum_{n=0}^{N(L)-1} g(n) + \sum_{j=0}^{L-1} \sum_{n=N(j+1)}^{N(j)-1} g(n). \tag{3.4}$$

Then we have

$$\begin{aligned} \sum_{n=0}^{N(L)-1} g(n) &= \sum_{l=0}^{\varepsilon_L - 1} \sum_{n=lG_L}^{(l+1)G_L - 1} g(n) \\ &= \sum_{l=0}^{\varepsilon_L - 1} g(lG_L) \sum_{n=0}^{G_L - 1} g(n) \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} \sum_{n=N(j+1)}^{N(j)-1} g(n) &= g(N(j+1)) \sum_{n=0}^{\varepsilon_j G_j - 1} g(n) \\ &= g(N(j+1)) \sum_{l=0}^{\varepsilon_j - 1} g(lG_j) \sum_{n=0}^{G_j - 1} g(n). \end{aligned} \tag{3.6}$$

By (3.4), (3.5), and (3.6) we obtain

$$\left|\sum_{n=0}^{N-1} g(n)\right| \leqslant \sum_{j=0}^{L} \left|\sum_{l=0}^{\varepsilon_j-1} g(lG_j)\right| \left|\sum_{n=0}^{G_j-1} g(n)\right|$$
$$\leqslant \sum_{j=0}^{L} \varepsilon_j G_j \frac{1}{G_j} \left|\sum_{n=0}^{G_j-1} g(n)\right|$$
$$\leqslant \sum_{j=0}^{r-1} \varepsilon_j G_j + \sum_{j=r}^{L} \varepsilon_j G_j \frac{1}{f(G_j)} \leqslant G_r + \frac{N}{f(G_r)}$$

for all $1 \leqslant r \leqslant L$.

Let t be the uniquely determined number satisfying $(t/\alpha) f(t/\alpha) = N$. We choose r such that $\alpha^{r+1} > t \geqslant \alpha^r$ and derive

$$G_r + \frac{N}{f(G_r)} \leqslant C\alpha^r + \frac{N}{f(\alpha^r)} \leqslant Ct + \frac{N}{f(t/\alpha)}$$
$$= \frac{C\alpha N}{f(t/\alpha)} + \frac{N}{f(t/\alpha)} \leqslant \frac{(C\alpha + 1)N}{f(\sqrt{N})}.$$

Thus the proof of Lemma 1 is complete.

Because of this lemma it is sufficient to consider sums of the type

$$S_k = \sum_{n=0}^{G_k-1} e^{2\pi i h s_G(n) x},$$

where we use the lemma with $g(\varepsilon_k G_k) = e^{2\pi i h \varepsilon_k x}$.

Furthermore we need the following elementary results.

LEMMA 2. *For real α and integral $q \geqslant 2$ we have*

$$\left|\sum_{j=0}^{q-1} e^{2\pi i \alpha j}\right| \leqslant q - 2\pi \|\alpha\|^2,$$

where $\|\alpha\| = \min(\{\alpha\}. 1 - \{\alpha\})$.

LEMMA 3. *The sequence S_n satisfies the recursion*

$$S_{k+d} = A_1 S_{k+d-1} + \cdots + A_d S_k,$$

where

$$A_l = e^{2\pi i h(a_1 + \cdots + a_{l-1})x} \sum_{m=0}^{a_l-1} e^{2\pi i h m x}$$

with suitable initial values.

In the case that $a_d \geqslant 2$ we have, by Lemma 2 and Lemma 3,

$$|S_{k+d}| \leqslant (a_1 - 2\pi \|hx\|^2)|S_{k+d-1}| + \cdots + (a_d - 2\pi \|hx\|^2)|S_k|.$$

Therefore $|S_k|$ is less than a suitable solution of the recurrence

$$T_{k+d} = (a_1 - 2\pi \|hx\|^2)\, T_{k+d-1} + \cdots + (a_d - 2\pi \|hx\|^2)\, T_k.$$

Hence we have to consider the dominating roots of the equation

$$p(z) = z^d - (a_1 - 2\pi \|hx\|^2)\, z^{d-1} - \cdots - (a_d - 2\pi \|hx\|^2) = 0. \tag{3.7}$$

An application of Rouché's theorem shows the existence of a uniquely determined dominating real root $\beta < \alpha$. Thus

$$S_k = O(\beta^k),$$

and in the case $a_d \geqslant 2$ the assertion of Theorem 1 follows by Weyl's criterion and an application of Lemma 1. In the case that at least one coefficient of G is $\geqslant 2$ an easy modification of the above argument yields the result.

In the remaining case all coefficients are equal to 1.

The recursion of Lemma 3 is in this case given by

$$S_{k+d} = S_{k+d-1} + e^{2\pi i h x} S_{k+d-2} + \cdots + e^{2\pi i h (d-1) x} S_k.$$

In order to estimate $|S_k|$ we need more precise information on the roots of

$$z^d - z^{d-1} - e^{2\pi i h x} z^{d-2} - \cdots - e^{2\pi i h (d-1) x};$$

thus after the substitution $w = e^{-2\pi i h x} z$ we have to consider

$$p(w) = w^d - e^{2\pi i h x}(w^{d-1} + \cdots + 1) = 0.$$

Again by an application of Rouché's theorem there is a uniquely determined dominating root β with $|\beta| < \alpha$, hence

$$|S_k| = O(\beta^k).$$

Thus the assertion of Theorem 1 follows by Weyl's criterion and Lemma 1.

Remark 4. In [4] an estimate for the discrepancy is obtained in the case $a_1 \geqslant 2$:

$$D_N(x s_G(n)) = \frac{C(G, x, \varepsilon)}{(\log N)^{(1/2\eta) - \varepsilon}}$$

for all $N > 1$, $\varepsilon > 0$ and any real number x of approximation type η. The proof is essentially based on Erdös and Turán's inequality. For the usual q-ary digit system see [9]. In [4] also the Hausdorff dimension of sets defined by digital properties with respect to Parry's α-expansion is computed.

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