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# Singular parameters for the Birman–Murakami–Wenzl algebra

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ABSTRACT

In this paper, we classify the singular parameters for the Birman–Murakami–Wenzl algebra over an arbitrary field. Equivalently, we give a criterion for the Birman–Murakami–Wenzl algebra being Morita equivalent to the direct sum of the Hecke algebras associated to certain symmetric groups.



## 1. Introduction

The Birman–Murakami–Wenzl algebra $\mathscr{B}_n$ was introduced independently by Birman, Wenzl [1] and Murakami [11] in order to study the link invariants. It is cellular over a commutative ring [19] in the sense of [7]. Further, Xi classified its irreducible modules over an arbitrary field [19].

Recently, Enyang [6] constructed the Jucys–Murphy basis for each cell module of $\mathscr{B}_n$. We lifted Enyang's basis to get the Jucys–Murphy basis for $\mathscr{B}_n$ [14]. Over certain fields, we use Jucys–Murphy basis of $\mathscr{B}_n$ to construct its orthogonal basis. This enables us to compute the Gram determinant associated to each cell module of $\mathscr{B}_n$. Therefore, we determine explicitly the semi-simplicity of $\mathscr{B}_n$ over an arbitrary field [14]. We remark that Wenzl has got some partial results in [18].

We also use our results on Gram determinants to classify the blocks of $\mathscr{B}_n$ over certain fields [15]. Via such results, we determine explicitly whether the Gram determinant associated to a cell module is equal to zero or not [15]. This is equivalent to saying that a cell module of $\mathscr{B}_n$ is equal to its simple head or not.

Morton and Wassermann [10] proved that $\mathscr{B}_n$ is isomorphic to the Kauffman tangle algebra [8]. Further, by specialization, they proved that the Kauffman tangle algebra is isomorphic to the Brauer algebra [2]. Therefore, the Brauer algebra [2] can be considered as the classical limit of $\mathscr{B}_n$.

In [9], König and Xi [9] proved that the Brauer algebra can be obtained from some inflations of the group algebras of certain symmetric groups along certain vector spaces. They introduced the notion of singular parameters and proved that the Brauer algebra is Morita equivalent to the direct sum of such group algebras if the defining parameter is not singular. However, there is no criterion to determine whether the defining parameter is singular or not.

The aim of this paper is to give a criterion to determine the singular parameters for the Brauer algebra. In fact, we will deal with $\mathscr{B}_n$ instead of the Brauer algebra.

We introduce the notion of singular parameters for $\mathscr{B}_n$ over an arbitrary field. Via some results on the inflations in [9], we prove that $\mathscr{B}_n$ is Morita equivalent to the direct sum of Hecke algebras associated to certain symmetric groups if the defining parameters are not singular. Further, we give an explicit criterion on the singular parameters for $\mathscr{B}_n$ over an arbitrary field. By specialization, we also obtain the explicit criterion on the singular parameters for Brauer algebras over an arbitrary field.

We organize our paper as follows. In Section 2, after recalling the inflation of an algebra along a vector space in [9], we give Theorems 2.17 and 2.19, the main results of this paper, which are the criterions on the singular parameters for $\mathscr{B}_n$ and

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the Brauer algebra, respectively. In Section 3, we recall some of our results on the representations of $\mathscr{B}_n$ over an arbitrary field. We will use them to prove Theorem 2.17 in Section 4.

## 2. The main results

In this section, we state the results on the classification of singular parameters for $\mathscr{B}_n$ and Brauer algebras over an arbitrary field. We start by recalling some results on the inflation of an algebra along a vector space in [9].

Throughout, we assume that $\kappa$ is a field with characteristic $\mathrm{char}(\kappa)$ either zero or $p$ with $p > 0$. When $\mathrm{char}(\kappa) = 0$, we set $p = \infty$. Let $q \in \kappa$ be an invertible element with $q^2 \neq 1$. We define the quantum characteristic

$$e = \min\{i \geq 2 \mid 1 + q^2 + \cdots + q^{2i-2} = 0\} \tag{2.1}$$

if $q^2$ is a root of unity and $\infty$, otherwise.

Given a finite dimensional $\kappa$-space $V$, a $\kappa$-algebra $B$, and a $\kappa$-bilinear form $\phi : V \otimes V \to B$, König and Xi [9] define a $\kappa$-algebra $A$ which is equal to $V \otimes V \otimes B$ as $\kappa$-vector space. The multiplication of $A$ is defined on basis elements as follows:

$$(a \otimes b \otimes x) \cdot (c \otimes d \otimes y) = a \otimes d \otimes x\phi(b, c)y. \tag{2.2}$$

König and Xi [9] called this $A$ the inflation of $B$ along $V$ if there is a $\kappa$-linear involution $\sigma$ on $B$ with $\sigma(\phi(b, c)) = \phi(c, b)$ such that this $\sigma$ can be extended to the $\kappa$-linear involution $\tau$ on $A$ satisfying

$$\tau(a \otimes b \otimes x) = b \otimes a \otimes \sigma(x). \tag{2.3}$$

In the remainder of this paper, we will use $\sigma$ instead of $\tau$ if there is no confusion. It has been pointed out in [9] that $A$ may not have a unit.

If $B$ is a simple $\kappa$-algebra, there is a unique irreducible $B$-module which is denoted by L. Let $v_1, v_2, \ldots, v_\ell$ be a $\kappa$-basis of $V$ with $\dim V = \ell$. König and Xi [9] considered the left $A$-module

$$P(L, i) = V \otimes v_i \otimes L, \tag{2.4}$$

for some basis element $v_i$ of $V$ and its $\kappa$-subspace

$$N_\phi(L, i) = \left\{ \sum_{v \in V, l \in L} v \otimes v_i \otimes l \in P(L, i) \;\middle|\; \sum_{v,l} \phi(w, v)l = 0, \forall w \in V \right\}. \tag{2.5}$$

They proved that $N_\phi(L, i)$ is an $A$-submodule of $P(L, i)$ and the corresponding quotient module $P(L, i)/N_\phi(L, i)$ is irreducible [9, Lemma 3.2]. If $N_\phi(L, i) \neq 0$, the bilinear form $\phi$ is called singular [9]. The following definition can be found in [9].

**Definition 2.6** (*[9]*). Given a $\kappa$-algebra $B$, let $\mathrm{Rad}B$ be its Jacobson radical. Let $\overline{\phi} = \pi \circ \phi$ where $\pi : B \to B/\mathrm{Rad}B$ is the canonical epimorphism. The bilinear form $\phi$ is called singular if $N_{\overline{\phi}}(L, i) \neq 0$ for some irreducible $B$-module $L$ and some basis element $v_i \in V$.

The key point is the following theorem, which follows from Corollary 3.5 and Proposition 4.2 in [9].

**Theorem 2.7** (*[9]*). *Given the $\kappa$-algebra $V \otimes V \otimes B$ which is the inflation of the $\kappa$-algebra $B$ along the $\kappa$-vector space $V$. If $\phi : V \otimes V \to B$, the corresponding bilinear form, is non-singular, then $V \otimes V \otimes B \overset{Morita}{\sim} B$.*

We are going to state our main result on $\mathscr{B}_n$. Throughout, we assume that $R$ is a commutative ring which contains the multiplicative identity $1_R$ and invertible elements $q$, $r$ and $q - q^{-1}$.

**Definition 2.8** (*[1,11]*). The Birman–Murakami–Wenzl algebra $\mathscr{B}_n$ with defining parameters $r$ and $q$ is the unital associative $R$-algebra generated by $T_i$, $1 \leq i < n$ subject to the relations:

(a) $(T_i - q)(T_i + q^{-1})(T_i - r^{-1}) = 0$, for $1 \leq i < n$,
(b) $T_iT_j = T_jT_i$ if $|i - j| > 1$,
(c) $T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}$, for $1 \leq i < n - 1$,
(d) $E_iT_i = r^{-1}E_i = T_iE_i$, for $1 \leq i \leq n - 1$,
(e) $E_iT_j^{\pm}E_i = r^{\pm}E_i$, for $j = i \pm 1$,

where $E_i = 1 - (q - q^{-1})^{-1}(T_i - T_i^{-1})$ for $1 \leq i \leq n - 1$.

It is known that there is an $R$-linear anti-involution $* : \mathscr{B}_n \to \mathscr{B}_n$ which fixes $T_i$. If we denote by $\langle E_1\rangle$ the two-sided ideal of $\mathscr{B}_n$ generated by $E_1$, then $\mathscr{B}_n/\langle E_1\rangle$ is isomorphic to the Hecke algebra $\mathscr{H}_n$ associated to the symmetric group $\mathfrak{S}_n$. We denote by

$$\varepsilon_n : \mathscr{H}_n \to \mathscr{B}_n/\langle E_1\rangle \tag{2.9}$$

the corresponding isomorphism.

Note that $\mathscr{H}_n$ is the $R$-algebra with generators $g_i$, $1 \leq i \leq n - 1$ subject to the defining relations

$$\begin{cases} (g_i - q)(g_i + q^{-1}) = 0, & \text{for } 1 \leq i \leq n - 1, \\ g_ig_j = g_jg_i, & \text{if } |i - j| > 1, \\ g_ig_jg_i = g_jg_ig_j, & \text{if } |i - j| = 1. \end{cases}$$

It is known that there is an $R$-linear anti-involution $*$ on $\mathscr{H}_n$ which fixes $g_i$, $1 \leq i \leq n - 1$.

Recall that the symmetric group $\mathfrak{S}_n$ in $n$ letters is the Coxeter group with distinguished generators $s_i$, $1 \le i \le n-1$ subject to the defining relations

$$\begin{cases} s_i^2 = 1, & \text{if } 1 \le i \le n-1, \\ s_i s_j = s_j s_i, & \text{if } |i-j| > 1, \\ s_i s_j s_i = s_j s_i s_j, & \text{if } j = i \pm 1. \end{cases}$$

It is known that $s_i$ can be identified with the basic transposition $(i, i+1)$. For each $w \in \mathfrak{S}_n$ with reduced expression $s_{i_1} \cdots s_{i_k}$, let $T_w := T_{i_1} T_{i_2} \cdots T_{i_k} \in \mathscr{B}_n$. It is well known that $T_w$ is independent of a reduced expression of $w$.

Given a non-negative integer $f \le \lfloor n/2 \rfloor$, let $\mathscr{B}_n^f$ be the two sided ideal of $\mathscr{B}_n$ generated by $E^{f,n}$, where

$$E^{f,n} = E_{n-1} E_{n-3} \cdots E_{n-2f+1}. \tag{2.10}$$

When $f = 0$, we denote $E^{0,n}$ by the identity of $\mathscr{B}_n$. It is known that

$$\mathscr{B}_n = \mathscr{B}_n^0 \supset \mathscr{B}_n^1 \supset \cdots \supset \mathscr{B}_n^{\lfloor n/2 \rfloor} \supset 0$$

is a filtration of two-sided ideals of $\mathscr{B}_n$. Let

$$s_{i,j} = \begin{cases} s_i s_{i+1} \cdots s_{j-1}, & \text{if } i < j, \\ s_{i-1} s_{i-2} \cdots s_j, & \text{if } i > j, \\ 1, & \text{if } i = j. \end{cases}$$

Enyang [5] proved that $\mathscr{B}_n^f / \mathscr{B}_n^{f+1}$ is free over $R$ with basis $S$ where

$$S = \{T_u^* E^{f,n} T_w T_v \mod \mathscr{B}_n^{f+1} \mid u, v \in D_{f,n}, w \in \mathfrak{S}_{n-2f}\}, \tag{2.11}$$

and

$$D_{f,n} = \left\{ s_{n-2f+1, i_f} s_{n-2f+2, j_f} \cdots s_{n-1, i_1} s_{n, j_1} \;\middle|\; \begin{smallmatrix} 1 \le i_f < \cdots < i_1 \le n; \\ 1 \le i_k < j_k \le n-2k+2; 1 \le k \le f \end{smallmatrix} \right\}. \tag{2.12}$$

Let $\kappa$ be a field which is an $R$-algebra. Let

$$\mathscr{B}_{n,\kappa} = \mathscr{B}_n \otimes_R \kappa.$$

By abusing notation, we will use $\mathscr{B}_n$ instead of $\mathscr{B}_{n,\kappa}$ in the remaining part of this paper.

Via the $\kappa$-basis $S$ for $\mathscr{B}_n^f / \mathscr{B}_n^{f+1}$ in (2.11), we will prove that $\mathscr{B}_n^f / \mathscr{B}_n^{f+1}$ is the inflation of $\mathscr{H}_{n-2f}$ along the vector space $V_f$ spanned by $T_u$, $u \in D_{f,n}$. We remark that the corresponding anti-involution $\sigma$ on $\mathscr{H}_{n-2f}$ is the anti-involution $*$ on $\mathscr{H}_n$. Further, $\sigma$ can be extended to $\mathscr{B}_n^f / \mathscr{B}_n^{f+1}$, which is the same as the anti-involution induced by $*$ on $\mathscr{B}_n$. The bilinear form

$$\phi_f : V_f \otimes V_f \to \mathscr{H}_{n-2f} \tag{2.13}$$

can be defined via the multiplication of $\mathscr{B}_n$. Details will be given in Proposition 3.10.

The following definition is motivated by König and Xi's work on Brauer algebras in [9].

**Definition 2.14.** The defining parameters $r$ and $q$ are said to be singular if there is a positive integer $f \le \lfloor n/2 \rfloor$ such that the bilinear form $\phi_f$ in (2.13) is singular in the sense of Definition 2.6.

The following result follows from Theorem 2.7 and [9, Lemma 7.1], immediately.

**Proposition 2.15.** *Let $\mathscr{B}_n$ be the Birman–Murakami–Wenzl algebra over $\kappa$. Then*

$$\mathscr{B}_n \overset{Morita}{\sim} \oplus_{0 \le f \le \lfloor n/2 \rfloor} \mathscr{H}_{n-2f}$$

*if the defining parameters $r$ and $q$ are not singular.*

Recall that $e$ is the quantum characteristic in (2.1). The following result, which is the main result of this paper, gives the classification of singular parameters $r$ and $q$ for $\mathscr{B}_n$ over an arbitrary field $\kappa$. We define

$$\mathscr{S} = \cup_{k=3}^{n} \{q^{3-2k}, \pm q^{3-k}, -q^{2k-3}, \pm q^{k-3}\}, \quad \text{for } n \ge 2. \tag{2.16}$$

**Theorem 2.17.** *Let $\mathscr{B}_n$ be the Birman–Murakami–Wenzl algebra over the field $\kappa$.*

(a) *Suppose $e > n-2$.*
  (1) *If $r \notin \{q^{-1}, -q\}$, then $r$ and $q$ are singular if and only if $r \in \mathscr{S}$,*
  (2) *If $r \in \{q^{-1}, -q\}$, then $r$ and $q$ are singular if and only if one of the following conditions holds:*
    (i) *$n$ is even or odd with $n \ge 7$,*
    (ii) *$n = 3$, and $q^4 + 1 = 0$.*
    (iii) *$n = 5$, and $2(q^4+1)(q^6+1)(q^8+1) = 0$.*

(b) *If $e \le n-2$, then $r$ and $q$ are singular if and only if $r \in \{q^a, -q^b \mid a, b \in \mathbb{Z}\}$.*

Recall that Morton and Wassermann proved that $\mathscr{B}_n$ is isomorphic the Kauffman tangle algebra [10]. Morton and Wassermann defined $\mathscr{B}_n$ over the commutative ring $\mathbb{Z}[r^{\pm}, \omega, \delta]/\langle r - r^{-1} - \omega(\delta - 1)\rangle$. In their definition, they do not need the invertibility of $\omega$. By specializing $\omega$ and $r$ to 0 and 1, respectively, they proved that Kauffman tangle algebra is isomorphic to the Brauer algebra $B_n(\delta)$ [2], which is the associative algebra over $\mathbb{Z}[\delta]$ generated by $s_i, e_i$, $1 \le i \le n-1$

subject to the following relations:

(a) $s_i^2 = 1$, for $1 \le i < n$.
(b) $s_i s_j = s_j s_i$ if $|i-j| > 1$,
(c) $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$, for $1 \le i < n-1$,
(d) $e_i^2 = \delta e_i$, for $1 \le i < n$.
(e) $s_i e_j = e_j s_i$, if $|i-j| > 1$,
(f) $e_i e_j = e_j e_i$, if $|i-j| > 1$.
(g) $e_i s_i = e_i = s_i e_i$, for $1 \le i \le n-1$,
(h) $s_i e_{i+1} e_i = s_{i+1} e_i$, $e_{i+1} e_i s_{i+1} = e_{i+1} s_i$, for $1 \le i \le n-2$.
(i) $e_{i+1} e_i e_{i+1} = e_{i+1}$ and $e_i e_{i+1} e_i = e_i$, for $1 \le i \le n-2$.

Note that the relationship between Morton–Wassermann's notations and our notations is

$$\delta = \frac{(q+r)(qr-1)}{r(q+1)(q-1)}.$$

Therefore, the Brauer algebra can be obtained from $\mathscr{B}_n$ by specializing $q, r$ to 1.

We have $\text{limit}_{q\to 1}\delta \in \mathcal{Z}$ if $r \in \mathcal{S}$ and $n \ge 2$ where

$$\mathcal{Z} = \{1, 2, \dots, n-2\} \cup \{-2, -4, \dots, 4-2n\} \cup \{-1, -2, \dots, 4-n\}. \tag{2.18}$$

When $n \le 2$, both $\mathcal{S}$ and $\mathcal{Z}$ are empty since we are assuming $\delta \ne 0$ for both $\mathscr{B}_n$ and $B_n(\delta)$ in Theorem 2.17(a)(1) and Theorem 2.19(a)(i). If $r = \pm q^a$, we have $\text{limit}_{q\to 1}\delta \in \mathbb{Z}$. Therefore, by Theorem 2.17, we have the following result immediately.

**Theorem 2.19.** *Let $B_n(\delta)$ be the Brauer algebra over the field $\kappa$. Let $p = \text{char}\,\kappa$ if $\text{char}\,\kappa > 0$ and let $p = \infty$ otherwise.*

(a) *Suppose $p > n-2$.*
 (i) *If $\delta \ne 0$, then $\delta$ is singular if and only if $\delta = a \cdot 1_\kappa$ and $a \in \mathcal{Z}$,*
 (ii) *If $\delta = 0$, then $\delta$ is singular if and only if one of the following conditions holds:*
  (1) *$n$ is either even or odd with $n > 7$,*
  (2) *$n = 3$ and $p = 2$.*
(b) *Suppose $p \le n-2$. Then $\delta$ is singular if and only if $\delta = a \cdot 1_\kappa$ and $a \in \mathbb{Z}$.*

By comparing Theorems 2.17–2.19, we find that 0 is not singular for $B_5(0)$ if $p > 3$. The reason is that $\text{limit}_{q\to 1} 2(q^4+1)(q^6+1)(q^8+1) \ne 0$ in $\kappa$ if $p > 3$.

König and Xi pointed out that the singular parameter $\delta$ is dependent of $\delta$ only [9, p1502]. This is not correct. One can find the counter-example by considering the cell module $\Delta(1,(1))$ when $p = 2$ and $\delta = 0$.

Finally, we remark that Theorem 2.19 can be proved by similar arguments for $\mathscr{B}_n$. In order to give the detailed proof, we need results which are similar to Theorem 3.3, Definition 3.5, Theorem 3.6 and Lemma 4.3 etc, which can be found in [3,7,12,13,16] etc. We leave the details to the reader.

## 3. Representations of Birman–Murakami–Wenzl algebras

In this section, we recall some results on the representations of $\mathscr{B}_n$ over a field . We will use them to prove Theorem 2.17 in Section 4. We start by recalling some combinatorics.

Recall that a partition of $n$ is a weakly decreasing sequence of non-negative integers $\lambda = (\lambda_1, \lambda_2, \dots)$ such that $|\lambda| := \lambda_1 + \lambda_2 + \cdots = n$. In this case, we write $\lambda \vdash n$. Let $\Lambda^+(n) = \{\lambda \mid \lambda \vdash n\}$. Then $\Lambda^+(n)$ is a poset with dominance order $\unlhd$ as the partial order on it. More explicitly, $\lambda \unlhd \mu$ for $\lambda, \mu \in \Lambda^+(n)$ if $\sum_{j=1}^{i}\lambda_j \le \sum_{j=1}^{i}\mu_j$ for all possible $i$. Write $\lambda \lhd \mu$ if $\lambda \unlhd \mu$ and $\lambda \ne \mu$.

Suppose that $\lambda$ and $\mu$ are two partitions. We say that $\mu$ is obtained from $\lambda$ by adding a box if there exists an $i$ such that $\mu_i = \lambda_i + 1$ and $\mu_j = \lambda_j$ for $j \ne i$. In this situation we will also say that $\lambda$ is obtained from $\mu$ by removing a box and we write $\lambda \to \mu$ and $\mu \setminus \lambda = (i, \lambda_i + 1)$. We will say that the pair $(i, \lambda_i + 1)$ is an addable node of $\lambda$ and a removable node of $\mu$. Note that $|\mu| = |\lambda| + 1$.

The Young diagram $[\lambda]$ for a partition $\lambda = (\lambda_1, \lambda_2, \dots)$ is a collection of boxes arranged in left-justified rows with $\lambda_i$ boxes in the $i$-th row of $[\lambda]$. A $\lambda$-tableau $\mathbf{s}$ is obtained by inserting $i$, $1 \le i \le n$ into $[\lambda]$ without repetition. The symmetric group $\mathfrak{S}_n$ acts on $\mathbf{s}$ by permuting its entries. Let $\mathbf{t}^\lambda$ be the $\lambda$-tableau obtained from the Young diagram $Y(\lambda)$ by adding $1, 2, \dots, n$ from left to right along the rows. For example, for $\lambda = (4, 3, 1)$,

$$\mathbf{t}^\lambda = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 & 7 \\ \cline{1-3} 8 \\ \cline{1-1}\end{array}$$

If $\mathbf{t}^\lambda w = \mathbf{s}$, write $w = d(\mathbf{s})$. Note that $d(\mathbf{s})$ is uniquely determined by $\mathbf{s}$.

A $\lambda$-tableau $\mathbf{s}$ is standard if the entries in $\mathbf{s}$ are increasing both from left to right in each row and from top to bottom in each column. Let $\mathscr{T}_n^{std}(\lambda)$ be the set of all standard $\lambda$-tableaux.

For $\lambda \vdash n-2f$, let $\mathfrak{S}_\lambda$ be the Young subgroup of $\mathfrak{S}_{n-2f}$ generated by $s_j$, $1 \le j \le n-2f-1$ and $j \ne \sum_{k=1}^{i} \lambda_k$ for all possible $i$.

Let $\Lambda_n = \{ (f, \lambda) \mid \lambda \vdash n-2f, 0 \le f \le \lfloor \frac{n}{2} \rfloor \}$. Given $(k, \lambda), (f, \mu) \in \Lambda_n$, define $(k, \lambda) \trianglelefteq (f, \mu)$ if either $k < f$ or $k = f$ and $\lambda \trianglelefteq \mu$. Write $(k, \lambda) \lhd (f, \mu)$, if $(k, \lambda) \trianglelefteq (f, \mu)$ and $(k, \lambda) \ne (f, \mu)$.

Let $I(f, \lambda) = \mathscr{T}_n^{std}(\lambda) \times D_{f,n}$ where $D_{f,n}$ is defined in (2.12). Define

$$C^{(f,\lambda)}_{(\mathbf{s},u)(\mathbf{t},v)} = T_u^* T_{d(\mathbf{s})}^* \mathfrak{M}_\lambda T_{d(\mathbf{t})} T_v, \quad (\mathbf{s}, u), (\mathbf{t}, v) \in I(f, \lambda) \tag{3.1}$$

where $\mathfrak{M}_\lambda = E^{f,n} X_\lambda$, $E^{f,n} = E_{n-1}E_{n-3} \cdots E_{n-2f+1}$, $X_\lambda = \sum_{w \in \mathfrak{S}_\lambda} q^{l(w)} T_w$, and $l(w)$, the length of $w \in \mathfrak{S}_n$.

**Theorem 3.2** (*[5]*). *Let $\mathscr{B}_n$ be the Birman–Murakami–Wenzl algebra over R. Let $* : \mathscr{B}_n \to \mathscr{B}_n$ be the R-linear anti-involution which fixes $T_i$, $1 \le i \le n-1$. Then*

(a) $\mathscr{C}_n = \left\{ C^{(f,\lambda)}_{(\mathbf{s},u)(\mathbf{t},v)} \mid (\mathbf{s}, u), (\mathbf{t}, v) \in I(f, \lambda), \lambda \vdash n-2f, 0 \le f \le \lfloor \frac{n}{2} \rfloor \right\}$ *is a free R-basis of* $\mathscr{B}_n$.

(b) $*(C^{(f,\lambda)}_{(\mathbf{s},u)(\mathbf{t},v)}) = C^{(f,\lambda)}_{(\mathbf{t},v)(\mathbf{s},u)}$.

(c) *For any* $h \in \mathscr{B}_n$,

$$h \cdot C^{(f,\lambda)}_{(\mathbf{s},u)(\mathbf{t},v)} \equiv \sum_{(\mathbf{u},w) \in I(f,\lambda)} a_{\mathbf{u},w} C^{(f,\lambda)}_{(\mathbf{u},w)(\mathbf{t},v)} \mod \mathscr{B}_n^{\rhd (f,\lambda)}$$

*where $\mathscr{B}_n^{\rhd (f,\lambda)}$ is the free R-submodule generated by $C^{(k,\mu)}_{(\tilde{\mathbf{s}},\tilde{u})(\tilde{\mathbf{t}},\tilde{v})}$ with $(k, \mu) \rhd (f, \lambda)$ and $(\tilde{\mathbf{s}}, \tilde{u}), (\tilde{\mathbf{t}}, \tilde{v}) \in I(k, \mu)$. Moreover, each coefficient $a_{\mathbf{u},w}$ is independent of $(\mathbf{t}, v)$.*

Theorem 3.2 shows that $\mathscr{C}_n$ is a cellular basis of $\mathscr{B}_n$ in the sense of [7]. We remark that Xi [19] first proved that $\mathscr{B}_n$ is cellular in the sense of [7]. The cellular basis $\mathscr{C}_n$ was constructed by Enyang in [5].

In this paper, we will only consider left modules for $\mathscr{B}_n$. By general theory about cellular algebras in [7], we know that, for each $(f, \lambda) \in \Lambda_n$, there is a cell module $\Delta(f, \lambda)$ of $\mathscr{B}_n$, spanned by

$$\{ T_v^* T_{d(\mathbf{t})}^* \mathfrak{M}_\lambda \bmod \mathscr{B}_n^{\rhd (f,\lambda)} \mid (\mathbf{t}, v) \in I(f, \lambda) \}.$$

Further, there is an invariant form $\phi_{f,\lambda}$ on $\Delta(f, \lambda)$. Let

$$\mathrm{Rad}\Delta(f, \lambda) = \{ x \in \Delta(f, \lambda) \mid \phi_{f,\lambda}(x, y) = 0 \text{ for all } y \in \Delta(f, \lambda) \}.$$

Then $\mathrm{Rad}\Delta(f, \lambda)$ is a $\mathscr{B}_n$–submodule of $\Delta(f, \lambda)$. Let

$$D^{f,\lambda} = \Delta(f, \lambda)/\mathrm{Rad}\Delta(f, \lambda).$$

Recall that $e$ is the order of $q^2$ if $q^2$ is a root of unity. Otherwise, $e = \infty$. We say that the partition $\lambda$ is $e$-restricted if $\lambda_i - \lambda_{i+1} < e$ for all possible $i$.

**Theorem 3.3** (*[19]*). *Let $\mathscr{B}_n$ be the Birman–Murakami–Wenzl algebra over the field $\kappa$ which contains non-zero parameters $r, q$, and $q - q^{-1}$.*

(a) *Suppose $r \notin \{q^{-1}, -q\}$. The non-isomorphic irreducible $\mathscr{B}_n$-modules are indexed by $(f, \lambda)$ and $\lambda$ is e-restricted.*

(b) *Suppose $r \in \{q^{-1}, -q\}$.*

  (i) *If n is odd, then the non-isomorphic irreducible $\mathscr{B}_n$-modules are indexed by $(f, \lambda)$, $0 \le f \le \lfloor n/2 \rfloor$. $\lambda \in \Lambda^+(n-2f)$ and $\lambda$ is e-restricted.*

  (ii) *If n is even, then the non-isomorphic irreducible $\mathscr{B}_n$-modules are indexed by $(f, \lambda)$, $0 \le f < \lfloor n/2 \rfloor$. $\lambda \in \Lambda^+(n-2f)$, $\lambda$ is e-restricted.*

By general results on cellular algebras in [7], $\mathscr{B}_n$ is (split) semisimple over $\kappa$ if and only if $D^{f,\lambda} = \Delta(f, \lambda)$ for all $(f, \lambda) \in \Lambda_n$. We remark that the authors have given the necessary and sufficient conditions for $\mathscr{B}_n$ being seimisimple over an arbitrary field [14]. However, we will not need this result in this paper. What we need is the explicit criterion for $\Delta(f, \lambda)$ being equal to its simple head $D^{f,\lambda}$. In other words, we give a criterion to determine when the Gram determinant $\det G_{f,\lambda} \ne 0$. Here $G_{f,\lambda}$ is the Gram matrix associated to the invariant form $\phi_{f,\lambda}$. We need some combinatorics to state this result.

For each box $p = (i, j) \in [\lambda]$, define $p^+ = (i, j+1)$, $p^- = (i+1, j)$, and

$$c_\lambda(p) = rq^{2(j-i)}. \tag{3.4}$$

We will use $c(p)$ instead of $c_\lambda(p)$.

Given two partitions $\lambda$ and $\mu$, we write $\lambda \supset \mu$ if $\lambda_i \ge \mu_i$ for all possible $i$. In this case, the corresponding skew Young diagram $[\lambda/\mu]$ can be obtained from $[\lambda]$ by removing all nodes in $[\mu]$.

We recall the definition of $(f, \mu)$-admissible partition $\lambda$ in [15]. In the Definition 3.5, we assume that $q^2$ is not a root of unity and the ground field $\kappa$ is the complex field $\mathbb{C}$ although we give this definition with loose restrictions in [15].

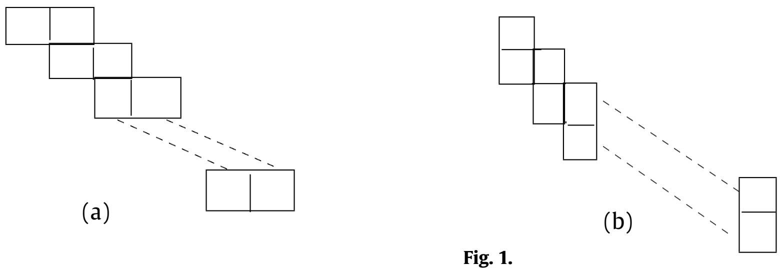


**Fig. 1.**

**Definition 3.5** ([15]). Given $\lambda \in \Lambda^+(n)$ and $\mu \in \Lambda^+(n-2f)$, we say $\lambda$ is $(f, \mu)$-admissible over the field $\kappa$ if

(a) $\lambda \supset \mu$,
(b) there is a pairing of nodes $p_i, \tilde{p}_i$, $1 \le i \le f$ in $[\lambda/\mu]$ such that $c(p_i)c(\tilde{p}_i) = 1$. We call $\{p_i, \tilde{p}_i\}$ an admissible pair. Further, there are two possible configurations of nodes in $[\lambda/\mu]$ as follows.
(c) the number of columns in Fig. 1(b) is even if $c(p) = q$ and $\{p, p^-\}$ is an admissible pair which is contained in Fig. 1(b).
(d) the number of rows in Fig. 1(a) is even if $c(p) = -q^{-1}$ and $\{p, p^+\}$ is an admissible pair which is contained in Fig. 1(a).

Given a $\lambda \in \Lambda^+(n)$ and a node $(i, j) \in [\lambda]$, let

$$h_{ij}^\lambda = \lambda_i - j + \lambda'_j - i + 1,$$

where $\lambda'$ is the dual partition of $\lambda$. This $h_{ij}^\lambda$ is known as the $(i, j)$-hook length in $[\lambda]$.

Recall that $e$ is the order of $q^2$ if it is a root of unity. Otherwise, $e = \infty$. Let $p = \mathrm{char}(\kappa)$ if $\mathrm{char}(\kappa) > 0$ and let $p = \infty$ if $\mathrm{char}(\kappa) = 0$. For each integer $h$, define

$$\nu_{e,p}(h) = \begin{cases} \nu_p(\frac{h}{e}), & \text{if } e < \infty \text{ and } e|h; \\ -1, & \text{otherwise} \end{cases}$$

where $\nu_p(h)$ is the largest power of $p$ dividing $h$ if $p$ is finite and $\nu_\infty(h) = 0$ if $p = \infty$,

In the following result, $\kappa$ is an arbitrary field.

**Theorem 3.6** ([15]). *Suppose $\kappa$ is a field which contains invertible $q, r$ and $q - q^{-1}$. For each $(f, \lambda) \in \Lambda_n$, let $\det G_{f,\lambda}$ be the Gram determinant associated to the cell module $\Delta(f, \lambda)$. Then $\det G_{f,\lambda} \neq 0$ if and only if the following conditions hold:*

(a) *$r \neq \pm q^a$ where the integer $a$ and the sign of $q^a$ are determined by $(f - \ell, \lambda)$-admissible partitions over $\mathbb{C}$ with $\mathbf{q} \in \mathbb{C}$, $o(\mathbf{q}^2) = \infty$ and $0 \le \ell \le f - 1$,*
(b) *$\lambda$ is e-restricted,*
(c) *$\nu_{e,p}(h_{ac}^\lambda) = \nu_{e,p}(h_{ab}^\lambda)$, $\forall (a, c), (a, b) \in [\lambda]$.*

We are going to prove that $\mathscr{B}_n$ can be obtained from $\mathscr{H}_{n-2f}$, $0 \le f \le \lfloor n/2 \rfloor$ by inflations along certain vector spaces $V_f$. When $f = \lfloor n/2 \rfloor$, we denote by $\mathscr{H}_{n-2f}$ the ground field $\kappa$.

It is proved in [1] that

$$E_{n-1}\mathscr{B}_n E_{n-1} = E_{n-1}\mathscr{B}_{n-2}. \tag{3.7}$$

Therefore, applying (3.7) repeatedly yields

$$E^{f,n}\mathscr{B}_n E^{f,n} = E^{f,n}\mathscr{B}_{n-2} \tag{3.8}$$

for all positive integers $f \le \lfloor n/2 \rfloor$.

Now, let $V_f$ be the free $\kappa$-module generated by $T_u$ with $u \in D_{f,n}$. By (3.8), for any $u, v \in V_f$, we have $E^{f,n}T_u T_v^* E^{f,n} = E^{f,n}h$ for some $h \in \mathscr{B}_{n-2f}$. Note that $E^{f,n}h = 0$ for $h \in \mathscr{B}_{n-2f}$ if and only if $h = 0$. This gives rise to a well-defined bilinear form

$$\phi_f : V_f \otimes V_f \to \mathscr{H}_{n-2f} \tag{3.9}$$

such that $\phi_f(T_u, T_v) = \varepsilon_{n-2f}^{-1}(h)$ and $\varepsilon_{n-2f}$ is given in (2.9).

**Proposition 3.10.** *Let $\mathscr{B}_n$ be the Birman–Murakami–Wenzl algebra over a field $\kappa$. For any non-negative integer $f \le \lfloor n/2 \rfloor$, we have $\mathscr{B}_n^f / \mathscr{B}_n^{f+1} \cong V_f \otimes V_f \otimes \mathscr{H}_{n-2f}$ as $\kappa$-algebras.*

**Proof.** For any $h \in \mathscr{B}_{n-2f}$, let $h'$ be the image of $h$ in $\mathscr{B}_{n-2f}/\langle E_1 \rangle$. Then $T'_w = g_w$ for all $w \in \mathfrak{S}_{n-2f}$. For all $u, v \in D_{f,n}$, by (2.11), the $\kappa$-linear map sending $T_v^* E^{f,n} T_w T_u \bmod \mathscr{B}_n^{f+1}$ to $T_v \otimes T_u \otimes g_w$ is the required isomorphism. The required anti-involution on $\mathscr{H}_{n-2f}$ is $*$ and the required anti-involution on $V_f \otimes V_f \otimes \mathscr{H}_{n-2f}$ can be defined as that for inflation of an algebra along a vector space in (2.3). $\square$

We are going to recall some results on the representations of the Hecke algebra $\mathscr{H}_n$ over an arbitrary field $\kappa$.

Via the Kazhdan–Lusztig basis for $\mathscr{H}_n$, Graham and Lehrer have proved that $\mathscr{H}_n$ is cellular over $\kappa$ in the sense of [7]. In this case, the corresponding poset is $\Lambda^+(n)$. We will use the cell module defined via Murphy basis $\{x_{\mathbf{ts}} \mid \mathbf{s}, \mathbf{t} \in \mathscr{T}_n^{std}(\lambda)\}$ for $\mathscr{H}_n$. We do not need the explicit construction of $x_{\mathbf{ts}}$. In fact, what we need is the fact that the cell module $\Delta(0, \lambda)$ for $\mathscr{B}_n$ with $\lambda \in \Lambda^+(n)$ can be considered as the cell module for $\mathscr{H}_n$ defined via $\{x_{\mathbf{ts}} \mid \mathbf{s}, \mathbf{t} \in \mathscr{T}_n^{std}(\lambda)\}$. We denote the corresponding cell module by $S^\lambda$. Its $\kappa$-basis elements are denoted by $\{x_{\mathbf{s}} \mid \mathbf{s} \in \mathscr{T}_n^{std}(\lambda)\}$. We remark that $S^\lambda$ is known as the dual Specht module for $\mathscr{H}_n$.

Let $\phi_\lambda$ be the invariant form on $S^\lambda$ and let $\mathrm{Rad}\, S^\lambda$ be the radical of $\phi_\lambda$. Then $\mathrm{Rad}S^\lambda$ is an $\mathscr{H}_n$-submodule of $S^\lambda$. It follows from general results on cellular algebras in [7] that $D^\lambda = S^\lambda/\mathrm{Rad}\, S^\lambda$ is either zero or absolutely irreducible. It is known that $D^\lambda \neq 0$ if and only if $\lambda$ is $e$-restricted.

By (2.4) and (2.5), the $\mathscr{B}_n^f/\mathscr{B}_n^{f+1}$-modules $P(D^\lambda, \ell)$ and $N_{\phi_f}(D^\lambda, \ell)$ are well defined where $D^\lambda$ is the simple head of $S^\lambda$, $1 \leq \ell \leq \dim V_f$, and $V_f$, $\phi_f$ are given in (3.9).

The following result which can be verified directly, sets up the relationship between $P(D^\lambda, \ell)$, $\Delta(f, \lambda)$ etc. Note that for each $(f, \lambda) \in \Lambda_n$, $\mathscr{B}_n^{f+1}$ acts on $\Delta(f, \lambda)$, trivially. Therefore, it can be considered as $\mathscr{B}_n^f/\mathscr{B}_n^{f+1}$-module.

**Proposition 3.11.** *Suppose* $0 < f \leq \lfloor n/2 \rfloor$ *and* $\lambda \in \Lambda^+(n-2f)$. *As* $\mathscr{B}_n^f/\mathscr{B}_n^{f+1}$*-modules, we have*

(a) $V_f \otimes v_\ell \otimes S^\lambda \cong \Delta(f, \lambda)$ *for each basis element* $v_\ell \in V_f$.
(b) $V_f \otimes v_\ell \otimes \mathrm{Rad}S^\lambda$ *is a submodule of* $V_f \otimes v_\ell \otimes S^\lambda$, $\forall$ *e-restricted partitions* $\lambda$. *The corresponding quotient module is isomorphic to* $P(D^\lambda, \ell)$.
(c) $P(D^\lambda, \ell)/N_{\phi_f}(D^\lambda, \ell) \cong D^{f,\lambda}$, *for all e-restricted partitions* $\lambda$.
(d) *If* $D^\lambda = S^\lambda$, *then* $N_{\phi_f}(D^\lambda, \ell) \cong \mathrm{Rad}\Delta(f, \lambda)$.

## 4. Proof of Theorem 2.17

In this section, we always assume that $r, q$ are defining parameters for $\mathscr{B}_n$. Recall that $e$ is the quantum characteristic in (2.1).

**Proposition 4.1.** *Suppose* $e > n - 2$.

(1) *If* $r \notin \{q^{-1}, -q\}$, *then r and q are singular if and only if* $r \in \mathcal{S}$ *given in* (2.16).
(2) *Suppose* $r \in \{q^{-1}, -q\}$. *Then r and q are singular if and only if one of the following conditions holds*
  (a) *n is either even or odd with* $n \geq 7$,
  (b) $n = 3$ *and* $q^4 + 1 = 0$.
  (c) $n = 5$ *and* $2(q^4 + 1)(q^6 + 1)(q^8 + 1) = 0$.

**Proof.** Since we are assuming $e > n-2$, $\mathscr{H}_{n-2f}$ is semisimple over $\kappa$ for any positive integer $f \leq [\frac{n}{2}]$. In particular, $S^\lambda = D^\lambda$, $\forall \lambda \in \Lambda^+(n-2f)$. By Proposition 3.11(d), $N_{\phi_f}(D^\lambda, i) \cong \mathrm{Rad}\Delta(f, \lambda)$ for some $i$, $1 \leq i \leq \dim V_f$. Therefore, $r, q$ are singular if and only if $\det G_{f,\lambda} = 0$ for some $(f, \lambda) \in \Lambda_n$ with $0 < f \leq \lfloor n/2 \rfloor$.

By [14, Prop. 5.1, Coro. 4.25], we have proved that $r \in S$ and $r \notin \{q^{-1}, -q\}$ if and only if $\det G_{1,(k-2)} \det G_{1,(1^{k-2})} = 0$ for some integer $k$ with $2 \leq k \leq n$. Further, in the proof of [14, Prop. 5.6], we have proved that there is an $(f, \lambda) \in \Lambda_n$ with $f > 0$ such that $\det G_{f,\lambda} = 0$ provided $\det G_{1,(k-2)} \det G_{1,(1^{k-2})} = 0$ for some integer $k$ with $2 \leq k \leq n$. Therefore, $r$ and $q$ are singular. This proves (1).

Suppose $r \in \{-q, q^{-1}\}$. In [14, p177], we have given the explicit formulae on $\det G_{1,\lambda}$ for $\lambda \in \{(1), (3), (1^3), (2, 1)\}$, and $\det G_{2,(1)}$ up to some invertible elements in $\kappa$. We list these formulae as follows. One can use the Gap program [17] to verify them easily.

(1) $\det G_{1,(1)} = (q^4 + 1)$ if $r \in \{q^{-1}, -q\}$.
(2) $\det G_{1,(3)} = 2^5[2]^{10}[3]^{14}(1 + q^8)$ if $r = -q$,
(3) $\det G_{1,(3)} = -[2]^{10}[3]^{11}(1 + q^4)^6$ if $r = q^{-1}$,
(4) $\det G_{1,(1,1,1)} = [3](1 + q^4)^6$ if $r = -q$,
(5) $\det G_{1,(1,1,1)} = 2^5[3]^4(1 + q^8)$ if $r = q^{-1}$,
(6) $\det G_{1,(2,1)} = -[2]^4[3]^{15}(1 + q^6)^4$ if $r \in \{q^{-1}, -q\}$.
(7) $\det G_{2,(1)} = -2^6(1 + q^2)(1 + q^4)^{10}(1 + q^6)$ if $r \in \{q^{-1}, -q\}$,

where $[a] = \frac{q^a - q^{-a}}{q - q^{-1}}$ for $a \in \mathbb{Z}$.

Since we are assuming that $e > n - 2$, we have $e > 3$ if $n = 5$ and $e \geq 2$ if $n = 3$. In each case, $S^\lambda = D^\lambda$. Therefore, (2)(b)–(c) follows from these explicit formulae, immediately. In [14, P177], we have also proved that $\det G_{n/2,0} = 0$ for even $n$. This proves the first part of (2)(a).

Suppose that $n$ is odd, with $n \geq 7$. We have $e > 5$ and hence $S^\lambda = D^\lambda$ for $\lambda = (3, 2)$. At the end of the proof of [14, Prop. 5.8], we have proved $\det G_{\frac{n-5}{2},(3,2)} = 0$, which forces $r$ and $q$ being singular. □

We are going to deal with the case $e \leq n-2$. The following result may be well known. We include a proof here.

**Lemma 4.2.** *Suppose that $A_n$ are $\kappa$-algebras for all positive integers n such that $A_{n-1}$ is a subalgebra of $A_n$. If $A_n$-modules M and N have filtrations of $A_{n-1}$-modules*

$$0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_{\ell-1} \subseteq M_\ell = M$$
$$0 = N_0 \subseteq N_1 \subseteq N_2 \subseteq \cdots \subseteq N_{k-1} \subseteq N_k = N,$$

*and if $Hom_{A_n}(M, N) \neq 0$, then there exist some integers i and j, $0 \leq i \leq \ell - 1$ and $0 \leq j \leq k-1$ such that $Hom_{A_{n-1}}(M_{i+1}/M_i, N_{j+1}/N_j) \neq 0$.*

**Proof.** Applying the left exact functor $\mathrm{Hom}_{A_{n-1}}(M, -)$ to the short exact sequence

$$0 \to N_{i-1} \to N_i \to N_i/N_{i-1} \to 0$$

and using induction on *i*, we can find some integer *j*, $0 \leq j \leq \ell - 1$ such that $\mathrm{Hom}_{A_{n-1}}(M, N_{j+1}/N_j) \neq 0$. Applying the left exact functor $\mathrm{Hom}_{A_{n-1}}(-, N_{j+1}/N_j)$ to the short exact sequence

$$0 \to M_{i-1} \to M_i \to M_i/M_{i-1} \to 0$$

and using induction on *i*, we can find an *i*, $0 \leq i \leq k-1$ such that $\mathrm{Hom}_{A_{n-1}}(M_{i+1}/M_i, N_{j+1}/N_j) \neq 0$, as required. □

Let $\mathscr{B}_n$-mod be the category of left $\mathscr{B}_n$-modules, by (3.7), $E_{n-1}\mathscr{B}_n E_{n-1} = E_{n-1}\mathscr{B}_{n-2}$. Therefore, one can define two functors

$$\mathcal{F}_n : \mathscr{B}_n\text{-mod} \to \mathscr{B}_{n-2}\text{-mod}, \text{ and } \mathcal{G}_{n-2} : \mathscr{B}_{n-2}\text{-mod} \to \mathscr{B}_n\text{-mod}$$

such that

$$\mathcal{F}_n(M) = E_{n-1}M \text{ and } \mathcal{G}_{n-2}(N) = \mathscr{B}_n E_{n-1} \otimes_{\mathscr{B}_{n-2}} N,$$

for all left $\mathscr{B}_n$-modules *M* and left $\mathscr{B}_{n-2}$-modules *N*. For the Brauer algebras, such two functors have been studied by Doran et al. in [4].

For the simplification of notation, we will use $\mathcal{F}$, $\mathcal{G}$ instead of $\mathcal{F}_n$ and $\mathcal{G}_{n-2}$, respectively. Note that $\mathcal{F}$ is an exact functor and $\mathcal{G}$ is a right exact functor. We will denote $\mathscr{B}_n^f/\mathscr{B}_n^{f+1}$-module $P(D^\lambda, \ell)$ by $P_n(D^\lambda, \ell)$.

**Lemma 4.3.** *Suppose that $(f, \lambda) \in \Lambda_n$.*

(a) $\mathcal{F}\mathcal{G} = 1$.
(b) $\mathcal{G}(\Delta(f, \lambda)) = \Delta(f+1, \lambda)$.
(c) $\mathcal{F}(\Delta(f, \lambda)) = \Delta(f-1, \lambda)$.
(d) *Let $\lambda$ be e-restricted. Then $\mathcal{G}(P_n(D^\lambda, \ell)) = P_{n+2}(D^\lambda, \ell)$ for any integer $\ell$, $1 \leq \ell \leq \dim V_f$.*
(e) *Let $\lambda$ be e-restricted. Then $\mathcal{F}(P_n(D^\lambda, \ell)) = P_{n-2}(D^\lambda, \ell)$ for any integer $\ell$, $1 \leq \ell \leq \dim V_f$.*
(f) *For each $\mathscr{B}_n$-module N, if $\phi : \Delta(f, \lambda) \to N$ is non-zero, and if $f \geq 1$, then $\mathcal{F}(\phi) \neq 0$.*

**Proof.** We have proved (a)–(c) and (f) in [15, 5.1] for right modules. One can use similar arguments to prove (a)–(c) and (f) for left modules. We leave the details to the reader.

Let $\varepsilon_{n-2f} : \mathscr{H}_{n-2f} \to \mathscr{B}_{n-2f}/\langle E_1 \rangle$ be the isomorphism in (2.9). Let $\sigma_f : \mathscr{H}_{n-2f} \to \mathscr{B}_n^f/\mathscr{B}_n^{f+1}$ be the $\kappa$-linear map defined by

$$\sigma_f(h) = E^{f,n}\varepsilon_{n-2f}(h) + \mathscr{B}_n^{f+1}.$$

Enyang [6, Coro. 3.4] proved that $\sigma_f(hg_w) = \sigma_f(h)T_w$ for all $h \in \mathscr{H}_{n-2f}$ and $w \in \mathfrak{S}_{n-2f}$. Enyang [6] used $E_1E_3 \cdots E_{2f-1}$ to define $\mathscr{B}_n^f$. We use $E_{n-1}E_{n-3} \cdots E_{n-2f+1}$ to define $\mathscr{B}_n^f$. Therefore, we have to make some modification.

It is well known that $x_{\mathbf{s}\mathbf{t}^\lambda} + \mathscr{H}_{n-2f}^{\rhd\lambda}$ can be considered as a $\kappa$-basis of $S^\lambda$ for all $\mathbf{s} \in \mathscr{T}_{n-2f}^{std}(\lambda)$. By abusing of notation, we write $x_\mathbf{s} = x_{\mathbf{s}\mathbf{t}^\lambda} \in \mathscr{H}_{n-2f}$. Then $\varepsilon_{n-2f}(x_\mathbf{s}) \in \mathscr{B}_{n-2f}/\langle E_1 \rangle$.

Let $\mathrm{Rad}\, S^\lambda$ be the Jacobson radical of $S^\lambda$. Since we are assuming that $\lambda$ is *e*-restricted, $\mathrm{Rad}\, S^\lambda$ is the same as the radical of the invariant form $\phi_\lambda$ on $S^\lambda$. By abusing notation, we denote by $\varepsilon_{n-2f}(\mathrm{Rad}\, S^\lambda) \subset \mathscr{B}_{n-2f}/\langle E_1 \rangle$ all elements $\varepsilon_{n-2f}(h)$ such that $h = \sum a_\mathbf{s} x_\mathbf{s}$ and $h + \mathscr{H}_{n-2f}^{\rhd\lambda} \in \mathrm{Rad}\, S^\lambda$. Similarly, we define $\varepsilon_{n-2f}(S^\lambda) \subset \mathscr{B}_{n-2f}/\langle E_1 \rangle$.

Let *M* (resp. $M_1$) be the $\kappa$-subspace generated by $T_u^* E^{f,n}\varepsilon_{n-2f}(h) + \mathscr{B}_n^{\rhd(f,\lambda)}$, $u \in D_{f,n}$ and $h \in \mathscr{H}_{n-2f}$ such that $\varepsilon_{n-2f}(h) \in \varepsilon_{n-2f}(S^\lambda)$ (resp. $\varepsilon_{n-2f}(h) \in \varepsilon_{n-2f}(\mathrm{Rad}\, S^\lambda)$). It is routine to check that $M \cong \Delta(f, \lambda)$ and $M_1$ is a $\mathscr{B}_n$-submodule of *M* such that $M/M_1 \cong P_n(D^\lambda, \ell)$.

Let *N* (resp. $N_1$) be the $\kappa$-subspace generated by $T_u^* E^{f+1,n+2}\varepsilon_{n-2f}(h) + \mathscr{B}_{n+2}^{\rhd(f,\lambda)}$, $u \in D_{f+1,n+2}$ and $h \in \mathscr{H}_{n-2f}$ such that $\varepsilon_{n-2f}(h) \in \varepsilon_{n-2f}(S^\lambda)$ (resp. $\varepsilon_{n-2f}(h) \in \varepsilon_{n-2f}(\mathrm{Rad}\, S^\lambda)$). It is routine to check that $N \cong \Delta(f+1, \lambda)$ and $N_1$ is a $\mathscr{B}_{n+2}$-submodule of *N* such that $N/N_1 \cong P_{n+2}(D^\lambda, \ell)$. In order to prove (d), we have to prove

$$\mathscr{B}_{n+2}E_{n+1} \otimes_{\mathscr{B}_n} M/M_1 \cong N/N_1.$$

We remark that $x_{\mathbf{t}^\lambda} + \mathscr{H}_{n-2f}^{\rhd\lambda} \not\in \operatorname{Rad} S^\lambda$ since $S^\lambda$ is the cyclic $\mathscr{H}_{n-2f}$-module generated by $x_{\mathbf{t}^\lambda} + \mathscr{H}_{n-2f}^{\rhd\lambda}$. This implies that $M/M_1$ (resp. $N/N_1$) is also the cyclic module generated by $E^{f,n}\varepsilon_{n-2f}(x_{\mathbf{t}^\lambda}) + M_1$ (resp. $E^{f+1,n+2}\varepsilon_{n-2f}(x_{\mathbf{t}^\lambda}) + N_1$).

By standard arguments, we define the $\mathscr{B}_{n+2}$-homomorphism

$$\psi : \mathscr{B}_{n+2}E_{n+1} \otimes_{\mathscr{B}_n} M/M_1 \to N/N_1$$

such that

$$\psi(hE_{n+1} \otimes_{\mathscr{B}_n} E^{f,n}\varepsilon_{n-2f}(x_{\mathbf{t}^\lambda}) + M_1) = hE^{f+1,n+2}\varepsilon_{n-2f}(x_{\mathbf{t}^\lambda}) + N_1.$$

Since $N/N_1$ is the cyclic module generated by $E^{f+1,n+2}\varepsilon_{n-2f}(x_{\mathbf{t}^\lambda}) + N_1$, $\psi$ is surjective. Write

$$E = E_{n-2}E_{n-4}\cdots E_{n-2f}.$$

Then $E^{f,n} = E^{f,n}EE^{f,n}$. Therefore,

$$\mathscr{B}_{n+2}E_{n+1} \otimes_{\mathscr{B}_n} M/M_1 = \mathscr{B}_{n+2}E^{f+1,n+2} \otimes EE^{f,n}\varepsilon_{n-2f}(x_{\mathbf{t}^\lambda}) + M_1.$$

By [20, 2.7] for $\mathscr{B}_{n+2}$, each element in $\mathscr{B}_{n+2}E^{f+1,n+2}$ can be written as a linear combination of $T_v^* E^{f+1,n+2}\mathscr{B}_{n-2f}$ and $v \in \mathscr{D}_{f+1,n+2}$ (Yu proved this result for cyclotomic Birman–Murakami–Wenzl algebras $\mathscr{B}_{m,n}$ of type $G(m, 1, n)$. What we need is the special result for $m = 1$). So,

$$\dim \mathcal{G}(P_n(D^\lambda, \ell)) \le \dim P_{n+2}(D^\lambda, \ell),$$

forcing $\psi$ to be injective. This proves (d). Finally, (e) follows from (a) and (d). □

**Proposition 4.4.** *Suppose $e \le n-2$. If $r$, $q$ are singular, then $r = \pm q^a$ for some $a \in \mathbb{Z}$.*

**Proof.** If $r$, $q$ are singular, then $N_{\phi_f}(D^\lambda, \ell) \ne 0$ for some positive integer $f \le \lfloor n/2 \rfloor$, some irreducible $\mathscr{H}_{n-2f}$-module $D^\lambda$ and some basis element $v_\ell \in V_f$. In fact, the singularity of $r$ and $q$ are independent of $v_\ell$. By Proposition 3.11(b), $\operatorname{Rad}\Delta(f, \lambda) \ne 0$. Let $D^{\ell,\mu}$ be a composition factor of $N_{\phi_f}(D^\lambda, \ell) \ne 0$. Then $(\ell, \mu) \lhd (f, \lambda)$. Further, there is a submodule $M$ of $P(D^\lambda, \ell)$ such that $\operatorname{Hom}(\Delta(\ell, \mu), P(D^\lambda, \ell)/M) \ne 0$.

Suppose $\ell = f$. Applying the exact functor $\mathcal{F}$ to both $\Delta(f, \mu)$ and $P(D^\lambda, \ell)/M$ repeatedly and using Lemma 4.3, we get a non-zero homomorphism from $\Delta(0, \mu)$ to $D^\lambda/\mathcal{F}^f(M)$. Since we are assuming that $D^\lambda$ is an irreducible $\mathscr{H}_{n-2f}$-module, we have $F^f(M) = 0$. In other words, there is a non-zero epimorphism from $\Delta(0, \mu)$ to $D^\lambda$, forcing $\mu \unrhd \lambda$. This contradicts $(\ell, \mu) \lhd (f, \lambda)$. Therefore, $\ell < f$.

We use induction on $n$ to prove the result. We have $n \ge 4$ since $2 \le e \le n-2$. The case $n = 4$ can be verified directly. Suppose $n > 4$. By assumption, we have a non-zero homomorphism from $D^{\ell,\mu}$ to $\Delta(f, \lambda)/N$ for some submodule $N \subset \Delta(f, \lambda)$. Applying the exact functor $\mathcal{F}$ and using Lemma 4.3, we can assume $\ell = 0$.

Suppose $f = 1$. Enyang [6] defined the Jucys–Murphy elements $L_i$, $1 \le i \le n$ for $\mathscr{B}_n$ such that $L_1 = r$ and $L_i = T_{i-1}L_{i-1}T_{i-1}$. Further, he proved that $L_1, \ldots, L_n$ commute each other and $\prod_{i=1}^n L_i$ is a central element of $\mathscr{B}_n$. Therefore, $\prod_{i=1}^n L_i$ acts on each cell module (and hence its non-zero quotient modules) as a scalar. We use it to act on both $D^{0,\mu}$ to $\Delta(1, \lambda)/N$. Since there is a nonzero homomorphism from $D^{0,\mu}$ to $\Delta(1, \lambda)/N$, and since any cell module of a cellular algebra is indecomposable, both modules belong to the same block. By [15, Theorem 2.2],

$$\prod_{p\in[\lambda]} c(p) = \prod_{p\in[\mu]} c(p), \tag{4.5}$$

where $c(p)$ is defined in (3.4). Since $\lambda \in \Lambda^+(n-1)$ and $\mu \in \Lambda^+(n)$, we can use the definition of $c(p)$ to solve for $r$, which shows that $r \in \{q^a, -q^a\}$ for some integer $a$, as required.

Suppose $f > 1$. As $\mathscr{B}_{n-1}$-module, $D^{0,\mu}$ may be reducible. Further, each composition factor is of form $D^{0,\eta}$ for some partition $\eta$ with $|\eta| = |\mu| - 1$. As $\mathscr{B}_{n-1}$-module, $\Delta(f, \lambda)$ has $\Delta$-filtration [6, Coro. 5.8]. By Lemma 4.2, $D^{0,\eta}$ has to be a composition factor of $\Delta(f_1, \tilde{\lambda})$ for some suitable $(f_1, \tilde{\lambda}) \in \Lambda_{n-1}$ such that $f_1$ is either $f$ or $f-1$. In particular, $f_1 \ne 0$. Further, in the first case, $\tilde{\lambda}$ can be obtained from $\lambda$ by adding an addable node. In the second case, $\tilde{\lambda}$ can be obtained from $\lambda$ by adding an addable node. By induction assumption on $n-1$, $r = \pm q^a$ for some $a \in \mathbb{Z}$. □

**Proposition 4.6.** *If $e \le n-2$ and $r = \pm q^a$ for some integer $a$, then $r$ and $q$ are singular.*

**Proof.** Since $e \le n-2$, we can assume $r = \pm q^a$ for some non-negative integer $a$ with $a < e$. What we want to do is find some suitable $\lambda \in \Lambda^+(n-2f)$ with $f > 0$ such that $S^\lambda = D^\lambda$. We remark that this can be verified by Theorem 3.6 for $f = 0$. We will define another $(\ell, \mu) \in \Lambda_n$. One can use the Definition 3.5 to verify that $\mu$ is $(f-\ell, \lambda)$-admissible. By Theorem 3.6 and Proposition 3.11, $\det G_{f,\lambda} = 0$ and $N_{\phi_f}(D^\lambda, \ell) \cong \operatorname{Rad}\Delta(f, \lambda) \ne 0$ for some $\ell$, $1 \le \ell \le \dim V_f$. So, $r$ and $q$ are singular.

In the remaining part of this proof, we will use Theorem 3.6 to construct $(\ell, \mu)$ and $(f, \lambda)$ explicitly. We define $b = a+1$. So, $1 \le b \le e \le n-2$. Since $o(q^2) = e$, we have $q^e = -1$. We will use this fact frequently in the remaining part of the proof.

**Case 1.** $r = \pm q^a$ **and** $r \not\in \{q^{-1}, -q\}$:

We define $(f,\lambda)=(\frac{n-b}{2},(1^b))$, and $(\ell,\mu)=(\frac{n-b-2}{2},(2,1^b))$ if $n-b$ is even. Otherwise, there are several subcases we have to discuss.

**Subcase 1.** $b\notin\{e-1,e-2\}$.

If $b\neq n-3$, then $b<n-3$. Otherwise, $b=e=n-2$ forcing $2\mid n-b$, a contradiction. Therefore, $\frac{n-b-3}{2}\geq 1$. We define $(\ell,\mu)=(\frac{n-b-5}{2},(3,2,1^b))$, and $(f,\lambda)=(\frac{n-b-3}{2},(2,2,1^{b-1}))$.

If $b=n-3$, we have $e=b=n-3$ and $r=\pm q^a=\pm q^{-1}$. Otherwise, $e=n-2$ forcing $b=e-1$, a contradiction. If $\mathrm{char}(\kappa)=2$, there is nothing to be proved. Otherwise, since we are assuming $r\neq q^{-1}$, we have $r=-q^{-1}$.

If $n$ is even, then $b$ and $e$ have to be odd. So, $e>2$. If $n$ is odd, then $b$ is even forcing $b\geq 2$ and $n\geq 5$. We define

$$(\ell,\mu)=\begin{cases}\left(\frac{n-4}{2},(3,1)\right), & \text{if } 2\mid n,\\ \left(\frac{n-5}{2},(2^2,1)\right), & \text{otherwise,}\end{cases}$$

and

$$(f,\lambda)=\begin{cases}\left(\frac{n-2}{2},(2)\right), & \text{if } 2\mid n,\\ \left(\frac{n-3}{2},(1^3)\right), & \text{otherwise.}\end{cases}$$

**Subcase 2.** $b=e-2$. We have $r=\pm q^{-3}$, $e\geq 3$ and $n\geq 5$.

If $n$ is even, then $n\geq 6$ and $2\nmid e$. We define

$$(\ell,\mu)=\begin{cases}\left(\frac{n-6}{2},(5,1)\right), & \text{if } e>4,\\ \left(\frac{n-3}{2},(2,1)\right), & \text{if } e=3,\end{cases}$$

and

$$(f,\lambda)=\begin{cases}\left(\frac{n-4}{2},(4)\right), & \text{if } e>4,\\ \left(\frac{n-1}{2},(1)\right), & \text{if } e=3.\end{cases}$$

If $n$ is odd, then $n\geq 7$. Otherwise, $n=5$ and $e=3$, which contradicts $2\nmid n-b$. We define

$$(\ell,\mu)=\begin{cases}\left(\frac{n-7}{2},(4,2,1)\right), & \text{if } e\neq 5,\\ \left(\frac{n-5}{2},(2,1^3)\right), & \text{if } e=5,\end{cases}$$

and

$$(f,\lambda)=\begin{cases}\left(\frac{n-5}{2},(3,1^2)\right), & \text{if } e\neq 5,\\ \left(\frac{n-3}{2},(1^3)\right), & \text{if } e=5.\end{cases}$$

**Subcase 3.** $b=e-1$. We have $r=\pm q^{-2}$, $e\geq 2$ and $n\geq 4$.

Suppose $n$ is odd. We have $2\nmid e$ and $n\geq 5$. We define

$$(\ell,\mu)=\begin{cases}\left(\frac{n-5}{2},(3,2)\right), & \text{if } e\neq 3,\\ \left(\frac{n-5}{2},(2^2,1)\right), & \text{if } e=3,\end{cases}$$

and

$$(f,\lambda)=\begin{cases}\left(\frac{n-3}{2},(2,1)\right), & \text{if } e\neq 3,\\ \left(\frac{n-1}{2},(1)\right), & \text{if } e=3.\end{cases}$$

Suppose $n$ is even. We define

$$(\ell,\mu)=\begin{cases}\left(\frac{n-6}{2},(3^2)\right), & \text{if } n\geq 6,\\ (0,(2^2)), & \text{if } n=4,\end{cases}$$

and

$$(f,\lambda)=\begin{cases}\left(\frac{n-2}{2},(1^2)\right), & \text{if } n\geq 6,\\ (2,(0)), & \text{if } n=4.\end{cases}$$

This completes the proof of our result for $r=\pm q^a$ and $r\notin\{q^{-1},-q\}$.

**Case 2.** $r\in\{q^{-1},-q\}$:

**Subcase 1.** $e=2$. Then $n\geq 4$ and $r\in\{q,-q\}$.

If $n$ is even, we define $(\ell,\mu)=(\frac{n-4}{2},(2,1^2))$ and $(f,\lambda)=(\frac{n-2}{2},(1^2))$.

If $n$ is odd, then $n\geq 5$. We define $(\ell,\mu)=(\frac{n-5}{2},(2^2,1))$ and $(f,\lambda)=(\frac{n-1}{2},(1))$.

**Subcase 2.** $e > 2$. Then $n \geq 5$.

Suppose $n$ is even. Then $n \geq 6$. We define

$$(\ell, \mu) = \begin{cases} \left(\frac{n-4}{2}, (3, 1)\right), & \text{if } r = q^{-1}, \\ \left(\frac{n-4}{2}, (2, 1^2)\right), & \text{if } r = -q, \end{cases}$$

and

$$(f, \lambda) = \begin{cases} \left(\frac{n-2}{2}, (2)\right), & \text{if } r = q^{-1}, \\ \left(\frac{n-2}{2}, (1^2)\right), & \text{if } r = -q. \end{cases}$$

Suppose $n$ is odd and $r = q^{-1}$. We define

$$(\ell, \mu) = \begin{cases} \left(\frac{n-7}{2}, (3^2, 1)\right), & \text{if } n \geq 7, e \neq 5, \\ \left(\frac{n-7}{2}, (3, 2^2)\right), & \text{if } n \geq 7, e = 5, \\ (0, (2, 1^3)), & \text{if } n = 5, e = 3, \end{cases}$$

and

$$(f, \lambda) = \begin{cases} \left(\frac{n-5}{2}, (3, 1^2)\right), & \text{if } n \geq 7 \text{ and } e \neq 5, \\ \left(\frac{n-5}{2}, (2^2, 1)\right), & \text{if } n \geq 7 \text{ and } e = 5, \\ (1, (1^3)), & \text{if } n = 5 \text{ and } e = 3. \end{cases}$$

Suppose $r = -q$. If $n$ is even, we define $(\ell, \mu) = (\frac{n-4}{2}, (2, 1^2))$ and $(f, \lambda) = (\frac{n-2}{2}, (1^2))$.
If $n$ is odd and $n \geq 7$, we define

$$(\ell, \mu) = \begin{cases} \left(\frac{n-7}{2}, (3, 2^2)\right), & \text{if } e \neq 5, \\ \left(\frac{n-7}{2}, (3^2, 1)\right), & \text{if } e = 5, \end{cases}$$

and

$$(f, \lambda) = \begin{cases} \left(\frac{n-5}{2}, (3, 1^2)\right), & \text{if } e \neq 5, \\ \left(\frac{n-5}{2}, (3, 2)\right), & \text{if } e = 5. \end{cases}$$

If $n = 5$, then $e = 3$. We define $(\ell, \mu) = (0, (1^5))$ and $(f, \lambda) = (1, (1^3))$. This completes the proof of our result for $r \in \{q^{-1}, -q\}$. □

**Proof of Theorem 2.17.** Theorem 2.17 follows from Propositions 4.1 and 4.4–4.6, immediately. □

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