Connections between Prime Divisors of Conjugacy Classes and Prime Divisors of $|G|$

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INTRODUCTION

All groups considered in this paper are finite. If $A$ is a non-empty finite set, let $|A|$ denote the number of distinct elements in $A$. If $n$ is a positive integer and $\prod_{i=1}^{k} p_i^{a_i} = n$ is the factorization of $n$ into distinct prime powers, let $w(n) = k$. If $x \in G$, $h_x = |G|/|G(x)|$ denotes the number of elements in the conjugacy class of $x$. For a group $G$, let $\alpha(G) = \max\{w(h_x) \mid x \in G\}$ and $\beta(G) = \{p \mid p$ a prime and $p \mid h_x$ for some $g \in G\}$. Let $\sigma(G) = \max\{w(\chi(1)) \mid \chi$ an irreducible character of $G\}$, and $\rho(G) = \{p \mid p$ a prime and $p \mid \chi(1)$ for some irreducible character $\chi$ of $G\}$.

Huppert has conjectured that if $G$ is a finite solvable group, then $|\beta(G)| \leq 2\sigma(G)$. This conjecture has been verified when $\sigma(G) = 1$ by Manz [4] and when $\sigma(G) = 2$ by Gluck [1]. It has also been shown [2] that $|\rho(G)| \leq 3\sigma(G) + 32$.

Recently there have been theorems showing a parallelism between results for characters and results for conjugacy classes. At the 1989 International Group Theory Conference in Bressanone, Professor Huppert asked whether this parallelism extended to relating $\beta(G)$ and $\alpha(G)$ in a way analogous to results relating $\rho(G)$ and $\sigma(G)$. There is some indication that this is possible since Chillag and Herzog [5] have shown $|\beta(G)| \leq 2\alpha(G)$ if $\alpha(G) = 1$. In this paper we find a parallel to the work of Gluck by showing $|\beta(G)| \leq 2\alpha(G)$ if $\alpha(G) = 2$. In particular, the following theorem is proved.

**THEOREM A.** Assume $G$ is a finite solvable group; then $|\beta(G)| \leq 4$ if $\alpha(G) = 2$.

It is well known that if $G$ is a finite group then $p \mid h_x$ for some $x \in G$ if and only if $p \mid |G/Z(G)|$. Therefore, Theorem A can be restated in the following form.
COROLLARY B. Assume \( G \) is a finite solvable group and \( h_x \) is divisible by at most two distinct primes for all \( x \in G \); then \( |G/Z(G)| \) has at most four distinct prime divisors.

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If \( \gamma \) is a set of primes and \( G \) is a solvable group, let \( G_{\gamma} \) and \( G_{\gamma'} \) denote Hall \( \gamma \)- (respectively Hall \( \gamma' \))-subgroups of \( G \). If \( H \triangleleft G \) and \( A \) is a non-empty subset in \( G \), let \( \bar{A} \) denote the homomorphic image of \( A \) in \( \bar{G} = G/H \). Since \( C_G(x) \leq C_G(\bar{x}) \) for \( x \in G \), \( h_x | h_{\bar{x}} \).

**Lemma 1.** Assume that \( G \) is a solvable group and \( r \) is a prime divisor of \( |G| \).

(i) If \( g \) is an \( r' \)-element which normalizes a non-trivial \( r \)-subgroup \( R \), then either \( r | h_g \) or \( [R, g] = 1 \).

(ii) If \( R \) is a minimal normal \( r \)-subgroup of \( G \) and \( v \) is a prime such that \( O_v(G/C_G(R)) \neq 1 \), then \( v | h_x \) for all \( x \in R^* \).

(iii) If \( H, K \) are subgroups such that \( R \leq C_G(H) \cup C_G(K) \), where \( R \) is an \( r \)-group, then \( R \leq C_G(H) \) or \( R \leq C_G(K) \).

**Proof.** (i) Since \( G \) is solvable, \( G = N_G(R_1) O_r(G) \), where \( R_1 \) is a Sylow \( r \)-subgroup of \( O_r(G) \) and \( C_G(R_1) \leq R_1 O_r(G) \) [3, Theorem 6.3.3]. Let \( \bar{G} = G/O_r(G) \) and \( \bar{A} \) denote the image of a set \( A \) in \( \bar{G} \). If \( \bar{g} \neq 1 \), then \( g \in N_G(\bar{R}_1) - C_G(\bar{R}_1) \). Therefore, \( \bar{R}_1 \triangle \bar{G} \) yields \( \bar{G}_r \leq C_G(\bar{g}) \) for any \( G_r \) of \( G \). Hence, \( r | h_\bar{g} \) and \( h_\bar{g} | h_g \) yields \( r | h_g \). If \( |g| = 1 \), then \( g \in O_r(G) \) and \( [g, R] = 1 \).

(ii) Let \( V \) be a Sylow \( v \)-subgroup of \( K \), where \( K/C_G(R) = O_v(G/C_G(R)) \); then \( K = VC_G(R) \) and \( G = N_G(V) C_G(R) \). It follows that \( C_K(V) \leq G \). Now \( C_K(V) \leq R \) and the minimality of \( R \) yield \( C_K(V) = 1 \). Thus, \( v \neq r \) and \( v | h_x \) for all \( x \in R^* \).

(iii) The result follows since for any group \( G \) if \( G = A \cup B \), where \( A \) and \( B \) are subgroups, it is well known that \( G \leq A \) or \( G \leq B \).

The proof of Theorem A relies heavily on the observation that if \( x \) and \( y \) are elements of co-prime order in \( G \) which commute, then \( h_{xy} = |G|/|C_G(x) \cap C_G(y)| \) implies that every prime divisor of \( h_x \) or \( h_y \) divides \( h_{xy} \).

We will say \( G \) satisfies Hypothesis A, if \( G \) is a counter-example of minimal order to Theorem A.

**Lemma 2.** Assume \( G \) satisfies Hypothesis A; then \( Z(G) = 1 \).
Proof. Suppose $p \mid |Z(G)|$ and let $1 \neq P_1$ be a minimal normal subgroup of $Z(G)$. Let $H \supset P_1$ be the subgroup of $G$ such that $H/P_1 = Z(G/P_1)$. Since $|G/P_1| < |G|$ and $G/P_1$ satisfies the Hypothesis of Theorem A, $|G/H|$ is divisible by at most four primes. Let $g$ be a $p'$-element of $H$; then $P_1 \leq Z(G)$ yields $\langle g \rangle \triangle G$. Now $[g, G] \leq \langle g \rangle \cap P_1$ implies that $g \in Z(G)$. Hence $H = Z(G) \times H_p$. Since $p \mid |H|$ for some $x \in G$ if and only if $p \mid |G/Z(G)|$, we may assume $H_p = G_p$ but $H_p \neq Z(G)$. Therefore, $G_p \triangle G$. Now $[G_p, G_{p'}] \leq P_1$ and $[P_1, G_{p'}] = 1$ yield $G = G_p \times G_{p'}$ [3, Theorem 5.5]. Hence, there is an $x \in G_p$ such that $p \mid |H_x|$. If $y \in G_{p'}$, then $h_{xy} = [G_p / C_{G_p} (y)]$. Hence, $ph_{xy} \mid h_{x_{xy}}$ and $h_{xy}$ is a prime power. By [5], $|G_{p'}| / |Z(G)_{p'}|$ has at most two prime divisors. Now $Z(G_{p'}) \leq Z(G)$ yields $w(|G/Z(G)|) \leq 3$, a contradiction.

Lemma 3. Assume $G$ satisfies Hypothesis A and $P$ is a non-trivial minimal normal $p$-subgroup of $G$; then $G_p = P$.

Proof. Let $\bar{G} = G/P$ and $\bar{A}$ denote the image of a set in $\bar{G}$. Since $|\bar{G}| < |G|$, $|\bar{G}/Z(\bar{G})|$ is divisible by at most four primes. Thus, there is a prime $t$ such that $\bar{G}_t = Z(\bar{G})$, but $t \mid |G/Z(G)| = |G|$. If $t = p$, then $G_p \triangle G$ and $G = G_p \times G_{p'}$. Now $Z(G) = 1$ and $P \leq Z(G_p)$ yield $[G_{p'}, G_p] = P$ and $G_p = C_{G_p}(G_{p'}) \times P$. Hence, $Z(C_{G_p}(G_{p'})) \leq Z(G) = 1$ yields $P = G_p$. Thus, we may assume $t \neq p$.

We next show $G = C_{G_p}(G_{p'}) \bar{G}_t \times \bar{G}_t$, $G = G_p \times G_{p'}$ yields $G = C_{G_p}(G_{p'})$. Now $Z(G) = 1$ implies that $(|C_{G_p}(G_{p'})|, t) = 1$. Since $G_{p'} C_{G_p}(G_{p'}) / C_{G_p}(G_{p'}) \leq Z(G/C_{G_p}(G_{p'}))$, if $G \neq C_{G_p}(G_{p'})$, then there is a prime $v \neq t$ such that $O_v(G/C_{G_p}(G_{p'})) \neq 1$. By Lemma 1(ii), $tv \mid h_{x_p}$ for $x_p \in P^*$. Thus, $(tv, p) = 1$. Let $r$ be any prime, such that $r \notin \{t, v, p\}$; then $tv \mid h_{x_p}$ implies that some $G_r \leq C_{G_r}(x_p)$. If $g \in G_r^*$, then $h_{x_p} \mid h_{x_{xy}}$ yields $h_{xy} = v^b t^a$. Since $G = C_{G_r}(G_p, a = 0$ and $h_{xy} = v^b > 1$. It follows that $P \leq G_p \leq C_{G_r}(G_p)$. Therefore, $C_{G_r}(P) \geq G_{(r, s)}$, where $r, s$ are distinct primes, $r, s \notin \{v, p, t\}$.

If $y$ is any $p'$-element of $C_{G_p}(G_r)$, then $h_{x_p} \mid h_{x_{xy}}$, $h_{x_{xy}}$ for $x_p \in P^*$ implies $h_{xy} = v^b t^a$. Now $G = C_{G_r}(G_r, P)$ again implies $h_{xy} = v^b > 1$. Hence, if $v \mid |C_{G_p}(G_r)|$, let $V$ be a Sylow $v$-subgroup of $C_{G_p}(G_r)$. Now $P \triangle G$ implies $C_{G_r}(P) \triangle G$. Hence $V$ is a normal subgroup of some $G_r$. Thus, there is an element $y \in Z(G_r) \cap V$. Now $h_{xy} = v^b > 1$ is a contradiction. Therefore $(|C_{G_p}(G_r)|, v) = 1$. Now $G_{(r, s)} \leq C_{G_r}(P) \triangle G$ yields $G_v \leq N_0(G_r) \cup N_0(G_s)$ for some $G_r, G_s$ of $G$. Since $G_r \cap C_{G_p}(P) = 1$, $p \mid h_{x_p}$ for $g \in G_r^*$. Hence, by Lemma 1(i), $g \in C_C(G_r) \cup C_{C_{G_r}(G_r)}$. Thus, $G_v \leq C_{C_{G_r}(G_r)} \cup C_{C_{G_r}(G_r)}$ and Lemma 1(iii) implies that we may assume $G_v \leq C_{G_r}(G_r)$. Hence, $v^b = h_{xy}$ for $y \in G_r^*$ yields $b = 0$. This contradicts $Z(G) = 1$. Therefore, $G = C_{G_p}(G_r)$.

$G = C_{G_p}(G_r) P = C_{G_p}(G_r) G_r$ yields $G \leq C_{G_p}(G_r) \times PG_r$. Let $u$ be any prime dividing $|C_{G_p}(G_r)|$; then every Sylow $u$ subgroup of $G$ centralizes $P$. It follows that if $u \mid |C_{G_p}(G_r) / C_{C_{G_r}(G_r)}(g)|$ for $g \in C_{G_r}(G_r, P)$, then $u \mid h_{xy}$ for
x ∈ P*. Now g centralizes all Sylow t-subgroups of G so ut|h_g. Hence, the number of elements in any $C_{\alpha}(G, P)$-class of $C_{\alpha}(G, P)^*$ is a prime power. By [5], at most two primes divide $|C_{\alpha}(G, P)/Z(C_{\alpha}(G, P))|$. However, $Z(C_{\alpha}(G, P)) \leq Z(G) = 1$ yields a contradiction.

**Proof of Theorem A.** Let γ denote the set of prime divisors of $|F(G)|$. By Lemma 3, $F(G) = \prod_{p \in \gamma} G_p$ is an abelian Hall γ-subgroup of G. Let $\overline{G} = G/F(G)$ and $\overline{A}$ denote the image of A in G. We first show $w(|\overline{G}|) \leq 2$.

If $\overline{x} \in \overline{G}^\times$, then $x \notin C_{\alpha}(F(G)) = F(G)$ implies that $p | h_x$ for some $p \in \gamma$ by Lemma 1. Now $|\overline{G}|, |F(G)| = 1$ so that $ph_\overline{x} | h_x$. It follows that $h_x$ is a prime-power whence $w(|\overline{G}/Z(\overline{G})|) \leq 2$. Therefore, $w(|\overline{G}|) \geq 3$ implies that there are distinct primes $u, t$ such that $u, t | |F(G)|$. Since $t, u \notin \gamma$, there are primes $p, q$ (conceivably $p = q$) in γ such that $O_t(G/C_{\alpha}(G_p)) \neq 1$ and $O_u(G/C_{\alpha}(G_q)) \neq 1$. Since $G_p$ and $G_q$ are minimal normal subgroups of G by Lemma 3, Lemma 1(ii) implies $t | h_x^q_p$ for every $x_p \in G_p^*$ and $u | h_x^q_q$ for every $x_q \in G_q^*$. If $p = q$, then $tu | h_{x_p}$. In either case, there is an $x \in F(G)$ such that $tu | h_x$. Let $v$ be a prime divisor of $|\overline{G}|$, where $v \notin \{t, u\}$. Since $tu | h_x$, $G_v \leq C_G(x)$ for some Sylow v-subgroup. Let $g \in G_v$.

Now $(u, v) = 1$ implies every prime divisor of $h_x$ and $h_x$ divides $h_{x_p}$. Thus, $tu | h_{x_p}$ implies that $h_x = tu^h$. Therefore, $|F(G)|, |C_G(g)|$ and $F(G)$ the Hall γ-subgroup implies $F(G) \leq C_G(g)$. This is a contradiction. Hence $w(|\overline{G}|) \leq 2$ and $|F(G)|$ is divisible by at least three primes.

Let $u$ be a prime divisor of $|\overline{G}|$; then there is a prime $p | |F(G)|$ such that $G_u \leq C_G(G_p)$. Let $r \in \gamma - \{p\}$, if $G_u \leq C_G(G_p) \cup C_G(G_r)$, then Lemma 1(i) and Lemma 1(iii) imply there is a $g \in G_u^*$ with $pr | h_g$. Therefore, $G_s \leq C_G(G_s)$, where $s \in \gamma - \{p, r\}$. Since $(s, u) = 1$, $pr | h_{xs}$ and $h_x | h_{xs}$ for $x \in G_s^*$ imply that $h_x$ is a $\{p, r\}$ number. However, $x \in F(G)$ and $F(G)$ abelian already yield $G_p G_r \leq C_G(x)$. Thus, $x \in Z(G) = 1$, which is a contradiction. Hence, $G_u \leq C_G(G_p) \cup C_G(G_r)$ and Lemma 1(iii) implies $G_u \leq C_G(G_r)$. Since $p$ was an arbitrary prime in $\gamma - \{p\}$, $G_u \leq C_G(F(G)_p)$.

If $w(|\overline{G}|) = 2$, there is a prime $v \neq u$ dividing $|\overline{G}|$. The same argument yields a prime $q \in \gamma$ such that $G_q \leq C_G(F(G)_q)$. Since $|\gamma| \geq 3$, there is a prime $r$ such that $G_r \leq F(G)_q \cap F(G)_p$. Now $F(G)$ abelian and $w(|\overline{G}|) \leq 2$ yield $G_r \lneq Z(G)$, a contradiction. If $w(|\overline{G}|) = 1$, then $F(G)_q \lneq Z(G)$ by the same argument, which is again a contradiction.

**REFERENCES**