JOURNAL OF ALGEBRA 15, 68–71 (1970)

Closure Homomorphisms

S. BURRIS

University of Waterloo, Waterloo, Ontario, Canada

Communicated by P. M. Cohn

Received April 28, 1969

The theory of closure homomorphisms will be patterned after the usual theory of homomorphisms for algebraic systems (cf. [3], [4]). In this paper we will see that the closure congruences submit to a simple axiomatization, and also if the closure space is algebraic, then we have a fundamental representation theorem.

A *closure space* (see [2]) is an ordered pair $(\mathbf{C}, S)$ where $\mathbf{C}$ is a closure operator on the set S (i.e., $\mathbf{C}$ is an extensive, monotone, and idempotent set mapping). If θ is a homomorphism from an algebra $(S, \mathscr{F})$ into an algebra $(S_1, \mathscr{F}_1)$ (for the definitions see [3], [4], etc.) and $(\mathbf{C}, S)$ and $(\mathbf{C}_1, S_1)$ are the *canonical* closure spaces associated with $(S, \mathscr{F})$ and $(S_1, \mathscr{F}_1)$ respectively (i.e., $\mathbf{C}(P)$ is the smallest subalgebra containing P, etc.), then we have the identity

$$\theta[\mathbf{C}(P)] = \mathbf{C}_1[\theta(P)] \quad \text{for all} \quad P \quad \text{in} \quad S.$$

We abstract and define a *closure homomorphism* to be a mapping θ from a closure space $(\mathbf{C}, S)$ to $(\mathbf{C}_1, S_1)$ satisfying the identity $\mathbf{C}_1\theta = \theta\mathbf{C}$. [It is interesting to note that had we started out with relational structures $(S, \mathscr{R})$ (see [3]) instead of algebras $(S, \mathscr{F})$, the corresponding condition would have been $\theta\mathbf{C} \subset \mathbf{C}_1\theta$, which for additive closures is precisely the study of continuous functions).]

I. Axiomatization of Closure Congruences

For any mapping θ from S into S_1 we know that $\theta^{-1}\theta$, considered as a subset of $S \times S$, is an equivalence relation. The equivalence relations on a closure space $(\mathbf{C}, S)$ which we form in this manner from closure homomorphisms will be called *closure congruences*, and the family of closure congruences for $(\mathbf{C}, S)$ will be denoted by $\mathfrak{K}(\mathbf{C}, S)$.

In the study of a general algebra $(S, \mathscr{F})$ we find that congruences are intrinsically characterized as equivalence relations which are subalgebras of the product algebra $(S, \mathscr{F}) \times (S, \mathscr{F})$ (see [3]). The following theorem will give an intrinsic characterization of closure congruences. [Where we consider an equivalence relation E to induce a set mapping $E(P) = \{y : (x, y) \in E$ for some x in $P\}$.]

LEMMA 1. *Let E be the equivalence relation associated with the closure homomorphism $\theta : (\mathbf{C}, S) \to (\mathbf{C}_1, S_1)$, (i.e.,$E = \theta^{-1}\theta$). Then $E\mathbf{C}E = E\mathbf{C}$.*

Proof. $E\mathbf{C}E = \theta^{-1}\theta\mathbf{C}\theta^{-1}\theta = \theta^{-1}\mathbf{C}_1\theta\theta^{-1}\theta = \theta^{-1}\mathbf{C}_1\theta = \theta^{-1}\theta\mathbf{C} = E\mathbf{C}$.

By simple examples we can show that not every equivalence relation will satisfy such an identity, so the above lemma gives a genuine restriction on the equivalence relations which are congruences.

LEMMA 2. *If $(\mathbf{C}, S)$ is a closure space and E is an equivalence relation on S, then the following are equivalent:*

(i) $E\mathbf{C}E = E\mathbf{C}$

(ii) $\mathbf{C}E\mathbf{C} = E\mathbf{C}$

(iii) $\mathbf{C}E \subset E\mathbf{C}$

Proof. From $E\mathbf{C}E = E\mathbf{C}$ follows $E\mathbf{C}E\mathbf{C} = E\mathbf{C}\mathbf{C} = E\mathbf{C}$, and since $E\mathbf{C} \subset \mathbf{C}E\mathbf{C} \subset E\mathbf{C}E\mathbf{C}$, we see that (i) $\Rightarrow$ (ii). Clearly (ii) $\Rightarrow$ (iii), and from $\mathbf{C}E \subset E\mathbf{C}$ follows $E\mathbf{C}E \subset EE\mathbf{C} = E\mathbf{C}$, so (iii) $\Rightarrow$ (i).

LEMMA 3. *Let $(\mathbf{C}, S)$ be a closure space and E an equivalence relation on S such that $E\mathbf{C}E = E\mathbf{C}$. Let $S_1 = S/E$ and define $\mathbf{C}_1$ to be $\mu\mathbf{C}\mu^{-1}$, where μ is the canonical map from S onto S_1. Then $\mathbf{C}_1$ is the unique closure on S_1 such that μ is a closure homomorphism with congruence E.*

Proof. $\mathbf{C}_1$ is clearly extensive and isotone. Also $\mathbf{C}_1^2 = \mu\mathbf{C}\mu^{-1}\mu\mathbf{C}\mu^{-1} = \mu\mathbf{C}E\mathbf{C}\mu^{-1} = \mu E\mathbf{C}\mu^{-1} = \mu\mu^{-1}\mu\mathbf{C}\mu^{-1} = \mu\mathbf{C}\mu^{-1} = \mathbf{C}_1$, so $\mathbf{C}_1$ is idempotent. To show that μ is a homomorphism we note that $\mu\mathbf{C} = \mu\mu^{-1}\mu\mathbf{C} = \mu E\mathbf{C} = \mu E\mathbf{C}E = \mu\mathbf{C}E = (\mu\mathbf{C}\mu^{-1})\mu = \mathbf{C}_1\mu$. The uniqueness of $\mathbf{C}_1$ is straightforward.

THEOREM 1. *If $(\mathbf{C}, S)$ is a closure space and E is an equivalence relation on S, then $E \in \mathfrak{K}(\mathbf{C}, S)$ if and only if $E\mathbf{C}E = E\mathbf{C}$.*

II. A REPRESENTATION THEOREM

LEMMA 4. *Let $E \in \mathfrak{K}(\mathbf{C}, S)$ and P, Q be subsets of S satisfying $E(P) \subset E(Q)$; then for any $x \in \mathbf{C}(P)$ there is a $y \in \mathbf{C}(Q)$ such that xEy.*

Proof. Note that $\mathbf{C}(P) \subset \mathbf{C}E(P) \subset \mathbf{C}E(Q) \subset E\mathbf{C}(Q)$.

LEMMA 5. *Suppose E is a congruence for the closure space $(\mathbf{C}, S)$ where $\mathbf{C}$ is algebraic* (i.e., $\mathbf{C}(P) = \bigcup\{\mathbf{C}(Q) : Q \subset P, Q \text{ finite}\}$). *Then there is an abstract algebra $(S, \mathscr{F})$ whose canonical closure is $\mathbf{C}$ and such that E is a congruence for $(S, \mathscr{F})$ in the sense of general algebra, and $\mathscr{F}$ consists solely of symmetrical operations.* [*An operator $f(x_1, ..., x_n)$ is symmetrical if its value is invariant under permutations of the arguments.*]

Proof. First, well-order S in such a manner that if $x < y$ and $x \not{E} y$, then for $x_1 E x$ and $y_1 E y$ we have $x_1 < y_1$. For each finite set Q in S and each $x \in \mathbf{C}(Q)$ define the map $f_{Q,x}$ by:

$$\begin{aligned} f_{Q,x}(x_1, ..., x_n) &= x \quad \text{if} \quad \{x_1, ..., x_n\} = Q \qquad [\text{where } n = \operatorname{Card}(Q)] \\ &= y \quad \text{if} \quad \{x_1, ..., x_n\} \neq Q,\ x_i E q_i, \text{ where} \\ &\qquad\quad \{q_1, ..., q_n\} = Q, \text{ and } y \text{ is an element} \\ &\qquad\quad \text{in } \mathbf{C}(\{x_1, ..., x_n\}) \text{ which is equivalent to } x \\ &= \inf\{x_1, ..., x_n\} \quad \text{otherwise.} \end{aligned}$$

In this definition we consider $\{x_1, ..., x_n\}$ to be a set (rather than an n-tuple). Also there is usually more than one candidate for y—select any one (the existence of such a y is guaranteed by Lemma 4). Clearly each $f_{Q,x}$ is a symmetric operator, and if each argument is replaced by an equivalent element, the values of the operator are equivalent (with respect to E). Then, considering the elements of $\mathbf{C}(\phi)$ as nullary operations, we obtain the desired algebra by letting

$$\mathscr{F} = \{f_{Q,x} : Q \subset S, x \in Q\} \cup \mathbf{C}(\phi).$$

THEOREM 2. *Let θ be a homomorphism from the algebraic closure space* $(\mathbf{C}, S)$ *onto the closure space* $(\mathbf{C}_1, S_1)$. *Then we can "fit" the spaces* $(\mathbf{C}, S)$ *and* $(\mathbf{C}_1, S_1)$ *with abstract algebras $(S, \mathscr{F})$ and $(S_1, \mathscr{F}_1)$ such that* (i) *θ is a homomorphism from $(S, \mathscr{F})$ to $(S_1, \mathscr{F}_1)$, and* (ii) *the algebraic structures induce the respective closures.*

Proof. Consider the congruence $\theta^{-1}\theta$ on $(\mathbf{C}, S)$ and apply Lemma 5 to obtain an algebra $(S, \mathscr{F})$ which induces $(\mathbf{C}, S)$ and for which $\theta^{-1}\theta$ is a congruence for $(S, \mathscr{F})$. Then define an algebra $(S_1, \mathscr{F}_1)$ by requiring that $g \in \mathscr{F}_1$ iff there is an $f \in \mathscr{F}$ such that

$$\theta f(x_1, ..., x_n) = g[\theta(x_1), ..., \theta(x_n)]$$

for all $x_1, ..., x_n \in S$. With this the theorem follows easily.

Acknowledgment

The author wishes to thank A. S. Davis for suggesting the possibility of an intrinsic characterization of closure congruences.

References

1. S. Burris, "Theory of Pre-Closures." Dissertation, University of Oklahoma, Norman, Oklahoma 73069, 1968.
2. S. Burris, Representation theorems for closure spaces. *Colloq. Math.* **XIX**, Fasc. 2 (1968).
3. P. M. Cohn, "Universal Algebra." Harper and Row, New York, 1965.
4. A. G. Kurosh, "Lectures on General Algebra." Chelsea, New York, 1963.